

UHF algebras by an example

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Contents

1 Introduction and preliminaries	1
2 A motivating example: the CAR algebra	2
3 UHF algebras: definitions and first properties	4
4 The supernatural number of a UHF algebra	6
References	10

1 Introduction and preliminaries

In his PhD thesis in 1960 ([2]), James Glimm introduced a class of C^* -algebras called *uniformly hyperfinite* (UHF); these are C^* -algebras that can be written as the closed union of an increasing chain of matrix subalgebras. There turn out to be interesting examples of these: the CAR algebra is a UHF algebra of interest in quantum mechanics. A particularly remarkable property of UHF algebras is that there is an invariant that completely captures the isomorphism class of a UHF algebra, called the *supernatural number*.

In this paper, I give an exposition of UHF algebras, the CAR algebra, and the supernatural number of a UHF algebra. My presentation is a simplified version of [1, Chapter III], which develops the more general class of approximately finite C^* -algebras, and views UHF algebras as examples of these.

Ideals are closed and two-sided; $0 \in \mathbb{N}$.

A preliminary definition and theorem:

Definition 1.1. Suppose \mathfrak{A} is a C^* -algebra; suppose $F_{ij} \in \mathfrak{A}$ for $i, j \in \{1, \dots, n\}$ satisfy:

1. $F_{ij}F_{k\ell} = \delta_{jk}F_{i\ell}$.
2. $F_{ij}^* = F_{ji}$.
3. $\sum_{i=1}^n F_{ii} = 1$.

Then the F_{ij} are $n \times n$ matrix units \mathfrak{A} .

Lemma 1.2. Suppose \mathfrak{A} is a C^* -algebra; suppose $F_{ij} \in \mathfrak{A}$ are $n \times n$ matrix units for \mathfrak{A} . Then $C^*(\{F_{ij} : i, j \in \{1, \dots, n\}\})$ is $*$ -isomorphic to $M_n(\mathbb{C})$.

Proof. It is clear from the given conditions that $\text{span}\{F_{ij} : i, j \in \{1, \dots, n\}\}$ is a unital $*$ -subalgebra of \mathfrak{A} of dimension n^2 ; hence since it is finite-dimensional it is closed, and is thus $C^*(\{F_{ij} : i, j \in \{1, \dots, n\}\})$. Furthermore, an identical proof to the case of $M_\ell(\mathbb{C})$ shows that $C^*(\{F_{ij} : i, j \in \{1, \dots, n\}\})$ is simple. But up to isomorphism the only simple finite-dimensional C^* -algebra of dimension n^2 is $M_n(\mathbb{C})$; the result follows. \square

[Lemma 1.2](#)

2 A motivating example: the CAR algebra

Before defining UHF algebras, we first study a prototype: the CAR algebra. This is a C^* -algebra that arises naturally in the context of quantum mechanics (as noted in [1, Example III.5.4]). We will show that it is the closed union of a chain of matrix subalgebras; generalizing this property will yield the definition of a UHF algebra.

Fix a separable Hilbert space \mathcal{H} and some continuous linear map $\alpha: \mathcal{H} \rightarrow \mathcal{B}(\mathcal{H})$ such that for all $x, y \in \mathcal{H}$ we have:

$$\alpha(x)\alpha(y) + \alpha(y)\alpha(x) = 0 \quad (\text{AC1})$$

$$\alpha(x)\alpha(y)^* + \alpha(y)^*\alpha(x) = \langle x, y \rangle I \quad (\text{AC2})$$

(These are called the *canonical anticommutation relations*; hence the name ‘‘CAR algebra’’.)

Fact 2.1 ([3, Lemma 6.5]). *Such \mathcal{H} and α exist.*

The *CAR algebra* is defined to be $\mathfrak{A} = C^*(\alpha(\mathcal{H}))$. (In principle this might depend on α and \mathcal{H} ; we will see in [Corollary 4.11](#) that it does not.)

Fix an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ for \mathcal{H} . For $n \in \mathbb{N}$ define $\mathfrak{A}_n = C^*(\alpha(e_0), \dots, \alpha(e_{n-1}))$.

Proposition 2.2. *There is a $*$ -isomorphism $\mathfrak{A}_n \cong M_{2^n}(\mathbb{C})$ (and hence a continuous $*$ -isomorphism, since they’re finite dimensional).*

The following proof appears in [1, Example III.5.4], though I have added some (perhaps unnecessary) details. Said details are somewhat unpleasant; the uninterested reader is encouraged to skip the proofs of the claims.

Proof. For $n \in \mathbb{N}$ we let $W_n = I - 2\alpha(e_n)^*\alpha(e_n)$. Note that AC1 implies $\alpha(x)^2 = 0$ for all $x \in \mathcal{H}$; hence by AC2 we have

$$(\alpha(e_n)^*\alpha(e_n))^2 = \alpha(e_n)^*\alpha(e_n)(I - \alpha(e_n)\alpha(e_n)^*) = \alpha(e_n)^*\alpha(e_n) - \alpha(e_n)^*\underbrace{\alpha(e_n)^2}_{=0}\alpha(e_n)^* = \alpha(e_n)^*\alpha(e_n)$$

Claim 2.3. *We have*

$$\begin{aligned} W_n\alpha(e_m) &= (-1)^{\delta_{mn}+1}\alpha(e_m)W_n \\ W_n\alpha(e_m)^* &= (-1)^{\delta_{mn}+1}\alpha(e_m)^*W_n \end{aligned}$$

and W_n is self-adjoint with $W_n^2 = I$.

Proof. Note by AC2 that $W_n = I - 2(\langle e_n, e_n \rangle I - \alpha(e_n)\alpha(e_n)^*) = 2\alpha(e_n)\alpha(e_n)^* - I$. Thus

$$\begin{aligned} W_n\alpha(e_n) &= (I - 2\alpha(e_n)^*\alpha(e_n))\alpha(e_n) \\ &= \alpha(e_n) - 2\alpha(e_n)^*\alpha(e_n)^2 \\ &= \alpha(e_n) \\ \alpha(e_n)W_n &= \alpha(e_n)(2\alpha(e_n)\alpha(e_n)^* - I) \\ &= -\alpha(e_n) \\ &= -W_n\alpha(e_n) \end{aligned}$$

Also if $m \neq n$ then $\langle e_m, e_n \rangle = 0$; so

$$\begin{aligned} W_n\alpha(e_m) &= (I - 2\alpha(e_n)^*\alpha(e_n))\alpha(e_m) \\ &= \alpha(e_m) - 2\alpha(e_n)^*\alpha(e_n)\alpha(e_m) \\ &= \alpha(e_m) + 2\alpha(e_n)^*\alpha(e_m)\alpha(e_n) \quad (\text{AC1}) \\ &= \alpha(e_m) + 2(\langle e_n, e_m \rangle I - \alpha(e_m)\alpha(e_n)^*)\alpha(e_n) \quad (\text{AC2}) \\ &= \alpha(e_m) - 2\alpha(e_m)\alpha(e_n)^*\alpha(e_n) \\ &= \alpha(e_m)(I - 2\alpha(e_n)^*\alpha(e_n)) \\ &= \alpha(e_m)W_n \end{aligned}$$

Also it is clear that $W_n^* = W_n$; hence

$$W_n \alpha(e_m)^* = (-1)^{\delta_{mn}+1} \alpha(e_m)^* W_n$$

follows by taking adjoints. Finally we have

$$W_n^2 = (I - 2\alpha(e_n)^* \alpha(e_n))(I - 2\alpha(e_n)^* \alpha(e_n)) = I - 4\alpha(e_n)^* \alpha(e_n) + 4(\alpha(e_n)^* \alpha(e_n))^2 = I$$

as desired. □ [Claim 2.3](#)

For $n \in \mathbb{N}$ let $V_n = W_0 \cdots W_{n-1} \in C^*(\alpha(e_1), \dots, \alpha(e_{n-1}))$.

Claim 2.4. *The elements*

$$\begin{aligned} E_{11}^{(n)} &= \alpha(e_n)^* \alpha(e_n) \\ E_{12}^{(n)} &= V_n \alpha(e_n) \\ E_{21}^{(n)} &= V_n \alpha(e_n)^* \\ E_{22}^{(n)} &= \alpha(e_n) \alpha(e_n)^* \end{aligned}$$

are 2×2 matrix units for \mathfrak{A} ; furthermore if $m \neq n$ then $E_{ij}^{(m)}$ commutes with $E_{kl}^{(n)}$ for all $i, j, k, \ell \in \{1, 2\}$.

Proof. Note by [Claim 2.3](#) that V_n commutes with $\alpha(e_n)$ and $\alpha(e_n)^*$, and that $V_n^* V_n = V_n V_n^* = I$. We now verify the properties of matrix units:

1. This is routine; we do two sample computations and leave the rest to the interested reader.

$$\begin{aligned} E_{11}^{(n)} E_{21}^{(n)} &= \alpha(e_n)^* \alpha(e_n) V_n \alpha(e_n)^* \\ &= V_n (I - \alpha(e_n) \alpha(e_n)^*) \alpha(e_n)^* \\ &= V_n \alpha(e_n)^* \\ &= E_{21}^{(n)} \\ E_{11}^{(n)} E_{12}^{(n)} &= \alpha(e_n)^* \alpha(e_n) V_n \alpha(e_n) \\ &= V_n \alpha(e_n)^* \alpha(e_n)^2 \\ &= 0 \end{aligned}$$

2. We have $(E_{12}^{(n)})^* = (V_n \alpha(e_n))^* = V_n \alpha(e_n)^* = E_{21}^{(n)}$, and we already saw that $(E_{11}^{(n)})^2 = (\alpha(e_n)^* \alpha(e_n))^2 = \alpha(e_n)^* \alpha(e_n) = E_{11}^{(n)}$; a similar proof yields that $(E_{22}^{(n)})^2 = E_{22}^{(n)}$.

3. This is just AC2.

For the last claim, we show that $V_n \alpha(e_n)$ commutes with $C^*(\alpha(e_0), \dots, \alpha(e_{n-1}))$; this will suffice, since $E_{ij}^{(n)} \in C^*(V_n \alpha(e_n))$ and $E_{ij}^{(k)} \in C^*(\alpha(e_0), \dots, \alpha(e_{n-1}))$ if $k < n$. But by [Claim 2.3](#) we get for $k < n$ that

$$\begin{aligned} V_n \alpha(e_n) \alpha(e_k) &= -W_0 \cdots W_{n-1} \alpha(e_k) \alpha(e_n) \\ &= \alpha(e_k) W_0 \cdots W_{n-1} \alpha(e_n) \\ &= \alpha(e_k) V_n \alpha(e_n) \end{aligned}$$

and similarly $V_n \alpha(e_n) \alpha(e_k)^* = \alpha(e_k)^* V_n \alpha(e_n)$. □ [Claim 2.4](#)

Claim 2.5. *The $E_{i_0 j_0}^{(0)} \cdots E_{i_{n-1} j_{n-1}}^{(n-1)}$ form $2^n \times 2^n$ matrix units for \mathfrak{A} .*

Proof. We verify the properties of matrix units.

1. By [Claim 2.4](#) we have

$$\begin{aligned} E_{i_0 j_0}^{(0)} \cdots E_{i_{n-1} j_{n-1}}^{(n-1)} \cdot E_{k_0 \ell_0}^{(0)} \cdots E_{k_{n-1} \ell_{n-1}}^{(n-1)} &= E_{i_0 j_0}^{(0)} E_{k_0 \ell_0}^{(0)} \cdots E_{i_{n-1} j_{n-1}}^{(n-1)} E_{k_{n-1} \ell_{n-1}}^{(n-1)} \\ &= \delta_{j_0 k_0} \cdots \delta_{j_{n-1} k_{n-1}} E_{i_0 \ell_0}^{(0)} \cdots E_{i_{n-1} \ell_{n-1}}^{(n-1)} \\ &= \delta_{(j_0, \dots, j_{n-1}), (k_0, \dots, k_{n-1})} E_{i_0 \ell_0}^{(0)} \cdots E_{i_{n-1} \ell_{n-1}}^{(n-1)} \end{aligned}$$

as desired.

2. This follows directly from [Claim 2.4](#).

3. We do this by induction on n . The base case is [Claim 2.4](#); for the induction step notice that

$$\sum_{i_0=1}^2 \cdots \sum_{i_n=1}^2 E_{i_0 i_0}^{(0)} \cdots E_{i_n i_n}^{(n)} = \left(\sum_{i_0=1}^2 \cdots \sum_{i_{n-1}=1}^2 E_{i_0 i_0}^{(0)} \cdots E_{i_{n-1} i_{n-1}}^{(n-1)} \right) \sum_{i_n=1}^2 E_{i_n i_n}^{(n)} = \sum_{i_n=1}^2 E_{i_n i_n}^{(n)} = I$$

by the induction hypothesis. □ [Claim 2.5](#)

Thus by [Lemma 1.2](#) we get that $C^*(\{E_{i_0 j_0}^{(0)} \cdots E_{i_{n-1} j_{n-1}}^{(n-1)}\})$ is *-isomorphic to $M_{2^n}(\mathbb{C})$. But $E_{ij}^{(k)} \in C^*(\alpha(e_0), \dots, \alpha(e_k))$; also

$$\alpha(e_\ell) = V_\ell^* V_\ell \alpha(e_\ell) = (I - 2E_{11}^{(\ell)}) E_{12}^{(\ell)}$$

and

$$E_{ij}^{(\ell)} = \sum_{i_0=1}^2 \cdots \sum_{i_{\ell-1}=1}^2 \sum_{i_{\ell+1}=1}^2 \cdots \sum_{i_{n-1}=1}^2 E_{i_0 i_0}^{(0)} \cdots E_{i_{\ell-1} i_{\ell-1}}^{(\ell-1)} E_{ij}^{(\ell)} E_{i_{\ell+1} i_{\ell+1}}^{(\ell+1)} \cdots E_{i_{n-1} i_{n-1}}^{(n-1)} \in C^*(\{E_{i_0 j_0}^{(0)} \cdots E_{i_{n-1} j_{n-1}}^{(n-1)}\})$$

So $C^*(\{\alpha(e_0), \dots, \alpha(e_{n-1})\}) = C^*(\{E_{i_0 j_0}^{(0)} \cdots E_{i_{n-1} j_{n-1}}^{(n-1)}\})$; so $\mathfrak{A}_n = C^*(\{\alpha(e_0), \dots, \alpha(e_{n-1})\})$ is *-isomorphic to $M_{2^n}(\mathbb{C})$, as desired. □ [Proposition 2.2](#)

Note, however, that

$$\mathfrak{A} = C^*(\alpha(\mathcal{H})) = \overline{\bigcup_{n \in \mathbb{N}} \text{alg}\{\alpha(e_0), \dots, \alpha(e_{n-1})\}} \subseteq \overline{\bigcup_{n \in \mathbb{N}} C^*(\alpha(e_0), \dots, \alpha(e_{n-1}))} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}$$

We have thus shown that there is a chain $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}$ of matrix subalgebras whose union is dense; this is the condition we will generalize when defining UHF algebras. This is a strong finiteness condition; we will see in [Theorem 4.7](#) that the behaviour of algebras with such a chain is tightly controlled by the behaviour within the chain.

3 UHF algebras: definitions and first properties

Definition 3.1. A *UHF algebra* is a unital C*-algebra \mathfrak{A} for which there exists a chain $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \cdots \subseteq \mathfrak{A}$ of unital subalgebras (i.e. subalgebras containing the unit of \mathfrak{A}) such that

- each \mathfrak{A}_k is *-isomorphic to a matrix algebra $M_{m_k}(\mathbb{C})$, and
- $\mathfrak{A} = \overline{\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k}$.

So in particular the CAR algebra is a UHF algebra.

Some properties:

Remark 3.2. Since linear maps, and in particular *-homomorphisms, between finite dimensional spaces are continuous, we get that the isomorphism $\mathfrak{A}_k \cong M_{m_k}(\mathbb{C})$ is continuous; so we may assume that topological properties are preserved as well.

Remark 3.3. UHF algebras are separable. Indeed, suppose

$$\mathfrak{A} = \overline{\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k}$$

is a UHF algebra. Let $A_k \subseteq \mathfrak{A}_k$ be the image of $M_{m_k}(\mathbb{Q})$ under the given isomorphism $M_{m_k}(\mathbb{C}) \rightarrow \mathfrak{A}_k$; so A_k is countable and dense in \mathfrak{A}_k . Then

$$A = \bigcup_{k \in \mathbb{N}} A_k$$

is dense in

$$\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k$$

and hence also in its closure \mathfrak{A} . But A is countable; so \mathfrak{A} is separable.

Proposition 3.4. *UHF algebras are simple.*

Proof. Suppose

$$\mathfrak{A} = \bigcup_{k \in \mathbb{N}} \mathfrak{A}_k$$

is a UHF algebra; suppose $\mathfrak{J} \subsetneq \mathfrak{A}$ is an ideal.

Claim 3.5. $\mathfrak{J} \cap \mathfrak{A}_k = \{0\}$ for all $k \in \mathbb{N}$.

Proof. We first check that $\mathfrak{J} \cap \mathfrak{A}_k$ is an ideal of \mathfrak{A}_k . It is clear that $\mathfrak{J} \cap \mathfrak{A}_k$ is a vector subspace of \mathfrak{A}_k that is closed in the relative topology. Suppose $a, b \in \mathfrak{A}_k$ and $c \in \mathfrak{J} \cap \mathfrak{A}_k$. Then $acb \in \mathfrak{A}_k$ since \mathfrak{A}_k is a subalgebra and $acb \in \mathfrak{J}$ since \mathfrak{J} is an ideal; so $acb \in \mathfrak{J} \cap \mathfrak{A}_k$, and $\mathfrak{J} \cap \mathfrak{A}_k$ is an ideal of \mathfrak{A}_k .

But $\mathfrak{A}_k \cong M_{m_k}(\mathbb{C})$, and $M_{m_k}(\mathbb{C})$ is simple. So $\mathfrak{J} \cap \mathfrak{A}_k$ is either $\{0\}$ or all of \mathfrak{A}_k .

Case 1. Suppose there is some $k \in \mathbb{N}$ such that $\mathfrak{J} \cap \mathfrak{A}_k = \{0\}$. Then for all $\ell < k$ we immediately get that $\mathfrak{J} \cap \mathfrak{A}_\ell \subseteq \mathfrak{J} \cap \mathfrak{A}_k = \{0\}$; also for $\ell > k$ with $\mathfrak{A}_\ell \supsetneq \mathfrak{A}_k$ we get that $\mathfrak{J} \cap \mathfrak{A}_\ell \subseteq (\mathfrak{A}_\ell \setminus \mathfrak{A}_k) \cup \{0\} \subsetneq \mathfrak{A}_\ell$, and thus $\mathfrak{J} \cap \mathfrak{A}_\ell = \{0\}$. The claim then follows.

Case 2. Suppose $\mathfrak{J} \cap \mathfrak{A}_k = \mathfrak{A}_k$ for all $k \in \mathbb{N}$. Then \mathfrak{J} is a closed set containing the dense set

$$\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k$$

So $\mathfrak{J} = \mathfrak{A}$, contradicting our assumption that \mathfrak{J} was proper. □ [Claim 3.5](#)

Consider the quotient map $q: \mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{J}$; we will show that q is injective. Note that $q \upharpoonright \mathfrak{A}_k$ is injective: indeed, $\ker(q \upharpoonright \mathfrak{A}_k) = \ker(q) \cap \mathfrak{A}_k = \mathfrak{J} \cap \mathfrak{A}_k = \{0\}$. We can thus define a continuous linear map $\varphi_k: q(\mathfrak{A}_k) \rightarrow \mathfrak{A}_k$ such that $\varphi_k \circ q = \text{id}_{\mathfrak{A}_k}$. This φ_k is uniquely determined, and thus the φ_k form a chain; so we can define

$$\varphi_\omega = \bigcup_{k \in \mathbb{N}} \varphi_k: \bigcup_{k \in \mathbb{N}} q(\mathfrak{A}_k) \rightarrow \bigcup_{k \in \mathbb{N}} \mathfrak{A}_k$$

But this φ_ω is a continuous linear map, and thus extends to

$$\varphi: \overline{\bigcup_{k \in \mathbb{N}} q(\mathfrak{A}_k)} \rightarrow \mathfrak{A}$$

But if $q(a) \in \mathfrak{A}/\mathfrak{J}$ and $\varepsilon > 0$ then there is some $k \in \mathbb{N}$ and $a_0 \in \mathfrak{A}_k$ such that $\|a - a_0\| < \varepsilon$; so $\|q(a) - q(a_0)\| \leq \|a - a_0\| < \varepsilon$. So

$$\bigcup_{k \in \mathbb{N}} q(\mathfrak{A}_k)$$

is dense in $\mathfrak{A}/\mathfrak{J}$; so $\varphi: \mathfrak{A}/\mathfrak{J} \rightarrow \mathfrak{A}$. But by construction $\varphi \circ q \upharpoonright \mathfrak{A}_k = \text{id}_{\mathfrak{A}_k}$; so by continuity of $\varphi \circ q$ we get that $\varphi \circ q = \text{id}_{\mathfrak{A}}$. So q is injective, and $\mathfrak{J} = \ker(q) = \{0\}$. □ [Proposition 3.4](#)

4 The supernatural number of a UHF algebra

In this section we will justify our earlier claim that the behaviour of a UHF algebra is tightly controlled by the behaviour of its matrix subalgebras; in fact in [Theorem 4.7](#) we will give an invariant that completely classifies the isomorphism type of a UHF algebra.

To understand the structure of a UHF algebra, it behooves us to study the structure of unital $*$ -embeddings between matrix algebras: the embeddings $M_{m_k}(\mathbb{C}) \rightarrow M_{m_{k+1}}(\mathbb{C})$ induced by the inclusions $\mathfrak{A}_k \hookrightarrow \mathfrak{A}_{k+1}$ will determine how the $M_{m_k}(\mathbb{C})$ “fit together” in \mathfrak{A} . Since unital $*$ -embeddings between matrix algebras can be viewed as non-degenerate representations, we first study the representation theory of matrix algebras.

Proposition 4.1. *All irreducible representations of $M_n(\mathbb{C})$ are unitarily equivalent to the identity representation $\text{id}: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$.*

Proof. We know from assignment 3 that an irreducible representation is a surjective homomorphism $M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})$, and the kernel is a maximal ideal of $M_n(\mathbb{C})$, and thus 0. So the only irreducible representations are $*$ -automorphisms $\rho: M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$.

Note that the E_{ii} must get sent to pairwise orthogonal rank 1 projections; we thus get an orthonormal basis f_1, \dots, f_n for \mathbb{C}^n such that $\rho(E_{ii}) = f_i f_i^* = F_{ii}$. But then also $\rho(E_{ij}) = \rho(E_{ii} E_{ij} E_{jj}) = F_{ii} \rho(E_{ij}) F_{jj}$, so since ρ preserves spectra we get $\rho(E_{ij}) = f_i f_j^* = F_{ij}$. Then the map $U e_i = f_i$ is unitary, and $\rho(E_{ij}) = F_{ij} = f_i f_j^* = U e_i e_j^* U^* = U E_{ij} U^*$ for all i, j ; so $\rho(A) = U A U^*$ for all A . So ρ is unitarily equivalent to the identity. \square [Proposition 4.1](#)

We can use this to classify all unital $*$ -embeddings between matrix algebras.

Corollary 4.2. *Suppose $\varphi: M_m(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is a unital $*$ -homomorphism. Then $m \mid n$ and there is a unitary $U \in M_n(\mathbb{C})$ such that $\varphi(A) = U^*(A \oplus \dots \oplus A)U$ for all $A \in M_m(\mathbb{C})$.*

Proof. Since φ is unital we get that φ defines a non-degenerate representation of a finite-dimensional C^* -algebra, and is thus completely reducible. (Indeed, one checks that the orthogonal complement of a subrepresentation is also a subrepresentation.) Hence by previous proposition we get that $\varphi = \rho_1 \oplus \dots \oplus \rho_k$ with each ρ_k unitarily equivalent to $\text{id}: M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C})$; so $n = km$ and $m \mid n$. If U_1, \dots, U_k are unitaries in $M_n(\mathbb{C})$ such that $\rho_i(A) = U_i A U_i^*$, then $U = U_1 \oplus \dots \oplus U_k \in M_n(\mathbb{C})$ is the desired unitary. \square [Corollary 4.2](#)

Suppose now that

$$\mathfrak{A} = \overline{\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k}$$

is a UHF algebra. Then the inclusions $\mathfrak{A}_k \rightarrow \mathfrak{A}_{k+1}$ induce unital $*$ -homomorphisms $M_{m_k}(\mathbb{C}) \rightarrow M_{m_{k+1}}(\mathbb{C})$; so $m_0 \mid m_1 \mid \dots$. Consider the prime factorizations of the m_k : let

$$m_k = \prod_{p \text{ prime}} p^{v_p(m_k)}$$

Then since the $m_k \mid m_{k+1}$ we get that the $(v_p(m_k))_k$ are increasing sequences of natural numbers; so each one is either eventually constant or diverges to infinity. Let $e_p = \sup_k v_p(m_k) \in \mathbb{N} \cup \{\infty\}$.

Definition 4.3. The *supernatural number* associated to \mathfrak{A} is the formal product

$$\delta(\mathfrak{A}) = \prod_{p \text{ prime}} p^{e_p}$$

(Of course $\delta(\mathfrak{A})$ is only a true natural number if $\mathfrak{A} \cong M_{\delta(\mathfrak{A})}(\mathbb{C})$.)

In principle our definition of $\delta(\mathfrak{A})$ may depend on the choice of chain $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \dots$; in fact it depends only on \mathfrak{A} .

Theorem 4.4. *$\delta(\mathfrak{A})$ is well-defined.*

We will need a lemma whose proof I omit; I direct the interested reader to [[1](#), Lemma III.3.2], of which the following is a weakening:

Lemma 4.5. *Suppose \mathfrak{D} is a C^* -algebra with a finite-dimensional C^* -subalgebra \mathfrak{A} . Then there is $\delta > 0$ such that if $\mathfrak{B} \subseteq \mathfrak{D}$ is a C^* -subalgebra with $\text{dist}(a, \mathfrak{B}) < \delta$ for all $a \in b_1(\mathfrak{A})$ then there is a unitary $u \in \mathfrak{D}$ such that $u^*\mathfrak{A}u \subseteq \mathfrak{B}$.*

Proof of Theorem 4.4. Suppose

$$\mathfrak{A} = \overline{\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k} = \overline{\bigcup_{\ell \in \mathbb{N}} \mathfrak{B}_\ell}$$

with

$$\begin{aligned} \mathfrak{A}_k &\cong M_{m_k}(\mathbb{C}) \\ \mathfrak{B}_\ell &\cong M_{n_\ell}(\mathbb{C}) \end{aligned}$$

Claim 4.6. *For all k there is ℓ such that $m_k \mid n_\ell$. (And vice-versa.)*

Proof. Fix k ; let δ be as in Lemma 4.5. Since \mathfrak{A}_k is finite-dimensional and contained in

$$\overline{\bigcup_{\ell \in \mathbb{N}} \mathfrak{B}_\ell}$$

there is ℓ such that $\text{dist}(a, \mathfrak{B}_\ell) < \delta$ for all $a \in \mathfrak{A}_k$. (Indeed, there is ℓ such that $\text{dist}(a, \mathfrak{B}_\ell) < m_k^{-2}\delta$ whenever $a \in \mathfrak{A}_k$ is the image of one of the standard matrix units in $M_{m_k}(\mathbb{C})$; the triangle inequality then yields the desired bound.) So by Lemma 4.5 there is unitary $u \in \mathfrak{A}$ such that $u^*\mathfrak{A}_k u \subseteq \mathfrak{B}_\ell$. But then the map $a \mapsto u^* a u$ is a unital $*$ -homomorphism $\mathfrak{A}_k \rightarrow \mathfrak{B}_\ell$, and thus induces a unital $*$ -homomorphism $M_{m_k}(\mathbb{C}) \rightarrow M_{n_\ell}(\mathbb{C})$; so by Corollary 4.2 we get that $m_k \mid n_\ell$. The “vice-versa” follows by symmetry. \square Claim 4.6

So repeatedly applying the above claim we get subsequences k_0, k_1, \dots and ℓ_0, ℓ_1, \dots such that $m_{k_0} \mid n_{\ell_0} \mid m_{k_1} \mid n_{\ell_1} \mid \dots$. Thus for any prime p we have

$$v_p(m_{k_0}) \leq v_p(n_{\ell_0}) \leq v_p(m_{k_1}) \leq v_p(n_{\ell_1}) \leq \dots$$

Thus

$$\sup_k v_p(m_k) = \sup\{v_p(m_{k_0}), v_p(m_{k_1}), \dots\} \leq \sup\{v_p(n_{\ell_0}), v_p(n_{\ell_1}), \dots\} = \sup_\ell v_p(n_\ell)$$

(where the equalities follow because $(v_p(m_k))_k$ and $(v_p(n_\ell))_\ell$ are increasing sequences). We likewise get $\sup_\ell v_p(n_\ell) \leq \sup_k v_p(m_k)$. So $\delta(\mathfrak{A})$ computed with respect to the \mathfrak{A}_k agrees with $\delta(\mathfrak{A})$ computed with respect to the \mathfrak{B}_ℓ . \square Theorem 4.4

We have shown that $\delta(\mathfrak{A})$ is an invariant depending only on \mathfrak{A} ; in fact it completely characterizes the isomorphism class of \mathfrak{A} .

Theorem 4.7. *If $\mathfrak{A}, \mathfrak{B}$ are UHF algebras with $\delta(\mathfrak{A}) = \delta(\mathfrak{B})$ then there is a continuous $*$ -isomorphism $\mathfrak{A} \rightarrow \mathfrak{B}$.*

Theorem 4.7 may seem unremarkable: it seems intuitive that the unions of “similar” chains of subalgebras should be isomorphic. This intuition hides the complexity that can arise from the arrangement of the subalgebras; indeed, the following fact shows that this intuition fails quite badly in even a slightly more general class of C^* -algebras.

Fact 4.8 ([1, Example III.3.7]). *There exist C^* -algebras \mathfrak{A} and \mathfrak{B} with chains of unital subalgebras*

$$\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \dots \subseteq \mathfrak{A}$$

and

$$\mathfrak{B}_0 \subseteq \mathfrak{B}_1 \subseteq \dots \subseteq \mathfrak{B}$$

such that

- each \mathfrak{A}_k and \mathfrak{B}_ℓ is finite-dimensional (and thus isomorphic to a direct sum of matrix algebras), and

$$\bullet \mathfrak{A} = \overline{\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k} \text{ and } \mathfrak{B} = \overline{\bigcup_{\ell \in \mathbb{N}} \mathfrak{B}_\ell}$$

with each $\mathfrak{A}_k \cong \mathfrak{B}_k$ but $\mathfrak{A} \not\cong \mathfrak{B}$.

Having hopefully convinced the reader that this is indeed a remarkable theorem, we proceed to its proof. This proof appears in [1, Theorem III.5.2].

Proof of Theorem 4.7. Write

$$\mathfrak{A} = \overline{\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k} \quad \mathfrak{B} = \overline{\bigcup_{\ell \in \mathbb{N}} \mathfrak{B}_\ell}$$

with each $\mathfrak{A}_k \cong M_{m_k}(\mathbb{C})$ and each $\mathfrak{B}_\ell \cong M_{n_\ell}(\mathbb{C})$.

Claim 4.9. *For all k there is ℓ such that $m_k \mid n_\ell$. (And vice-versa.)*

Proof. Suppose p is prime with $v_p(m_k) \neq 0$. Then since $\delta(\mathfrak{A}) = \delta(\mathfrak{B})$ we get

$$v_p(m_k) \leq \sup_k v_p(m_k) = \sup_\ell v_p(n_\ell)$$

So there is ℓ_p such that $v_p(m_k) \leq v_p(n_{\ell_p})$. Let ℓ be the maximum over such p of ℓ_p . Then for such p we have

$$v_p(m_k) \leq v_p(n_{\ell_p}) \leq v_p(n_\ell)$$

So $m_k \mid n_\ell$. The ‘‘vice versa’’ again follows by symmetry. □ Claim 4.9

Thus using the above claim we can drop to a subsequence of the \mathfrak{A}_k and a subsequence of the \mathfrak{B}_ℓ such that $m_0 \mid n_0 \mid m_1 \mid n_1 \mid \dots$. (Note that since the \mathfrak{A}_k form a chain dropping to a subsequence won’t change the union; likewise with the \mathfrak{B}_ℓ .)

Claim 4.10. *There exists unital *-homomorphisms $\varphi_k: \mathfrak{A}_k \rightarrow \mathfrak{B}_k$ and $\psi_k: \mathfrak{B}_k \rightarrow \mathfrak{A}_{k+1}$ such that $\psi_k \circ \varphi_k: \mathfrak{A}_k \rightarrow \mathfrak{A}_{k+1}$ and $\varphi_{k+1} \circ \psi_k: \mathfrak{B}_k \rightarrow \mathfrak{B}_{k+1}$ are the inclusion mappings. i.e. we require that the following diagram commute:*

$$\begin{array}{ccccccc} \mathfrak{A}_0 & \hookrightarrow & \mathfrak{A}_1 & \hookrightarrow & \mathfrak{A}_2 & \hookrightarrow & \dots \\ \downarrow \varphi_0 & \nearrow \psi_0 & \downarrow \varphi_1 & \nearrow \psi_1 & \downarrow \varphi_2 & \nearrow \psi_2 & \\ \mathfrak{B}_0 & \hookrightarrow & \mathfrak{B}_1 & \hookrightarrow & \mathfrak{B}_2 & \hookrightarrow & \dots \end{array}$$

Proof. Since $m_0 \mid n_0$ there is a natural unital *-homomorphism $M_{m_0}(\mathbb{C}) \rightarrow M_{n_0}(\mathbb{C})$, namely $A \mapsto A \oplus \dots \oplus A$; let $\varphi_0: \mathfrak{A}_0 \rightarrow \mathfrak{B}_0$ be induced by this map.

Suppose we have defined φ_k . Since $n_k \mid m_{k+1}$ there is a natural unital *-homomorphism $M_{n_k}(\mathbb{C}) \rightarrow M_{m_{k+1}}(\mathbb{C})$; let ψ'_k be induced by this map. Then $\psi'_k \circ \varphi_k$ is a unital *-homomorphism $\mathfrak{A}_k \rightarrow \mathfrak{A}_{k+1}$, as is the inclusion map. So by Corollary 4.2 there is a unitary $u \in \mathfrak{A}_{k+1}$ such that $u^* \psi'_k(\varphi_k(a))u = a$ for all $a \in \mathfrak{A}_k$. Define $\psi_k: \mathfrak{B}_k \rightarrow \mathfrak{A}_{k+1}$ by $\psi_k(b) = u^* \psi'_k(b)u$; then by construction we have $\psi_k \circ \varphi_k$ is the inclusion $\mathfrak{A}_k \rightarrow \mathfrak{A}_{k+1}$.

The definition of φ_{k+1} assuming ψ_k has been defined is identical. □ Claim 4.10

Hence since the following diagram commutes:

$$\begin{array}{ccccccc} \mathfrak{A}_0 & \hookrightarrow & \mathfrak{A}_1 & \hookrightarrow & \mathfrak{A}_2 & \hookrightarrow & \dots \\ \downarrow \varphi_0 & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \\ \mathfrak{B}_0 & \hookrightarrow & \mathfrak{B}_1 & \hookrightarrow & \mathfrak{B}_2 & \hookrightarrow & \dots \end{array}$$

we get a well-defined continuous *-homomorphism

$$\bigcup_{k \in \mathbb{N}} \varphi_k: \bigcup_{k \in \mathbb{N}} \mathfrak{A}_k \rightarrow \mathfrak{B}$$

Since this is continuous and linear, it extends to a map

$$\varphi: \overline{\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k} = \mathfrak{A} \rightarrow \mathfrak{B}$$

Similarly we get $\psi: \mathfrak{B} \rightarrow \mathfrak{A}$ extending the ψ_k .

But for $a \in \mathfrak{A}_k$ we have $\psi(\varphi(a)) = \psi(\varphi_k(a)) = \psi_k(\varphi_k(a)) = a$; so $\psi \circ \varphi$ agrees with $\text{id}_{\mathfrak{A}}$ on a dense subset, and by continuity we get $\psi \circ \varphi = \text{id}_{\mathfrak{A}}$. We likewise get $\varphi \circ \psi = \text{id}_{\mathfrak{B}}$. So φ is a continuous *-isomorphism $\mathfrak{A} \rightarrow \mathfrak{B}$. \square [Theorem 4.7](#)

Corollary 4.11. *The CAR algebra is independent of the choice of \mathcal{H} and α .*

Proof. If \mathfrak{A} is a CAR algebra (i.e. constructed from some \mathcal{H} and α as detailed in [Section 2](#)) then recall from [Section 2](#) that

$$\mathfrak{A} = \overline{\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k}$$

where $\mathfrak{A}_k \cong M_{2^k}(\mathbb{C})$; so $\delta(\mathfrak{A}) = 2^\infty$. So any two constructions of the CAR algebra have the same supernatural number; so by [Theorem 4.7](#) we get that there is a continuous *-isomorphism between any two constructions of the CAR algebra. \square [Corollary 4.11](#)

Remark 4.12. Suppose we are given a supernatural number

$$\mathbf{n} = \prod_{p \text{ prime}} p^{e_p}$$

We construct a UHF algebra \mathfrak{A} with $\delta(\mathfrak{A}) = \mathbf{n}$. Pick a bijection $\Phi: \mathbb{N} \rightarrow P \times \mathbb{N}$, where P is the set of primes. Let $m_0 = 1$. Suppose we have chosen m_k ; write $\Phi(k) = (p, e)$. If $e > e_p$, we let $m_{k+1} = m_k$; else let $m_{k+1} = \text{lcm}(m_k, p^e)$. Note in particular that $m_k \mid m_{k+1}$ for all k .

Fix a separable Hilbert space \mathcal{H} with orthonormal basis $\{e_0, e_1, \dots\}$. For $n \geq 1$ and $0 \leq j < n$ let $P_j^{(n)} \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection with range $\text{span}\{e_\ell : \ell \equiv j \pmod{n}\}$; for $\delta \in \mathbb{Z}$ let $S_\delta \in \mathcal{B}(\mathcal{H})$ be

$$S_\delta e_k = \begin{cases} e_{k+\delta} & \text{if } k + \delta \geq 0 \\ 0 & \text{else} \end{cases}$$

Then for $i, j \in \{0, \dots, n-1\}$ the map $E_{ij}^{(n)} = S_{i-j} P_j^{(n)}$ is given by

$$e_k \mapsto \begin{cases} e_{qn+i} & \text{if } k = qn + j \text{ for some } q \\ 0 & \text{else} \end{cases}$$

In fact the $E_{ij}^{(n)}$ act as $n \times n$ matrix units: we have $E_{ij}^{(n)} E_{i'j'}^{(n)} = \delta_{ji'} E_{ij'}$ and $(E_{ij}^{(n)})^* = E_{ji}^{(n)}$ and

$$\sum_{i=0}^{n-1} E_{ii}^{(n)} = I$$

Thus by [Lemma 1.2](#) we get that $\mathfrak{M}_n = C^*(\{E_{ij}^{(n)} : i, j \in \{0, \dots, n-1\}\})$ is *-isomorphic to $M_n(\mathbb{C})$. Note also that if $n \mid m$ then

$$P_i^{(n)} = \sum_{j=0}^{\frac{m}{n}-1} P_{i+jn}^{(m)}$$

We thus get that $E_{ij}^{(n)} = S_{i-j} P_j^{(n)} \in \mathfrak{M}_m$ (by expanding using the above sum and distributing), and hence $\mathfrak{M}_n \subseteq \mathfrak{M}_m$.

Now if we let $\mathfrak{A}_k = \mathfrak{M}_{m_k} \subseteq \mathcal{B}(\mathcal{H})$ then since $m_k \mid m_{k+1}$ we get that $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \dots$; so if

$$\mathfrak{A} = \overline{\bigcup_{k \in \mathbb{N}} \mathfrak{A}_k}$$

then \mathfrak{A} is a UHF algebra.

Claim 4.13. $\delta(\mathfrak{A}) = \mathbf{n}$.

Proof. Suppose p is prime, and recall that

$$\mathbf{n} = \prod_{p \text{ prime}} p^{e_p}$$

We must then check that $e_p = \sup_k v_p(m_k)$.

(\leq) Suppose $e \in \mathbb{N}$ has $e \leq e_p$. Let $k = \Phi^{-1}(p, e)$; then by construction we have $p^e \mid m_{k+1}$, and hence $e \leq v_p(m_{k+1}) \leq \sup_k v_p(m_k)$. Since this holds for all natural $e \leq e_p$, we get that $e_p \leq \sup_k v_p(m_k)$.

(\geq) One checks by induction on k that if $e_p < e$ then $v_p(m_k) < e$; this is simply the nature of the construction of the m_k . Hence in particular e_p is an upper bound for the $v_p(m_k)$, and $e_p \geq \sup_k v_p(m_k)$. \square [Claim 4.13](#)

So for every supernatural \mathbf{n} there is a UHF algebra \mathfrak{A} with $\delta(\mathfrak{A}) = \mathbf{n}$. So the continuous isomorphism classes of UHF algebras are in bijection with the supernatural numbers.

References

- [1] Kenneth R. Davidson. *C*-Algebras by Example*. Hindustan Book Agency, 1996 (cit. on pp. [1](#), [2](#), [6–8](#)).
- [2] James G. Glimm. “On a Certain Class of Operator Algebras”. In: *Transactions of the American Mathematical Society* 95.2 (1960), pp. 318–340 (cit. on p. [1](#)).
- [3] Paul Skoufranis. *Separable Exact C*-Algebras Embed Into the Cuntz Algebra*. July 8, 2016. URL: <http://pskoufra.info.yorku.ca/files/2016/07/Separable-Exact-C-Algebras-Embed-Into-the-Cuntz-Algebras.pdf> (cit. on p. [2](#)).