Course notes for PMATH 863

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Lectures by Stephen New

Contents

1 Introduction

Lectures by Stephen New, office MC 5419, office hours 2:30-3:20 MWF (also after about 5:30 MWF if you tell him ahead of time).

Course outline found on his website.

Collaboration encouraged but acknowledge help (aside from him and books). (Write your own assignment though.) Assignments will be challenging, exam easier. (Foreknowledge of topics will be given for the exam.)

A somewhat vague introduction (formality later):

Definition 1. A *Lie grape* is both a C^∞ manifold and a grape G with smooth grape operations. (i.e. multiplication $m\colon G \times G \to G$ and inversion $v\colon G \to G$ are smooth).

Example 2.

- $\mathbb{R}^n$ under $+$
- $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ under component-wise multiplication
- $M_n(\mathbb{R})$ under $+$
- $\mathrm{GL}_n(\mathbb{R})$ under matrix multiplication

Definition 3. A *Lie algebra* is a vector space $\mathfrak{g}$ with an operation $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is alternating, bilinear, and satisfies the Jacobi identity. i.e. for $X, Y, Z \in \mathfrak{g}$ we have

- $[X, Y] = -[Y, X]$ (or equivalently, in the presence of bilinearity, $[X, X] = 0$).
- $[X, Y + Z] = [X, Y] + [X, Z]$ and $[X, cY] = c[X, Y]$, etc.
- $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

Example 4.

- $M_n(\mathbb{R})$ with $[X, Y] = XY - YX$.
- The set of smooth vector fields on a manifold M with $[X, Y] = XY - YX$ as differential operators.
- When G is a Lie grape the set of left-invariant vector fields on G is a Lie algebra, which we identify with $\mathfrak{g} = T_eG$; we call this the *Lie algebra of G*.

There is a map called the *exponential map* from $\mathfrak{g} = T_eG$ to G given (roughly) by taking a tangent vector $X \in T_eG$, using it to induce a left-invariant vector field X on all of G, finding the integral curve α of X with $\alpha(0) = e$ and $\alpha'(t) = X(\alpha(t))$, and setting $\exp(X) = \alpha(1)$.

One can show that $\exp\colon \mathfrak{g} \to G$ is a local diffeomorphism. For the classical matrix Lie grapes

$$\begin{aligned}
\mathrm{GL}(n, \mathbb{R}) &= \{\, A \in M_n(\mathbb{R}) : \det(A) \neq 0 \,\} \\
\mathrm{SL}(n, \mathbb{R}) &= \{\, A \in \mathrm{GL}_n(\mathbb{R}) : \det(A) = 1 \,\} \\
\mathrm{O}(n, \mathbb{R}) &= \{\, A \in \mathrm{GL}_n(\mathbb{R}) : A^T A = I \,\} \\
\mathrm{SO}(n, \mathbb{R}) &= \{\, A \in \mathrm{O}(n, \mathbb{R}) : \det(A) = 1 \,\} \\
\mathrm{U}(n) &= \{\, A \in \mathrm{GL}(n, \mathbb{C}) : A^* A = I \,\}
\end{aligned}$$

etc. we can identify the Lie algebra $\mathfrak{g}$ with a matrix algebra

$$\begin{aligned}
\mathfrak{gl}(n, \mathbb{R}) &= M_n(\mathbb{R}) \\
\mathfrak{sl}(n, \mathbb{R}) &= \{\, A \in M_n(\mathbb{R}) : \mathrm{tr}(A) = 0 \,\}
\end{aligned}$$

etc., and then $\exp\colon \mathfrak{g} \to G$ is given by

$$\exp(A) = e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

Definition 5. A *representation* of a grape is a grape homomorphism $\rho\colon G \to \mathrm{Perm}(X)$ for some set X, or $\rho\colon G \to \mathrm{GL}(n, \mathbb{R})$ or $\rho\colon G \to \mathrm{GL}(V)$ for some vector space V.

This gives an action of G on X or $\mathbb{R}^n$ or V: for $a \in G$ and $x \in X$ or $\mathbb{R}^n$ or V we write $a \cdot x = \rho(a)(x)$; this gives a G-module structure on V.

Given a representation $\rho\colon G \to \mathrm{GL}(V)$ we get $\rho = \mathrm{d}\rho\colon T_eG \to T_e \mathrm{GL}(V)$; i.e. $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$.

A representation $\rho\colon G \to \mathrm{GL}(V)$ induces a character $\chi\colon G \to \mathbb{R}$ given by $\chi(a) = \mathrm{tr}(\rho(a))$; one can show that for a Lie grape a representation is determined by its character. In the finite-dimensional case, one can always decompose a representation into irreducible subrepresentations.

1.1 Stuff we probably won't get to

A compact Lie grape G has a maximal torus T (i.e. a torus subgrape of maximal dimension) that is unique up to conjugation; its dimension is called the rank of the grape.

Example 6. SU(3) has maximal torus $T = \{\, \mathrm{diag}(\exp(i2\pi t_1), \exp(i2\pi t_2), \exp(i2\pi t_3)) : \sum t_i = 0 \,\}$, which has Lie algebra

$$\mathfrak{t} = \{\, \mathrm{diag}(t_1, t_2, t_3) : t_i \in \mathbb{R} \,\}$$

with $\exp(t_1, t_2, t_3) = (\exp(i2\pi t_1), \exp(i2\pi t_2), \exp(i2\pi t_3))$. A given representation $\rho\colon G \to \mathrm{GL}(n, \mathbb{R})$ reduces

TODO 1. *Is this $\mathbb{R}$ or $\mathbb{C}$?*

to give $\rho\colon T \to \mathrm{GL}(n, \mathbb{C})$. The irreducible representations of T are known, and are all 1-dimensional. The irreducible representations are classified by *weights* in $\mathfrak{t}$. The set of weights $\Omega \subseteq \mathfrak{t}$ is an integral lattice in $\mathfrak{t}$ related to the kernel of the exponential map. For SU(3) we have $\Omega = \ker(\exp) = \{\, \mathrm{diag}(k_1, k_2, k_3) : \text{ each } k_i \in \mathbb{Z}, \sum k_i = 0 \,\}$.

For $u_1 = \mathrm{diag}(1, -1, 0)$ and $u_2 = \mathrm{diag}(0, 1, -1)$ the "angle" is given by

$$\Theta(u_1, u_2) = \cos^{-1} \frac{u_1 \cdot u_2}{|u_1||u_2|} = \frac{2\pi}{3}$$

The integral span of these (ignoring the diag) gives a lattice of equilateral triangles. The weights of the adjoint representation are called roots: for $a \in G$ we define $c_a\colon G \to G$ by $c_a(x) = axa^{-1}$; the gives a map $\mathrm{d}c_a\colon \mathfrak{g} \to \mathfrak{g}$, which gives the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ (with $\mathrm{Ad}(a) = \mathrm{d}c_a$).

In SU(3) the weights of Ad are $\pm u_1, \pm u_2, \pm(u_1 + u_2)$.

2 Manifolds

Definition 7. Suppose M is a topological space.

- We say M is *second-countable* if there is a countable basis for the topology on M.
- We say M is *Hausdorff* if for all $p, q \in M$ there are disjoint open sets $U, V \subseteq M$ with $p \in U$ and $q \in V$.
- We say M is *locally homeomorphic to* $\mathbb{R}^n$ if for all $p \in M$ there is an open $U \subseteq M$ containing p, open $V \subseteq \mathbb{R}^n$, and a homeomorphism $\varphi\colon U \to V$.

  Such φ are called *(local coordinate) charts* on M at p. A set of charts whose domains cover M is called an *atlas* on M.

Remark 8. Note that when $\varphi\colon U \subseteq M \to \varphi(U) \subseteq \mathbb{R}^n$ and $\psi\colon V \subseteq M \to \psi(V) \subseteq \mathbb{R}^n$ are charts at p (so $p \in U \cap V$) then $\psi \circ \varphi^{-1}\colon \varphi(U \cap V) \to \psi(U \cap V)$ is a homeomorphism between two open sets in $\mathbb{R}^n$; such a map $\psi \circ \varphi^{-1}$ is called a *change in coordinates* map or a *transition map*.

Definition 9. An *n-dimensional topological manifold* is a topological space M that is separable, Hausdorff, and locally homeomorphic to $\mathbb{R}^n$.

Definition 10. An *n-dimensional smooth (or $\mathcal{C}^\infty$) manifold* is an n-dimensional topological manifold which has an atlas whose transition maps are $\mathcal{C}^\infty$.

Example 11. Some $\mathcal{C}^\infty$ manifolds:

- $\mathbb{R}^n$ (with one chart, the identity map)
- $\mathbb{S}^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \}$ (using, for example, the $2n+2$ charts

  $$\varphi_k\colon \{ (x_1, \ldots, x_{n+1}) \in \mathbb{S}^n : x_k > 0 \} \to B = \{ y \in \mathbb{R}^n : |y| < 1 \}$$

  given by $\varphi_k(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1})$ and

  $$\psi_k\colon \{ (x_1, \ldots, x_{n+1}) \in \mathbb{S}^{n+1} : x_k < 0 \} \to B$$

  given by the same formula).
- $\mathbb{P}^n = (\mathbb{R}^{n+1} \setminus \{ 0 \})/(\mathbb{R} \setminus \{ 0 \}) = \{ [x] : x \in \mathbb{R}^{n+1} \setminus \{ 0 \} \}$ where $[x] = \{ tx : 0 \neq t \in \mathbb{R} \}$ using the $n+1$ charts $\varphi_k\colon U_k \to \mathbb{R}^n$ where $U_k = \{ [x_1, \ldots, x_{n+1}] : x_k \neq 0 \}$ and

  $$\varphi([x_1, \ldots, x_{n+1}]) = \left(\frac{x_1}{x_k}, \ldots, \frac{x_{k-1}}{x_k}, \frac{x_{k+1}}{x_k}, \ldots, \frac{x_{n+1}}{x_k} \right)$$

- Every open subset of a manifold is also a manifold (of the same dimension). If N, M are manifolds then so is $N \times M$. In particular, we get

  $$\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$$

Remark 12. If M is both an n-dimensional and an m-dimensional manifold then $m = n$. (One can see this by looking at the Jacobians of the transition maps.) If M is n-dimensional we write $\dim(M) = n$.

Definition 13. Suppose N and M are $\mathcal{C}^\infty$ manifolds with $\dim(N) = n$ and $\dim(M) = m$. Suppose $f\colon N \to M$. We say that f is *smooth* or $\mathcal{C}^\infty$ at $p \in M$ when there is some (and hence for all) charts $\varphi\colon U \subseteq N \to \varphi(N) \subseteq \mathbb{R}^n$ and $\psi\colon V \subseteq M \to \psi(V) \subseteq \mathbb{R}^m$ with $p \in U$ and $f(p) \in V$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth ($\mathcal{C}^\infty$) at $\varphi(p)$.

In this case we define the *rank* of f at p to be equal to the rank of $D(\psi f \varphi^{-1})(\varphi(p))$. We sometimes denote the matrix $D(\psi f \varphi^{-1})(\varphi(p))$ by $Df(p)$.

There are a few different sensible notions of submanifold.

Definition 14. Let M be a smooth manifold. A *regular submanifold* of M is a subset $N \subseteq M$ which is a manifold such that for all $p \in N$ there are charts $\varphi\colon U \subseteq N \to \varphi(U) \subseteq \mathbb{R}^n$ at p on N and $\psi\colon V \subseteq M \to \psi(V) \subseteq \mathbb{R}^m$ at p on M such that

- $U = V \cap N$
- $\varphi(p) = 0$ and $\psi(p) = 0$ (if you want)
- For $x = (x_1, \ldots, x_n) \in \varphi(U) \subseteq \mathbb{R}^n$ we have $\psi\varphi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$.

Definition 15. Suppose N and M are $\mathcal{C}^\infty$-manifolds with $\dim(N) = n$, $\dim(M) = m$, and $n \leq m$. A function $f\colon N \to M$ is called an *immersion* when f is smooth and has maximal rank everywhere (i.e. rank $Df(p) = n$ for all $p \in N$). An *immersed submanifold* of M is the image $f(N)$ of some injective immersion; we use the topology and charts induced by the map f (so that $f\colon N \to f(N)$ is a diffeomorphism).

Note that immersions need not be injective. Note also that the topology on an immersed submanifold $N \subseteq M$ need not be the subspace topology inherited from M.

Definition 16. Suppose N and M are $\mathcal{C}^\infty$ manifolds with $\dim(N) = n$ and $\dim(M) = m$; suppose $f\colon N \to M$. We say that f is an *embedding* (or a *regular immersion*) when f is an injective immersion and the topology on $f(N)$ induced by the map f agrees with the subspace topology on $f(N)$. The image $f(N)$ of such an embedding $f\colon N \to M$ is called an *embedded submanifold* of M.

Example 17. Consider the map $f\colon (-\pi, \pi) \to \mathbb{R}^2$ given by $f(t) = (\sin(t), \sin(2t))$ (image looks like an infinity sign). The image is an immersed submanifold but not an embedded submanifold because of the behaviour around the origin.

Example 18. Consider $f\colon \mathbb{R} \to \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ given by $f(t) = (\exp(iat), \exp(ibt))$ with $a, b \in \mathbb{R}$. If $a \neq 0$ and $\frac{b}{a} \notin \mathbb{Q}$ then the image of f is dense in $\mathbb{T}^2$; this is an immersed manifold but not an embedded manifold.

Theorem 19. *Suppose $f\colon N \to M$ is an injective immersion of smooth manifolds; suppose that N is compact. Then f is an embedding.*

Proof. Consider $f(N)$ with the subspace topology. Suppose $K \subseteq N$ is closed (and hence compact, since N is compact). Since K is compact and $f\colon N \to f(N) \subseteq M$ is continuous, we get that $f(K)$ is compact. Since $f(K)$ is compact and M is Hausdorff, we get that $f(K)$ is closed. So f sends closed sets to closed sets; so, since $f\colon N \to f(N)$ is bijective, we get that f is open, and thus a homeomorphism $N \to f(N)$. $\square$ Theorem 19

Theorem 20 (Rank theorem)**.** *Suppose N, M are $\mathcal{C}^\infty$ manifolds with $\dim(N) = n$ and $\dim(M) = m$. Suppose $f\colon N \to M$ is a smooth map of constant rank r around p (i.e. $\operatorname{rank}(Df(p)) = r$ for all x in some neighbourhood of p). Then there exist a chart $\varphi\colon U \subseteq N \to \varphi(U) \subseteq \mathbb{R}^n$ at p on N and a chart $\psi\colon V \subseteq M \to \psi(V) \subseteq \mathbb{R}^m$ at $f(p)$ on M such that $\varphi(p) = 0$ and $\psi(p) = 0$ (if you want) and for all $x = (x_1, \ldots, x_n) \in \varphi(U)$ we have $\psi\varphi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_r, 0, \ldots, 0)$. (In particular, if the rank is globally constant, then for all $q \in f(N)$ we have $K = f^{-1}(q) = \{\, x \in N : f(x) = q \,\}$ is a closed regular embedded submanifold of M.)*

Corollary 21. *Every injective immersion $f\colon N \to M$ of smooth manifolds is locally an embedding.*

Corollary 22. *If $f\colon N \to M$ is an embedding of smooth manifolds then $f(N)$ is a regular submanifold.*

Remark 23. One can define variations on the definition of a manifold. For example, an n-dimensional *complex* or $\mathbb{C}$*-manifold* is a $2n$-dimensional topological manifold with charts such that the transition maps are all holomorphic. One could also define $\mathcal{C}^k$ or analytic manifolds.

Definition 24. A *Lie grape* is a set G which is both a $\mathcal{C}^\infty$ manifold and a grape such that the grape operations multiplication $m\colon G \to G$ and inversion $v\colon G \to G$ are smooth.

Example 25.

- $\mathbb{R}^n$ under addition

- $M_n(\mathbb{R})$ under addition
- $\mathbb{R}^*$ or $\mathbb{C}^*$ or $\mathbb{S}^1$ under multiplication
- $\mathbb{T}^n$ under (component-wise) multiplication
- $\mathrm{GL}(n, \mathbb{R}) = \{ A \in M_n(\mathbb{R}) : A \text{ is invertible} \}$ is a Lie grape under multiplication. Indeed, $M_n(\mathbb{R})$ is diffeomorphic to (and can be identified with) $\mathbb{R}^{n^2}$ using the map $F\colon M_n(\mathbb{R}) \to \mathbb{R}^2$ given by

$$F(u_1, \dots, u_n) = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

where each $u_k \in \mathbb{R}^n$. The determinant map $\varphi\colon M_n(\mathbb{R}) \to \mathbb{R}$ given by $\varphi(X) = \det(X)$ is a polynomial in the entries of X (and so is $\mathcal{C}^\infty$). So $\mathrm{GL}(n, \mathbb{R}) = \varphi^{-1}(\mathbb{R} \setminus \{ 0 \})$ is open in $M_n(\mathbb{R})$, and is thus endowed with a smooth structure.

Note as well that the map $m(X, Y) = XY$ is polynomial in the entries of X and Y, and is thus smooth; also

$$v(x) = \frac{1}{\det(X)} \operatorname{Adj}(X)$$

is a quotient of a polynomial by a non-zero polynomial in the entries of X, and is thus smooth.

Definition 26. Suppose H and G are Lie grapes. A map $f\colon H \to G$ is called a *Lie grape homomorphism* when f is a smooth grape homomorphism. (*Isomorphisms* and *isomorphic* are defined accordingly.)

Example 27.

- The map $F\colon M_n(\mathbb{R}) \to \mathbb{R}^{n^2}$ above given by

$$F(u_1, \dots, u_n) = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

is an isomorphism of Lie grapes.

- The determinant map $\varphi\colon\ \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^2$ is a Lie grape homomorphism.
- For $a \in \mathbb{R}^n$ we can define $\varphi\colon \mathbb{R} \to \mathbb{T}^n$ given by $\varphi(t) = (\exp(ia_1 t), \dots, \exp(ia_n t))$ and $\psi\colon \mathbb{R} \to \mathbb{T}^n$ given by $\psi(t_1, \dots, t_n) = (\exp(ia_1 t_1), \dots, \exp(ia_n t_n))$ are Lie grape homomorphisms.

Definition 28. Suppose G is a Lie grape. An *(immersed) Lie subgrape* of G is the image $\varphi(H)$ of a Lie grape homomorphism $\varphi\colon H \to G$ which is an immersion. An *embedded* (or *regular immersed*) Lie subgrape of G is the image $\varphi(H)$ of some Lie grape homomorphism $\varphi\colon H \to G$ which is an embedding.

Theorem 29. *Suppose H and G are Lie grapes; suppose $\varphi\colon H \to G$ is a homomorphism of Lie grapes. Then φ has constant rank.*

Proof. For $a \in H$ we let $\ell_a\colon H \to H$ be left-multiplication by a (so $\ell_a(x) = ax$ for $x \in H$). For all $a, x \in H$ we have $\varphi(ax) = \varphi(a)\varphi(x)$; i.e. $\varphi(\ell_a(x)) = \ell_{\varphi(a)}(\varphi(x))$. So, implicitly fixing charts, we apply chain rule to the above to get that

$$D\varphi(ax) \cdot D\ell_a(x) = D\ell_{\varphi(a)}(\varphi(x)) \cdot D\varphi(x)$$

Since ℓ_a and $\ell_{\varphi(a)}$ are diffeomorphisms (with inverses $\ell_{a^{-1}}$ and $\ell_{(\varphi(a))^{-1}}$, respectively), the matrices $D\ell_a(x)$ and $D\ell_{\varphi(a)}(\varphi(x))$ are invertible. So

$$\operatorname{rank}(D\varphi(ax)) = \operatorname{rank}(D\varphi(x))$$

for all $a, x \in H$. In particular, taking $a = x^{-1}$ gives $\operatorname{rank}(D\varphi(x)) = \operatorname{rank}(D\varphi(e))$ for all $x \in H$. $\square$ Theorem 29

Example 30. Show that the grapes

$$\begin{aligned}
\mathrm{SL}(n,\mathbb{R}) &= \{\, A \in \mathrm{GL}(n,\mathbb{R}) : \det(A) = 1 \,\} && \text{(special linear grape)}\\
\mathrm{O}(n,\mathbb{R}) &= \{\, A \in \mathrm{GL}(n,\mathbb{R}) : A^T A = I \,\} && \text{(orthogonal grape)}\\
\mathrm{SO}(n,\mathbb{R}) &= \{\, A \in \mathrm{O}(n,\mathbb{R}) : \det(A) = 1 \,\} && \text{(special orthogonal grape)}\\
\mathrm{GL}(n,\mathbb{C}) &= \{\, A \in M_n(\mathbb{C}) : A \text{ is invertible} \,\} && \text{(general linear grape over } \mathbb{C})\\
\mathrm{U}(n) &= \{\, A \in \mathrm{GL}(n,\mathbb{C}) : A^* A = I \,\} && \text{(unitary grape)}\\
\mathrm{SU}(n) &= \{\, A \in \mathrm{U}(N) : \det(A) = 1 \,\} && \text{(special unitary grape)}\\
\mathrm{GL}(n,\mathbb{H}) &= \{\, A \in M_n(\mathbb{H}) : A \text{ is invertible} \,\} && \text{(general linear grape over the quaternions)}\\
\mathrm{Sp}(n) &= \{\, A \in \mathrm{GL}(n,\mathbb{H}) : A^* A = I \,\} && \text{(symplectic grape)}
\end{aligned}$$

are regular Lie subgrapes of $\mathrm{GL}(m,\mathbb{R})$ for some m. (Here $A^* = (\overline{A})^T$ and $\mathbb{H} = \{\, a+bi+cj+dk : a,b,c,d \in \mathbb{R} \,\}$ with $i^2 = j^2 = k^2 = -1$ with $i^2 = j^2 = k^2 = ijk = -1$ and $ij = k, jk = i, ki = jk$.)

We do some sample computations:

($\mathrm{SL}(n,\mathbb{R})$) The determinant map $\varphi\colon \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^*$ is a Lie grape homomorphism and $\mathrm{SL}(n,\mathbb{R}) = \ker(\varphi)$. So $\mathrm{SL}(n,\mathbb{R})$ is a closed, regular Lie subgrape.

Exercise 31 (Possibly worthwhile). Compute the Jacobian of φ and show directly that the rank is 1.

($\mathrm{O}(n,\mathbb{R})$) Consider the map $\varphi\colon \mathrm{GL}(n,\mathbb{R})$ given by $\varphi(X) = X^T X$.

Claim 32. *φ has constant rank.*

Proof. For $A \in \mathrm{GL}(n,\mathbb{R})$ we let $L_A, R_A\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ be left- and right-multiplication by A, respectively. Then for $A, X \in \mathrm{GL}(n,\mathbb{R})$ we have

$$\varphi(R_A(x)) = \varphi(XA) = A^T X^T X A = L_{A^T}(R_A(\varphi(x)))$$

So by the chain rule (again implicitly fixing charts) we have

$$D\varphi(XA) D_{R_A}(X) = D_{L_{A^T}}(X^T X A) D_{R_A}(\varphi(X))$$

Then since R_A and L_{A^T} are diffeomorphisms we get that $\operatorname{rank}(D\varphi(XA)) = \operatorname{rank}(D\varphi(X))$ for all X. In particular for $A = X^{-1}$ we get $\operatorname{rank}(D\varphi(X)) = \operatorname{rank}(D\varphi(I))$. □ Claim 32

Hence $\mathrm{O}(n,\mathbb{R})$ is a closed regular Lie subgrape of $\mathrm{GL}(n,\mathbb{R})$ because $\mathrm{O}(n,\mathbb{R}) = \varphi^{-1}(I)$.

($\mathrm{SO}(n,\mathbb{R})$) It's the kernel of the determinant map.

Exercise 33. Check the rest.

Remark 34. We can also define *complex Lie grapes.* Some examples include $\mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C}), \mathrm{O}(n,\mathbb{C}), \mathrm{SO}(n,\mathbb{C}) = \{\, A \in M_n(\mathbb{C}) : A^T A = I, \det(A) = 1 \,\}, \mathrm{Sp}(2n,\mathbb{C})$.

Fact 35. *There are no compact complex Lie grapes.*

Exercise 36. Which of the real Lie grapes exhibited above are compact?

Definition 37. Suppose M is a $\mathcal{C}^\infty$ manifold of dimension n and $p \in M$. A *tangent vector* on M at p is a set of ordered pairs (φ, u) with one pair for each chart φ at p and each $u \in \mathbb{R}^n$ obtained from the following procedure: pick a smooth curve $\alpha\colon (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$, and define $\alpha'(0)$ to be the set of pairs (φ, u) where given a chart φ at p we let $u = \beta'(0)$ where $\beta(t) = \varphi(\alpha(t))$. The space of tangent vectors on M at p is denoted by $T_p M$.

Remark 38. When ψ is another chart and (ψ, v) is another pair induced by α, we have $v = \gamma'(0)$ where

$$\gamma(t) = \psi(\alpha(t)) = \psi(\varphi^{-1}(\beta(t))) = (\psi\varphi^{-1})(\beta(t))$$

so $\gamma'(t) = D(\psi\varphi^{-1})(\beta(t)) \cdot \beta'(t)$, and $v = \gamma'(0) = D(\psi\varphi^{-1})(\varphi(p))u$. Thus u and v are related by

$$v = D(\psi\varphi^{-1})(\varphi(p)) \cdot u$$
$$v_k = \sum_{i=1}^{n} \frac{\partial(\psi\varphi^{-1})_k}{\partial x_i} u_i$$

Definition 39. Suppose M is a $\mathcal{C}^\infty$ manifold and $p \in M$. A *derivation* on M at p is a linear map $D\colon \mathcal{C}_p^\infty(M, \mathbb{R}) \to \mathbb{R}$ such that $D(fg) = D(f) \cdot g + f \cdot D(g)$ for $f, g \in \mathcal{C}_p^\infty(M, \mathbb{R})$ where $\mathcal{C}_p^\infty(M, \mathbb{R})$ is the space of locally smooth functions on M at p; i.e. smooth functions $g\colon U \subseteq M \to \mathbb{R}$ where U is open in M with $p \in U$, and two such functions $g\colon U \subseteq M \to \mathbb{R}$ and $h\colon V \subseteq M \to \mathbb{R}$ are considered equivalent when they agree in some open $W \subseteq U \cap V$ with $p \in W$.

Remark 40. A tangent vector $X \in T_pM$ acts as a derivation on M at p as follows: choose a locally smooth curve $\alpha\colon (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = X$. Then we define $X(g) = h'(0)$ where $h(t) = g(\alpha(t))$.

Note that if X is given locally in the chart φ at p by $u \in \mathbb{R}^n$ then $u = \beta'(0)$ where $\beta(t) = \varphi(\alpha(t))$; so $h(t) = g(\alpha(t)) = (g\varphi^{-1})(\beta(t))$, and $h'(t) = D(g\varphi^{-1})(\beta(t)) \cdot \beta'(t)$. So $X(g) = h'(0) = D(g\varphi^{-1})(\varphi(p)) \cdot u$.

If we write $g\varphi^{-1}$ simply as g and $x = \varphi(p)$ then

$$X(g) = D(g\varphi^{-1})(\varphi(p)) \cdot u = \sum_{i=1}^{n} \frac{\partial(g\varphi^{-1})}{\partial x_i}(x) u_i$$

i.e.

$$X(g) = \sum_{i=1}^{n} u_i \frac{\partial g}{\partial x_i}$$

Because of this formula, it is customary to write the standard basis vectors in $\mathbb{R}^n$ (with $u \in \mathbb{R}^n$) as $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$; so

$$u = \sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}$$

Definition 41. Suppose $f\colon N \to M$ is a smooth map of smooth manifolds. Then f induces a linear map $f_*\colon T_pN \to T_{f(p)}M$ for each $p \in N$. (The map f_* is also denoted $\mathrm{d}f$ or Df.) Indeed, given $X \in T_pN$ we choose $\alpha\colon (-\varepsilon, \varepsilon) \to N$ with $\alpha(0) = p$ and $\alpha'(0) = X$, and then define $f_*X = \beta'(0)$ where $\beta(t) = f(\alpha(t))$.

TODO 2. *Roman d?*

Remark 42. If X is given locally in φ by $u \in \mathbb{R}^n$ then $u = \gamma'(0)$ where $\gamma(t) = \varphi(\alpha(t))$. Then $\beta(t) = f(\alpha(t)) = (f\varphi^{-1})(\gamma(t))$, and $\beta'(t) = D(f\varphi^{-1})(\gamma(t)) \cdot \gamma'(t)$; so

$$f_*X = \beta'(0) = D(f\varphi^{-1})(\varphi(p)) \cdot u$$

Theorem 43.

1. *Suppose $f\colon M \to N$ and $g\colon N \to L$ are smooth maps of smooth manifolds. Then $(g \circ f)_* = g_* \circ f_*\colon T_pM \to T_{g(f(p))}L$.*

2. *Suppose $f\colon N \to M$ is smooth; suppose $g\colon U \subseteq M \to \mathbb{R}$ where $U \subseteq M$ is open with $f(p) \in U$. (Or suppose $g\colon M \to \mathbb{R}$ is smooth.) Then for $X \in T_pN$ we have $(f_*X)(g) = X(g \circ f)$.*

Definition 44. Suppose M is a smooth manifold. A *vector field on* M is a map $X\colon M \to \bigcup_{p \in M} T_pM$ such that $X(p) \in T_pM$ for all $p \in M$. We sometimes write X_p to denote $X(p)$. A vector field X on M is given locally (in a chart $\varphi\colon U \subseteq M \to \varphi(U) \subseteq \mathbb{R}^n$) by a vector $u = u(x) \in \mathbb{R}^n$ at each point $x \in \varphi(U)$. We say that X is *continuous (or smooth, or $\mathcal{C}^k$))* when for some (hence for every) chart φ the resulting function $u(x)$ is continuous (or smooth, or $\mathcal{C}^k$). The space of all smooth vector fields on M is denoted $\Gamma(M, TM)$.

Remark 45. When $f\colon N \to M$ is a smooth map of smooth manifolds and $X \in \Gamma(N, TN)$, we don't necessarily have a well-defined vector field on M: if f is not injective we might have $p \neq q$ in N wth $f(p) = f(q)$ but $f_* X_p \neq f_* X_q$ in $T_{f(p)}M = T_{f(q)}M$. If f is surjective then $f_* X$ is well-defined as a vector field on $f(N) \subseteq M$. If $f\colon N \to M$ is a diffeomorphism then f_* gives a well-defined map $\Gamma(N, TN) \to \Gamma(M, TM)$. If $f\colon N \to M$ is an injective immersion then f is a smooth diffeomorphism as a map $f\colon N \to f(N)$ (where the latter is endowed with the topology and smooth structure induced from N via f).

Theorem 46. *Suppose M is a smooth manifold; suppose $X, Y \in \Gamma(M, TM)$. Then there exists a (unique) smooth vector field Z on M such that $Z(g) = X(Y(g)) - Y(X(g))$ for all smooth maps $g\colon M \to \mathbb{R}$.*

Proof. Suppose X, Y are given locally in a chart $\varphi\colon U \to \varphi(U)$ by vectors $u, v \in \mathbb{R}^n$. Write $x = \varphi(p)$ and $g\varphi^{-1}$ as g. Then

$$\begin{aligned}
X(Y(g)) - Y(X(g)) &= \sum_{i=1}^n u_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n v_j \frac{\partial g}{\partial x_j}\right) - \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^n u_j \frac{\partial g}{\partial x_j}\right) \\
&= \sum_{i,j} u_i \left(\frac{\partial v_j}{\partial x_i}\frac{\partial g}{\partial x_j} + v_j \frac{\partial^2 g}{\partial x_i \partial x_j}\right) - \sum_{i,j} v_i \left(\frac{\partial u_j}{\partial x_i}\frac{\partial g}{\partial x_j} + u_j \frac{\partial^2 g}{\partial x_i \partial x_j}\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial v_j}{\partial x_i} u_i - \frac{\partial u_j}{\partial x_i} v_i\right) \frac{\partial g}{\partial x_j}
\end{aligned}$$

Thus $X(Y(g)) - Y(X(g)) = Z(g)$ where Z is the smooth vector field given locally in the chart φ by

$$w = \sum_{j=1}^n \sum_{i=1}^n \left(\frac{\partial v_j}{\partial x_i} u_i - \frac{\partial u_j}{\partial x_i} v_j\right) \frac{\partial}{\partial x_j}$$

(where again $\frac{\partial}{\partial x_j}$ is the j^{th} standard basis vector). i.e.

$$w = Dv \cdot u - Du \cdot v$$

□ Theorem 46

Exercise 47. Check that if you change coordinates that w satisfies the rule.

Fact 48. *A tangent vector is determined by its action as a derivation. Hence a smooth vector field is determined by its action on locally smooth functions. Using smooth bump functions this shows that a smooth vector field is determined by its action on global smooth functions.*

Definition 49. The vector field Z in the above theorem is called the *Lie bracket* of X and Y and is denoted $[X, Y]$.

Theorem 50. *Suppose $f\colon N \to M$ is a smooth map of smooth manifolds. Suppose $X, Y \in \Gamma(N, TN)$; suppose $U, V \in \Gamma(M, TM)$ satisfy*

$$\begin{aligned}
f_* X_p &= U_{f(p)} \\
f_* Y_p &= V_{f(p)}
\end{aligned}$$

for all $p \in N$. Then $(f_*[X, Y])_p = ([U, V])_p$ for all $p \in N$.

Exercise 51. Prove this. Hint: if $f, g\colon \mathbb{R} \to \mathbb{R}$ and $h = g \circ f$ then $h' = (g' \circ f) \cdot f'$.