Course notes for PMATH 810

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Contents

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1 Banach algebras

Definition 1.1. A *Banach algebra* is an associative algebra $\mathfrak{A}$ over $\mathbb{C}$ (or $\mathbb{R}$, but not for us) which has a norm that makes $(\mathfrak{A}, \|\cdot\|)$ a Banach space and satisfies

$$\|xy\| \leq \|x\|\|y\|$$

and if $\mathfrak{A}$ has a unit (which we will denote e or 1) then $\|e\| = 1$.

Remark 1.2. The above implies that multiplication is jointly continuous. Indeed, we have

$$x_1y_1 - x_2y_2 = x_1y_1 - x_2y_1 + x_2y_1 - x_2y_2 = (x_1 - x_2)y_1 + x_2(y_1 - y_2)$$

so

$$\|x_1y_1 - x_2y_2\| \le \|x_1 - x_2\|\|y_1\| + \|x_2\|\|y_1 - y_2\|$$

Hence if $x_n \to x$ and $y_n \to y$ then $x_ny_n \to x_1y_1$.

Example 1.3.

1. If $\mathfrak{X}$ is a Banach space then $\mathcal{B}(\mathfrak{X})$ is a Banach algebra (with $\|T\| = \sup\{\, \|Tx\| : \|x\| \le 1 \,\}$).

2. If X is a compact Hausdorff space then $C(X)$ is a Banach space where $\|f\|_\infty = \sup\{\, |f(x)| : x \in X \,\}$. If X is locally compact and Hausdorff then we define $C_0(X)$ to consist of the continuous functions f on X such that for all $\varepsilon > 0$ the set $\{\, x \in X : |f(x)| \ge \varepsilon \,\}$ is compact; we define $C_b(X)$ to consist of the bounded continuous functions. For both $C_0(X)$ and $C_b(X)$ the norm $\|f\|_\infty = \sup\{\, |f(x)| : x \in X \,\}$ confers a Banach algebra structure.

3. Consider the set $C^{(n)}[a,b]$ of functions on $[a,b]$ with n continuous derivatives. Our product rule is

   $$(fg)^{(k)} = \sum \binom{k}{j} f^{(j)} g^{(k-j)}$$

   The norm

   $$\|f\|_{C^n} = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\infty}{k!}$$

   makes $C^{(n)}[a,b]$ into a Banach algebra.

   Exercise 1.4. Check that $\|fg\|_{C^n} \le \|f\|_{C^n}\|g\|_{C^n}$.

4. Suppose G is a locally compact abelian grape (e.g. $\mathbb{R}^n, \mathbb{T}^k, \mathbb{T}^k \times \mathbb{R}^n, \ldots$). We get a Haar measure m on G: a regular Borel measure that is translation-invariant (i.e. $m(A + s) = m(A)$ for Borel $A \subseteq G$ and $s \in G$). We define $L^1(G)$ to be the set of measurable f on G such that

   $$\|f\|_1 = \int |f| \mathrm{d}m < \infty$$

   The product on $L^1(G)$ is given by convolution:

   $$(f * g)(t) = \int_G f(s)g(t-s)\mathrm{d}m(s)$$

   One can check that

   - $g * f = f * g$
   - $(f * g) * h = f * (g * h)$ (this follows form Fubini).

   For the norm bound, note that

   $$\begin{aligned}
   \|f * g\|_1 &= \int_G |(f * g)(t)| \mathrm{d}m(t) \\
   &= \int_G \left| \int_G f(s)g(t-s)\mathrm{d}m(s) \right| \mathrm{d}m(t) \\
   &\le \int_G \int_G |f(s)||g(\underbrace{t-s}_{u})| \mathrm{d}m(s)\mathrm{d}m(t) \\
   &= \int_G \int_G |f(s)||g(u)| \mathrm{d}m(s)\mathrm{d}m(u) \\
   &= \|f\|_1 \|g\|_1
   \end{aligned}$$

(since the Jacobian of $(s,t) \mapsto (s,u)$ is
$$\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$
).

5. Consider $A(\mathbb{D})$ the *disk algebra* consisting of $f(z)$ continuous on $\overline{\mathbb{D}}$ and analytic on $\mathbb{D} = \{\, z \in \mathbb{Z} : |z| < 1 \,\}$. Together with the norm
$$\|f\| = \sup_{|z| \le 1} |f(z)| = \sup_{|z|=1} |f(z)|$$
(where the second equality is by the maximum modulus principle) forms a Banach algebra. Then $A(\mathbb{D}) \subseteq C(\mathbb{D})$; in fact $A(\mathbb{D}) \subseteq C(\mathbb{T})$ where $\mathbb{T} = \{\, z : |z| = 1 \,\} = \partial\overline{\mathbb{D}}$. Indeed the map $f \mapsto f \restriction \mathbb{T}$ is isometric.

6. For $T \in \mathcal{B}(\mathfrak{X})$ where $\mathfrak{X}$ is a Banach space, we define $\mathcal{A}(T) = \overline{\{\, p(T) : p \in \mathbb{C}[z] \,\}}^{\|\cdot\|} \subseteq \mathcal{B}(\mathfrak{X})$. If $T \in \mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ a Hilbert space we define $C^*(T) = \overline{\operatorname{alg}\{\, I, T, T^* \,\}}^{\|\cdot\|}$. (Here alg is "the algebra generated by".)

7. If (X, μ) is a measure space we define $L^\infty(\mu)$ to be the set of measurable f such that f is essentially bounded (i.e. there is t such that $\mu(\{\, x : |f(x)| > t \,\}) = 0$) modulo $f \sim g$ if $f - g = 0$ almost everywhere. The norm is given by
$$\|f\|_\infty = \inf\{\, t : \mu(\{\, x : |f(x)| > t \,\}) = 0 \,\} = \operatorname{ess.sup}|f|$$
We have an embedding $L^\infty(\mu) \hookrightarrow \mathcal{B}(L^2(\mu))$ given by $f \mapsto M_f$ where $M_f(h) = fh$.

Remark 1.5. If $\mathfrak{A}$ is a Banach algebra without unit we define $\mathfrak{A}^+ = \{\, (a, \lambda) : a \in \mathfrak{A}, \lambda \in \mathbb{C} \,\}$; we write $(a, \lambda) = a + \lambda e$. We define
$$(a + \lambda e)(b + \mu e) = (ab + \lambda b + \mu a) + \lambda\mu e$$
$$\|a + \lambda e\| = \|a\| + |\lambda|$$
so
$$\|(a + \lambda e)(b + \mu e)\| \le \|a\|\|b\| + |\lambda|\|b\| + |\mu|\|a\| + |\lambda\mu| = (\|a\| + |\lambda|)(\|b\| + |\mu|)$$
In fact $\mathfrak{A}$ is a (closed) maximal ideal in $\mathfrak{A}^+$.

Proposition 1.6. *Every Banach algebra $\mathfrak{A}$ is isometrically isomorphic to a subalgebra of $\mathcal{B}(\mathfrak{X})$ for some Banach space $\mathfrak{X}$.*

Proof. We map $\mathfrak{A}$ into $\mathcal{B}(\mathfrak{A}^+)$ by $a \mapsto L_a$ where $L_a x = ax$. Then
$$\|a\| = \|ae\| \le \|L_a\| = \sup\{\, \|ax\| : x \in \mathfrak{A}^+, \|x\| \le 1 \,\} \le \sup\{\, \|a\|\|x\| : x \in \mathfrak{A}^+, \|x\| \le 1 \,\} = \|a\|$$
so this is indeed an isometry. ☐ Proposition 1.6

Definition 1.7. Suppose $\mathfrak{A}$ is a unital Banach algebra and $a \in \mathfrak{A}$.

- The *spectrum* of a is $\sigma_{\mathfrak{A}}(a) = \{\, \lambda \in \mathbb{C} : \lambda 1 - a \text{ is not invertible} \,\}$. (If the $\mathfrak{A}$ is clear from context we will sometimes omit it and write $\sigma(a)$.)
- The *resolvent* of a is $\rho(a) = \mathbb{C} \setminus \sigma(a)$.
- The *resolvent function* $R(a, \lambda) = (\lambda - a)^{-1}$ is defined on $\rho(a)$.

Definition 1.8. Suppose $T \in \mathcal{B}(\mathfrak{X})$ for some Banach space $\mathfrak{X}$.

- We define the *point spectrum* $\sigma_p(T)$ to be the set of eigenvalues of T: those λ for which there is $x \neq 0$ such that $Tx = \lambda x$.

- We define the *approximate point spectrum* $\sigma_\pi(T)$ to be the set of $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not bounded below. (An operator T is *bounded below* if there is $\varepsilon > 0$ such that $\|Tx\| \geq \varepsilon\|x\|$ for all $x \in \mathfrak{X}$.)
- We define the *compression spectrum* $\gamma(T)$ to be $\{\, \lambda : \overline{(\lambda I - T)\mathfrak{X}} \neq \mathfrak{X} \,\}$; i.e. the λ for which $\lambda I - T$ does not have dense range.

Theorem 1.9. *For $T \in \mathcal{B}(\mathfrak{X})$ with $\mathfrak{X}$ a Banach space, the following are equivalent:*

1. *T is invertible.*
2. *T maps $\mathfrak{X}$ bijectively to itself.*
3. *T is bounded below and has dense range.*
4. *T and T^* are bounded below ($T^* \in \mathcal{B}(\mathfrak{X}^*)$).*
5. *T^* is invertible in $\mathcal{B}(\mathfrak{X}^*)$.*

Proof.

(1) $\Longrightarrow$ (2) Immediate.

(2) $\Longrightarrow$ (1) Banach isomorphism theorem.

(1) $\Longrightarrow$ (3) Note that $x = T^{-1}(Tx)$; so $\|x\| \leq \|T^{-1}\|\|Tx\|$, and $\|Tx\| \geq (\|T^{-1}\|)^{-1}\|x\|$, and T is bounded below. (Surjectivity implies dense range.)

(3) $\Longrightarrow$ (2) If $x \neq 0$ then $\|Tx\| \geq \varepsilon\|x\| > 0$; hence $Tx \neq 0$, and T is injective. For surjectivity, suppose $y \in \mathfrak{X}$; then since T has dense range there are x_n such that $y_n = Tx_n \to y$. Then in particular the y_n are Cauchy; since

$$\|y_n - y_m\| = \|T(x_n - x_m)\| \geq \varepsilon\|x_n - x_m\|$$

we get that the x_n are also Cauchy, and thus have a limit $x \in \mathfrak{X}$. Then

$$Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n = y$$

and T is surjective.

(1) $\Longrightarrow$ (5) By hypothesis we have $I_{\mathfrak{X}} = T^{-1}T = TT^{-1}$; so

$$I_{\mathfrak{X}^*} = I_{\mathfrak{X}}^* = T^*(T^{-1})^* = (T^{-1})^*T^*$$

so T^* is invertible in $\mathcal{B}(X^*)$.

(5) $\Longrightarrow$ (4) If T^* is invertible then T^* is bounded below (by $1 \implies 3$); also ($1 \implies 5$) implies that T^{**} is invertible and thus bounded below. But $T = T^{**} \restriction \mathfrak{X}$; so T is bounded below.

(4) $\Longrightarrow$ (3) T is bounded below by hypothesis. Note that

$$(\operatorname{Ran} T)^\perp = \{\, f \in \mathfrak{X}^* : \underbrace{f(Tx)}_{(T^*f)(x)} = 0 \text{ for all } x \in \mathfrak{X} \,\} = \{\, f : T^*f = 0 \,\} = \ker(T^*) = \{\, 0 \,\}$$

(since T^* is bounded below). By the Hahn-Banach theorem if $\overline{\operatorname{Ran} T}$ were a proper subspace then there would be $0 \neq f \in \mathfrak{X}^*$ such that $f \restriction \overline{\operatorname{Ran} T} = 0$, a contradiction. So $\overline{\operatorname{Ran} T} = \mathfrak{X}$, and T has dense range. $\square$ Theorem 1.9

Corollary 1.10. *If $T \in \mathcal{B}(\mathfrak{X})$ then $\sigma(T) = \sigma_\pi(T) \cup \gamma(T)$.*

Proposition 1.11. *Suppose $\mathfrak{A}$ is a unital Banach algebra. If $\|a\| < 1$ then $1 - a$ is invertible.*

Proof. If $x \in \mathbb{C}$ and $|x| < 1$ then

$$\frac{1}{1-x} = \sum_{n=}^{\infty} x^n$$

If $\|a\| < 1$, define

$$b = \sum_{n=0}^{\infty} a^n$$

(where $a^0 = 1$). To see that this is well-defined, note that

$$\sum_{n=0}^{\infty} \|a^n\| \leq \sum_{n=0}^{\infty} \|a\|^n < \infty$$

So the sequence

$$b_k = \sum_{n=0}^{k} a^n$$

is a convergent sequence, and b is well-defined in $\mathfrak{A}$ as the limit of the b_k. Since multiplication is continuous we get that

$$(1-a)b = \lim_{k\to\infty}(1-a)b_k = \lim_{k\to\infty}(1-a)\sum_{n=0}^{k} a^n = \lim_{k\to\infty}(1-a^{k+1}) = 1$$

(since $\|a^{k+1}\| \leq \|a\|^{k+1} \to 0$). Also $(1-a)b_k = b_k(1-a)$, so $b(1-a) = (1-a)b = 1$, as desired. $\square$ Proposition 1.11

Corollary 1.12. *$\mathfrak{A}^{-1}$ is open and $a \mapsto a^{-1}$ is a continuous antihomomorphism $\mathfrak{A}^{-1} \to \mathfrak{A}^{-1}$. (Note that $\mathfrak{A}^{-1}$ is a grape under multiplication and $(ab)^{-1} = b^{-1}a^{-1}$.)*

Proof. The previous proposition says that $b_1(1) = \{\, a : \|1-a\| < 1 \,\} \subseteq \mathfrak{A}^{-1}$. Suppose $a \in \mathfrak{A}^{-1}$ and $b \in \mathfrak{A}$ with $\|b\| < \frac{1}{\|a^{-1}\|}$. Then $a - b = a(1-a^{-1}b)$ and $\|a^{-1}b\| \leq \|a^{-1}\|\|b\| < 1$. So $1 - a^{-1}b$ is invertible (in fact the inverse is

$$\sum_{n=0}^{\infty}(a^{-1}b)^n$$

). So $a - b$ is invertible with

$$(a-b)^{-1} = (1-a^{-1}b)^{-1}a^{-1} = \sum_{n=0}^{\infty}(a^{-1}b)^n a^{-1}$$

So $b_{\|a^{-1}\|^{-1}}(a) \subseteq \mathfrak{A}^{-1}$, and $\mathfrak{A}^{-1}$ is open.

$(ab)^{-1} = b^{-1}a^{-1}$ shows that $a \mapsto a^{-1}$ is an antihomomorphism; bijectivity follows from $a = (a^{-1})^{-1}$. It remains to check continuity. If $\|a\| < 1$ then

$$\|(1-a)^{-1} - 1\| = \left\| \sum_{n=0}^{\infty} a^n - 1 \right\| = \left\| \sum_{n=1}^{\infty} a^n \right\| \leq \sum_{n=1}^{\infty} \|a\|^n = \frac{\|a\|}{1-\|a\|}$$

As $a \to 0$ we have

$$\frac{\|a\|}{1-\|a\|} \to 0$$

(uniform estimate). Thus if $b_n \to 1$ then $a_n = 1 - b_n \to 0$, and $b_n^{-1} = (1-a_n)^{-1} \to 1$. So inversion is continuous at 1. So if $a \in \mathfrak{A}^{-1}$ and $a_n \in \mathfrak{A}^{-1}$ converge to a, eventually $\|a - a_n\| < \frac{1}{\|a^{-1}\|}$. Then write $a_n = a - b_n = a(1-a^{-1}b_n)$ so $a^{-1}b_n \to 0$. Then $a_n^{-1} = (1-a^{-1}b_n)^{-1}a^{-1} \to a^{-1}$, and inversion is indeed continuous. $\square$ Corollary 1.12

Proposition 1.13. *Suppose $\mathfrak{A}$ is a unital Banach algebra and $a \in \mathfrak{A}$. Then $\rho(a)$ is open and $\sigma(a)$ is a compact subset of $\{ \lambda \in \mathbb{C} : |\lambda| \leq \|a\| \}$.*

Proof. Note that

$$\rho(a) = \{ \lambda : \lambda 1 - a \text{ is invertible} \} = \varphi^{-1}(\underbrace{\mathfrak{A}^{-1}}_{\text{open}})$$

where $\varphi \colon \lambda \mapsto \lambda 1 - a$. Alternatively, if $\lambda_0 - a$ is invertible then

$$b_{\|(\lambda_0 - a)^{-1}\|^{-1}}(\lambda_0 - a)$$

is contained in $\mathfrak{A}^{-1}$ and $\{ \lambda : |\lambda - \lambda_0| < \|(\lambda_0 - a)^{-1}\|^{-1} \}$. So $\sigma(a) = \mathbb{C} \setminus \rho(a)$ is closed.

If $|\lambda| > \|a\|$ then

$$\lambda - a = \lambda(1 - \frac{a}{\lambda})$$

But $\|\frac{a}{\lambda}\| = \frac{\|a\|}{|\lambda|} < 1$, so $1 - \frac{a}{\lambda}$ is invertible. So $\lambda - a$ is invertible; so $\sigma(a) \subseteq \{ \lambda : |\lambda| \leq \|a\| \}$; so it is closed and bounded, and thus compact.

TODO 1. *Connectives?*

□ Proposition 1.13

Example 1.14.

1. Let $\mathcal{H} = L^2(0,1)$, $f \in \mathcal{H}$, and $M_f h = fh$ for $h \in L^2(0,1)$.

   Claim 1.15. $\|M_f\| = \|f\|_\infty = \operatorname{ess.sup}|f|$.

   Proof. Note that

$$\begin{aligned} \|M_f\|^2 &= \sup\{ \|fh\|_2^2 : \|h\|_2 \leq 1 \} \\ &= \sup\left\{ \int |fh|^2 : \|h\|_2 \leq 1 \right\} \\ &\leq \sup\left\{ \int \|f\|_\infty^2 |h|^2 : \|h\|_2 \leq 1 \right\} \\ &= \|f\|_\infty^2 \sup\{ \|h\|_2^2 : \|h\|_2 \leq 1 \} \\ &= \|f\|_\infty^2 \end{aligned}$$

   So $\|M_f\| \leq \|f\|_\infty$.

   For $\varepsilon > 0$, let $A_\varepsilon = \{ x : |f(x)| > \|f\|_\infty - \varepsilon \}$; then $m(A_\varepsilon) > 0$. Let $h_\varepsilon = \frac{\chi_{A_\varepsilon}}{m(A_\varepsilon)^{\frac{1}{2}}}$. Then

$$\|h_\varepsilon\|_2^2 = \int \frac{\chi_{A_\varepsilon}}{m(A_\varepsilon)} = 1$$

$$|fh_\varepsilon| \geq (\|f\|_\infty - \varepsilon) \frac{\chi_\varepsilon}{m(A_\varepsilon)^{\frac{1}{2}}}$$

   So

$$\|fh_\varepsilon\| \geq (\|f\|_\infty - \varepsilon)\|h_\varepsilon\| = \|f\|_\infty - \varepsilon$$

   and

$$\|Mf\| \geq \sup_{\varepsilon > 0} \|f\|_\infty - \varepsilon = \|f\|_\infty$$

   □ Claim 1.15

Note that $f \mapsto M_f$ is an algebra homomorphism of $L^\infty(0,1)$ into $B(L^2(0,1))$ which is isometric. What is M_f^*? Well, for $h, k \in L^2(0,1)$ we have

$$\begin{aligned}
\langle M_f^* h, k \rangle &= \langle h, M_f k \rangle \\
&= \langle h, fk \rangle \\
&= \int h\overline{fk} \\
&= \int (\overline{f}h)\overline{k} \\
&= \langle \overline{f}h, k \rangle \\
&= \langle M_{\overline{f}} h, k \rangle
\end{aligned}$$

So $M_f^* = M_{\overline{f}}$.

Claim 1.16. $\sigma(M_f) = \sigma_{L^\infty}(f) = \operatorname{ess.ran}(f) = \{\, \lambda : m(f^{-1}(b_\varepsilon(\lambda))) > 0$ *for all* $\varepsilon > 0 \,\}$.

Proof. Note that

$$\mathbb{C} \setminus \operatorname{ess.ran}(f) = \{\, \lambda : \exists \varepsilon > 0 \text{ such that } m(f^{-1}(b_\varepsilon(\lambda))) = 0 \,\}$$

If $\lambda \notin \operatorname{ess.ran}(f)$ then there is ε such that $|f(x) - \lambda| > \varepsilon$ almost everywhere; so $\frac{1}{f-\lambda} \in L^\infty$ (since $\left|\frac{1}{f-\lambda}\right| \leq \frac{1}{\varepsilon}$ almost everywhere). So $f - \lambda$ is invertible in L^∞.

Note that $I = M_1$ and $\lambda I - M_f = M_{\lambda - f}$. So

$$M_{\lambda-f} M_{\frac{1}{\lambda-f}} = M_{\frac{1}{\lambda-f}} M_{\lambda-f} = M_1 = I$$

So if $\lambda \notin \operatorname{ess.ran}(f)$ then $\lambda - f$ is invertible in L^∞ and $M_{\lambda-f}$ is invertible in $\mathcal{B}(L^2(0,1))$.

If $\lambda \in \operatorname{ess.ran}(f)$ then $\frac{1}{\lambda-f}$ is not essentially bounded and may take value $+\infty$ somewhere; so $\lambda - f$ is not invertible in L^∞.

For $\varepsilon > 0$ let $A_\varepsilon = \{\, x : |f(x) - \lambda| < \varepsilon \,\}$; then $m(A_\varepsilon) > 0$. Let $h_\varepsilon = \frac{\chi_{A_\varepsilon}}{m(A_\varepsilon)^{\frac{1}{2}}}$. Then $|M_{\lambda-f} h_\varepsilon| = |(\lambda - f)h_\varepsilon| < \varepsilon |h_\varepsilon|$; so $\|M_{\lambda-f} h_\varepsilon\| < \varepsilon$. So $M_{\lambda-f}$ is not bounded below, and $M_{\lambda-f}$ is not invertible. $\square$ Claim 1.16

Example 1.17. Consider M_x. We have $\overline{\operatorname{Ran}(x)} = \operatorname{ess.ran}(x) = [0,1]$ and $\sigma_p(M_x) = \emptyset$. If $M_x h = xh = \lambda h$ then $(x - \lambda)h = 0$ almost everywhere; since $x - \lambda \neq 0$ almost everywhere, we get that $h = 0$ almost everywhere.

If $\lambda \in [0,1]$, then $M_{\lambda-x}$ is not bounded below.

We have

$$\overline{\operatorname{Ran} M_{\lambda-x}} \supseteq \bigcup M_{\lambda-x} L^2([0, \lambda - \varepsilon] \cup [\lambda + \varepsilon, 1])$$

Since $|\lambda - x| \geq \varepsilon$ on $B_\varepsilon = [0, \lambda - \varepsilon] \cup [\lambda + \varepsilon, 1]$ and $M_{\lambda-f} \colon L^2(B_\varepsilon) \to L^2(B_\varepsilon)$, we get that $M_{\lambda-x}$ is invertible on $L^2(B_\varepsilon)$ and $M_{\lambda-x} L^2(B_\varepsilon) = L^2(B_\varepsilon)$. So

$$\overline{\operatorname{Ran} M_{\lambda-x}} \supseteq \overline{\bigcup_{\varepsilon>0} L^2(B_\varepsilon)} = L^2(0,1)$$

2. Let $\mathcal{H} = \ell_2$ with orthonormal basis $\{\, e_n : n \geq 0 \,\}$. If $(d_n : n \in \mathbb{N})$ is bounded we let $D = \operatorname{diag}((d_n : n \in \mathbb{N}))$ so

$$D\Big(\sum a_n e_n\Big) = \sum d_n a_n e_n$$

So $\|D\| = \sup |d_n|$, and $\sigma(D) = \overline{\{\, d_n \,\}}$.

3. Let S be the unilateral shift on ℓ_2 so

$$S \sum_{n \geq 0} a_n e_n = \sum_{n \geq 0} a_n e_{n+1}$$

The adjoint has

$$\begin{aligned}\left\langle S^* \sum a_n e_n, \sum b_n e_n \right\rangle &= \left\langle \sum a_n e_n, S \sum b_n e_n \right\rangle \\ &= \left\langle \sum a_n e_n, \sum b_n e_{n+1} \right\rangle \\ &= \sum_{n=0}^{\infty} a_{n+1} \overline{b_n} \\ &= \left\langle \sum_{n=0}^{\infty} a_{n+1} e_n, \sum b_n e_n \right\rangle\end{aligned}$$

So

$$S^* e_n = \begin{cases} e_{n-1} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$

is the backwards shift.

Proposition 1.18. *If $\mathcal{H}$ is a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ then $\sigma(T^*) = \sigma(T)^*$ (where the latter is pointwise complex conjugation).*

Proof. If $\lambda \notin \sigma(T)$ then $(\lambda I - T)(\lambda I - T)^{-1} = I = (\lambda I - T)^{-1}(\lambda I - T)$. Taking adjoints we find that

$$((\lambda I - T)^{-1})^*(\overline{\lambda} I - T^*) = I^* = I = (\overline{\lambda} I - T^*)((\lambda I - T)^{-1})^*$$

so $(\overline{\lambda} I - T^*)^{-1} = ((\lambda I - T)^{-1})^*$. Since $T = T^{**}$ this is reversible. So $\rho(T^*) = \rho(T)^*$. □ Proposition 1.18

Note that $S^* S = I$ but $SS^* = I - P_{\mathbb{C}e_0}$ where $P_{\mathbb{C}e_0} = e_0 e_0^*$.

Notation 1.19. If $x, y \in \mathcal{H}$ then $xy^* \in \mathcal{B}(\mathcal{H})$ of rank 1 is given by $(xy^*)(z) = x(y^* z) = \langle z, y \rangle x$.

So S, S^* are not invertible. We have that S is injective but not surjective, with $\mathrm{Ran}(S) = (\mathbb{C}e_0)^\perp$; also S^* is surjective but not injective with $S^* e_0 = 0$, and $\ker(S^*) = \mathbb{C}e_0$. So $0 \in \sigma(S)$.

We have $\|S\| = \|S^*\| = 1$ and S is an isometry ($\|Sx\| = \|x\|$ for all x). So $\sigma(S) \subseteq \overline{\mathbb{D}} = \{ \lambda : |\lambda| \leq 1 \}$.

If $S^* x = \lambda x$ where $x = (x_0, x_1, \ldots)$ then $x_{n+1} = \lambda x_n$ for all n; so $x = x_0(1, \lambda, \lambda^2, \ldots)$. Then

$$\|x\|_2^2 = |x_0|^2 \sum_{n=0}^{\infty} |\lambda|^{2n} = \begin{cases} \frac{|x_0|^2}{1 - |\lambda|^2} < \infty & \text{if } |\lambda| < 1 \\ 0 & \text{if } x_0 = 0 \\ \infty & \text{else} \end{cases}$$

So if $x_\lambda = (1, \lambda, \lambda^2, \ldots)$ for $|\lambda| < 1$ then $S^* x_\lambda = \lambda x_\lambda$. So $\sigma_p(S^*) = \mathbb{D}$. So $\sigma(S^*) = \overline{\mathbb{D}}$ and $\sigma(S) = \overline{\mathbb{D}}$.

If $Sx = \lambda x$ for $\lambda \neq 0$ then $x_0 = 0 = x_1 = x_2 = \cdots$; so $\lambda \notin \sigma_p(S)$. Also $0 \notin \sigma_p(S)$ because S is isometric. So $\sigma_p(S) = \emptyset$.

Suppose $|\lambda| = 1$; let $x_n = \frac{1}{\sqrt{n}}(1, \lambda, \lambda^2, \ldots, \lambda^{n-1}, 0, 0, 0 \ldots)$. Then

$$S^* x_n = \frac{1}{\sqrt{n}}(\lambda, \lambda^2, \ldots, \lambda^{n-1}, 0, 0, \ldots)$$

so

$$S^* x_n - \lambda x_n = \frac{1}{\sqrt{n}}(0, \ldots, 0, -\lambda^n, 0, 0, \ldots)$$

and $\|(S^* - \lambda)x_n\| = \frac{1}{\sqrt{n}} \to 0$, so $S^* - \lambda$ isn't bounded below. Also

$$Sx_n = \frac{1}{\sqrt{n}}(0, 1, \lambda, \lambda^2, \dots, \lambda^{n-2}, \lambda^{n-1}, 0, \dots)$$

and

$$\overline{\lambda} x_n \frac{1}{\sqrt{n}}(\overline{\lambda}, 1, \lambda, \dots, \lambda^{n-2}, 0, 0, \dots)$$

so

$$\|(S - \overline{\lambda} I)x_n\| = \left\| \frac{1}{\sqrt{n}}(-\overline{\lambda}, 0, \dots, 0, \lambda^{n-1}, 0, \dots) \right\| = \sqrt{\frac{2}{n}} \to 0$$

and $S - \overline{\lambda}$ is not bounded below.

Definition 1.20. Suppose $\Omega \subseteq \mathbb{C}$ is open and $\mathfrak{X}$ is a Banach space. We say $f \colon \Omega \to \mathfrak{X}$ is *strongly analytic* on Ω if for all $z_0 \in \Omega$ there is $r > 0$ and $(x_n : n \geq 0)$ in $\mathfrak{X}$ such that

$$f(z) = \sum_{n=0}^{\infty} x_n (z - z_0)^n$$

converges absolutely and uniformly on $\{ z : |z - z_0| \leq r \}$. We say f is *weakly analytic* if for all $\varphi \in \mathfrak{X}'$ we have that $\varphi \circ f \colon \Omega \to \mathbb{C}$ is analytic.

Exercise 1.21 (Homework). Weakly analytic implies strongly analytic. (I think he said something about Banach-Steinhaus?)

Theorem 1.22. *Suppose $\mathfrak{A}$ is a unital Banach algebra and $a \in \mathfrak{A}$.*

1. *For $\lambda, \mu \in \rho(a)$ we have*
$$\frac{R(a,\lambda) - R(a,\mu)}{\lambda - \mu} = -R(a,\lambda)R(a,\mu)$$
2. *$R(a,\lambda)$ is a strongly analytic function on $\rho(a)$.*
3. *$R'(a,\lambda) = -R(a,\lambda)^2$.*
4. *$\|R(a,\lambda)\| \to 0$ as $|\lambda| \to \infty$.*

Proof.

1. We have$(R(a,\lambda) - R(a,\mu)(\lambda - a)(\mu - a) = (\mu - a) - (\lambda - a) = \mu - \lambda$; multiply by $\frac{R(a,\lambda)-R(a,\mu)}{\lambda-\mu}$ to get the desired result.
2. If $\lambda_0 \in \rho(a)$ and $|\lambda - \lambda_0| < \frac{1}{\|(\lambda_0 - a)^{-1}\|}$.
$$\begin{aligned} \lambda - a &= (\lambda_0 - a) - (\lambda_0 - \lambda) \\ &= (\lambda_0 - a)(1 - (\lambda_0 - \lambda)(\lambda_0 - a)^{-1}) \end{aligned}$$
$$\|(\lambda_0 - \lambda)(\lambda_0 - a)^{-1}\| = |\lambda_0 - \lambda|\|(\lambda_0 - a)^{-1}\| < 1$$
So
$$(\lambda - a)^{-1} = \sum_{n=0}^{\infty} ((\lambda_0 - \lambda)(\lambda_0 - a)^{-1})^n (\lambda_0 - a)^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda_0 - a)^{-n-1} (\lambda - \lambda_0)^n$$
If $0 < R < \frac{1}{\|(\lambda_0 - a)^{-1}\|}$ then if $|\lambda - \lambda_0| \leq R$ then $\|(\lambda - \lambda_0)(\lambda_0 - a)^{-1}\| \leq \frac{R}{\|(\lambda_0 - a)^{-1}\|} = r < 1$. So
$$\sum \|((\lambda_) - \lambda)(\lambda_0 - a)^{-1}\| \|(\lambda_0 - a)^{-1}\| \leq \sum r^n \|(\lambda_0 - a)^{-1}\| = \frac{\|(\lambda_0 - a)^{-1}\|}{1 - r} < \infty$$
So convergence is absolute and uniform (by M-test) on $\{ \lambda : |\lambda - \lambda_0| \leq R \}$. So $R(a,\lambda)$ is strongly analytic.

3. We note that
$$R'(a,\mu) = \lim_{\lambda\to\mu} \frac{R(a,\lambda) - R(a,\mu)}{\lambda - \mu} = -R(a,\mu)^2$$
4. If $|\lambda| = 2\|a\|$ then $(\lambda - a)^{-1} = \lambda^{-1}(1 - \lambda^{-1}a)^{-1} = \lambda^{-1}\sum(\lambda^{-1}a)^n$. So
$$\|(\lambda - a)^{-1}\| \le \frac{1}{|\lambda|}\sum_{n=0}^{\infty}\|(\lambda^{-1}a)^n\| \le \frac{1}{|\lambda|}\sum \frac{1}{2^n} = \frac{2}{|\lambda|}$$
So $\|R(a,\lambda)\| \le \frac{2}{|\lambda|} \to 0$ as $|\lambda| \to \infty$. ☐ Theorem 1.22

Theorem 1.23 (Liouville)**.** *If $f\colon \mathbb{C} \to \mathfrak{X}$ is a weakly analytic entire function which is bounded then it is constant.*

Proof. For all $\varphi \in \mathfrak{X}'$ we have $\varphi \circ f\colon \mathbb{C} \to \mathbb{C}$ is entire and bounded. So $\varphi \circ f$ is constant by Liouville's theorem. By Hahn-Banach we have that f is constant: if $f(z_1) \ne f(z_2)$ then there would be φ such that $\varphi(f(z_1) - f(z_2)) \ne 0$. ☐ Theorem 1.23

Theorem 1.24. *Suppose $\mathfrak{A}$ is a unital Banach algebra. Then $\sigma(a)$ is not empty.*

Proof. If $\sigma(a) = \emptyset$ then $R(a,\lambda)$ is entire, strongly analytic, and has $\|R(a,\lambda)\| \to 0$ as $|\lambda| \to \infty$, and is thus bounded. So by Liouville's theorem it is constant, a contradiction since $R(a,0) = -a^{-1} \ne (1-a)^{-1} = R(a,1)$. ☐ Theorem 1.24

If $K \subseteq \mathbb{C}$ is compact we let $\mathrm{Rat}(K)$ consist of rational functions $\frac{p(x)}{q(x)}$ with $p, q \in \mathbb{C}[x]$ such that the poles (zeroes of q) lie in $\mathbb{C} \setminus K$. If $\sigma(a) = K$ and $\frac{p}{q} \in \mathrm{Rat}(K)$ then we may write $q(x) = (x - \alpha_1)\cdots(x - \alpha_m)$ with each $\alpha_i \notin K$; then $q(a) = (a - \alpha_1 1)\cdots(a - \alpha_m 1)$, and $q(a)^{-1} = (a - \alpha_1 1)^{-1}\cdots(a - \alpha_m 1)^{-1}$ is well-defined because $\alpha_i \notin K = \sigma(a)$. We can then define $\frac{p}{q}(a) = p(a)q(a^{-1})$. This is a well-defined algebra homomorphism of $\mathrm{Rat}(\sigma(a))$ into $\mathfrak{A}$.

Theorem 1.25 (Spectral mapping theorem for rational functions)**.** *If $a \in \mathfrak{A}$ and $f = \frac{p}{q} \in \mathrm{Rat}(\sigma(a))$ then $\sigma(f(a)) = f(\sigma(a))$.*

Proof. Write $f = \frac{p}{q}$ with
$$q(x) = \prod_{i=1}^{m}(x - \alpha_i)$$
If $\lambda \in \mathbb{C}$ then we may write $f(x) - \lambda 1 = \frac{p_1(x)}{q(x)}$ with
$$p_1(x) = \prod_{j=1}^{n}(x - \beta_j)$$
Then
$$f(a) - \lambda 1 = p_1(a)q(a)^{-1} = \prod_{j=1}^{n}(a - \beta_j 1)q(a)^{-1}$$
So
$$\begin{aligned} \lambda \in \sigma(f(a)) &\iff f(a) - \lambda 1 \text{ is not invertible} \\ &\iff \exists j \text{ such that } a - \beta_j 1 \text{ is not invertible} \\ &\iff \exists j \text{ such that } \beta_j \in \sigma(a) \end{aligned}$$
and
$$\begin{aligned} \lambda \in f(\sigma(a)) &\iff \exists \beta \in \sigma(a) \text{ such that } f(\beta) - \lambda = 0 \\ &\iff \exists x \in \sigma(a) \text{ such that } \prod_{j=1}^{n}(x - \beta_j)q(x) = 0 \\ &\iff \exists j \text{ such that } x = \beta_j \end{aligned}$$

TODO 2. *Typo here?*

But the last equivalences are the same. ☐ Theorem 1.25

Definition 1.26. The *spectral radius* of a is $\operatorname{spr}(a) = \sup\{\, |\lambda| : \lambda \in \sigma(a) \,\}$.

Theorem 1.27. *Suppose $\mathfrak{A}$ is a unital Banach algebra and $a \in \mathfrak{A}$. Then*

$$\operatorname{spr}(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$$

Proof. By the spectral mapping theorem we have $\sigma(a^n) = \sigma(a)^n$. Since $\operatorname{spr}(a) \leq \|a\|$ we have

$$\operatorname{spr}(a) = \operatorname{spr}(a^n)^{\frac{1}{n}} \leq \|a^n\|^{\frac{1}{n}}$$

thus

$$\operatorname{spr}(a) \leq \inf_{n\geq 1} \|a^n\|^{\frac{1}{n}}$$

Recall that $R(a,\lambda) = (\lambda - a)^{-1}$ is analytic on $\mathbb{C} \setminus \sigma(a)$. Hence for $|\lambda| > \|a\|$ we have

$$R(a,\lambda) = \sum_{n=0}^{\infty} a^n \lambda^{-n-1}$$

TODO 3. *why? Something about a power series around ∞?*

If $\varphi \in \mathfrak{A}'$ then

$$\varphi(R(a,\lambda)) = \sum_{n=0}^{\infty} \varphi(a^n)\lambda^{-n-1}$$

is scalar-valued and analytic on $\rho(a) \supseteq \mathbb{C} \setminus \{\, \lambda : |\lambda| \leq \operatorname{spr}(a) \,\}$; note that this last set is the biggest disk around $\mathbb{C}$ on which R is defined. In particular, convergence is absolute and uniform over $|\lambda| \geq r + \varepsilon$ (with $r = \operatorname{spr}(a)$). So

$$\sup_{n\geq 0} |\varphi(a^n)|(r+\varepsilon)^{-n-1} < \infty$$

(as the terms in the series approach 0). So

$$\sup_{n\geq 0} \left|\varphi\left(\left(\frac{a}{r+\varepsilon}\right)^n\right)\right| \leq \frac{C(\varphi)}{r+\varepsilon}$$

for some constant $C(\varphi)$ (depending on φ). Hence by the uniform boundedness principle we have

$$\sup_{n\geq 0} \left\|\left(\frac{a}{r+\varepsilon}\right)^n\right\| = C' < \infty$$

Thus $\|a^n\| \leq C'(r+\varepsilon)^n$, and hence $\|a^n\|^{\frac{1}{n}} \leq (C')^{\frac{1}{n}}(r+\varepsilon) \to r+\varepsilon$. So

$$\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq r \leq \inf \|a^n\|^{\frac{1}{n}}$$

TODO 4. *port limsup typesetting to essential range?*

So $r = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \inf \|a^n\|^{\frac{1}{n}}$. ☐ Theorem 1.27

Remark 1.28. $R(a,\lambda) = \sum_{n=0}^{\infty} a^n \lambda^{-n-1}$ converges absolutely and uniformly on $\{\, \lambda : |\lambda| \geq r + \varepsilon \,\}$.

Exercise 1.29. Check the details of this.

Proposition 1.30 (Mazur)**.** *If $\mathfrak{A}$ is a Banach field then $\mathfrak{A} = \mathbb{C}1$.*

Proof. If $a \in \mathfrak{A}$ then $\sigma(a) \neq 0$. Pick $\lambda \in \sigma(a)$; then $a - \lambda 1$ is not invertible, so since $\mathfrak{A}$ is a field we get that $a - \lambda 1 = 0$ and $a = \lambda 1$. ☐ Proposition 1.30

2 Riesz functional calculus

Suppose U is open and contains $\sigma(a)$. Suppose f is a holomorphic function on U and $\lambda \in \sigma(a)$. Cauchy's theorem tells us that to evaluate $f(\lambda)$ we can draw a rectifiable curve

TODO 5. *rectifiable?*

$\mathcal{C}$ such that $\mathcal{C} \subseteq U \setminus \sigma(a)$ and the winding number

$$\operatorname{ind}_{\mathcal{C}}(z) = \begin{cases} 0 & \text{if } z \in \mathbb{C} \setminus U \\ 1 & \text{if } z \in K \end{cases}$$

TODO 6. *K?*

Then by Cauchy's theorem we have

$$f(\lambda) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{z - \lambda} \mathrm{d}z$$

for $z \in \sigma(a)$.

We can try to define $f(a)$ by

$$f(a) = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(z1 - a)^{-1} \mathrm{d}z$$

Note that $(z1 - a)^{-1}$ is defined on $\mathbb{C} \setminus \sigma(a)$, and thus on $\mathcal{C}$; also $f(z)$ is defined and analytic on $U \supseteq \mathcal{C}$. So $f(z)(z - a)^{-1}$ is defined on $U \setminus \sigma(a)$; it is analytic, and thus continuous.

Theorem 2.1. *Suppose $\mathfrak{X}$ is a Banach space; suppose $\mathcal{C}$ is a rectifiable curve in $\mathbb{C}$ and $f \colon \mathcal{C} \to \mathfrak{X}$ is continuous. Then*

$$\int_{\mathcal{C}} f(z) \mathrm{d}z$$

makes sense as a Riemann integral.

Proof. Parametrize $\mathcal{C}$ by arc length s for $0 \leq s \leq L$. Take partitions Δ consisting of $0 = s_0 < s_1 < \cdots < s_n = L$ and Ξ consisting of $\xi_i \in \varphi([s_{i-1}, \ldots, s_i])$ for $1 \leq i \leq n$. If $\varphi \colon [0, L] \to \mathcal{C}$ is our parametrization then our Riemann sum is

$$J(\Delta, \Xi) = \sum_{i=1}^{n} f(\xi_i)(\varphi(s_i) - \varphi(s_{i-1}))$$

We define

$$\operatorname{mesh}(\Delta) = \max_{1 \leq i \leq n} (s_i - s_{i-1})$$

Claim 2.2. $\lim\limits_{\operatorname{mesh}(\Delta) \to 0} J(\Delta, \Xi)$ *converges; we call this limit* $\displaystyle\int_{\mathcal{C}} f(z) \mathrm{d}z$.

TODO 7. *I believe this is a limit of nets?*

Proof. Suppose $\varepsilon > 0$. By continuity of f there is $\delta > 0$ such that $|s - t| < \delta$ implies $\|f(\varphi(s)) - f(\varphi(t))\| < \varepsilon$. Suppose (Δ_1, Ξ_1) and (Δ_2, Ξ_2) both have mesh $< \delta$. Let $\Delta = \Delta_1 \cup \Delta_2 = \{\, 0 = s_0 < s_1 \cdots < s_n = L \,\}$, and for $p \in \{\, 1, 2 \,\}$ write $\Delta_p = \{\, s_i : i \in J_p \,\}$ with $\{\, 0, n \,\} \subseteq J_p \subseteq \{\, 0, \ldots, n \,\}$. Let $\Xi = \{\, \varphi(s_i) : 1 \leq i \leq n \,\}$. We compare $J(\Delta_p, \Xi_p$ to $J(\Delta, \Xi)$.

$$J(\Delta, \Xi) - J(\Delta_p, \Xi_p) = \sum_{i=1}^{n} f(\varphi(s_i))(\varphi(s_i) - \varphi(s_{i-1})) - \sum f(\xi_j)(\varphi(s_i) - \varphi(s_{i-1}))$$

where $j \in J_p$ satisfies $[s_{i-1}, s_i] \subseteq [s_j, s_{j'}]$ with $[s_j, s_{j'}]$ an interval in Δ_p. Hence

$$\begin{aligned}
\|J(\Delta, \Xi) - J(\Delta_p, \Xi_p)\| &= \left\| \sum_{i=1}^n f(\varphi(s_i))(\varphi(s_i) - \varphi(s_{i-1})) - \sum f(\xi_j)(\varphi(s_i) - \varphi(s_{i-1})) \right\| \\
&\leq \sum_{i=1}^n \|f(\varphi(s_i)) - f(\xi_j)\| |\varphi(s_i) - \varphi(s_{i-1})| \text{ (note } \varphi(s_i) \text{ and } \xi_j \text{ are within } \delta \text{ of each other)} \\
&< \sum_{i=1}^n \varepsilon(s_i - s_{i-1}) \\
&= \varepsilon L
\end{aligned}$$

So $\|J(\Delta_1, \Xi_1) - J(\Delta_2, \Xi_2)\| < (2L)\varepsilon$. So the Riemann sums are Cauchy, and thus converge. □ TODO 7

□ Theorem 2.1

Theorem 2.3 (Riesz functional calculus)**.** *Suppose $\mathfrak{A}$ a unital Banach algebra and $a \in \mathfrak{A}$. If $f \in \mathrm{Hol}(U)$ with $U \subseteq \mathbb{C}$ an open set containing $\sigma(a)$, we define*

$$f(a) = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(z-a)^{-1} \mathrm{d}z$$

where $\mathcal{C}$ is a curve in $U \setminus \sigma(a)$ such that

$$\mathrm{ind}_{\mathcal{C}}(z) = \begin{cases} 1 & \text{if } z \in \sigma(a) \\ 0 & \text{if } z \notin U \end{cases}$$

Then

1. *This definition is independent of the choice of $\mathcal{C}$; hence $f(a)$ is well-defined.*
2. *$(f+g)(a) = f(a) + g(a)$ and $(\lambda f)(a) = \lambda \cdot f(a)$.*
3. *$(fg)(a) = f(a)g(a)$. (Hence, combining all the above, we get that $f \mapsto f(a)$ is a homomorphism.)*
4. *If*
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$
is analytic in a disk $D_R(z_0) \supseteq \sigma(a)$ then
$$f(a) = \sum_{n=0}^{\infty} a_n (a - z_0 1)^n$$

Proof.

1. Suppose $\mathcal{C}_1, \mathcal{C}_2$ are permissable curves. Then $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$ (i.e. union of $\mathcal{C}_1$ and $\mathcal{C}_2$ with the orientation of $\mathcal{C}_2$ reversed) is a curve such that
$$\mathrm{ind}_{\mathcal{C}}(z) = \begin{cases} 0 & \text{if } z \in \mathbb{C} \setminus U \\ 0 & \text{if } z \in \sigma(a) \end{cases}$$
So $f(z)(z-a)^{-1}$ is analytic on $U \setminus \sigma(a)$, and $\mathcal{C} \subseteq U \setminus \sigma(a)$; so $\mathcal{C}$ is homologous to zero in $U \setminus \sigma(a)$. Taking $\varphi \in \mathfrak{A}'$ we have
$$\begin{aligned}
\varphi\left(\frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(z-a)^{-1} \mathrm{d}x \right) &= \frac{1}{2\pi i} \int_{\mathcal{C}} \underbrace{f(z)\varphi((z-a)^{-1})}_{\text{scalar-valued and analytic in } U \setminus \sigma(a)} \mathrm{d}z \\
&= 0
\end{aligned}$$

by Cauchy's theorem. But this holds for all $\varphi \in \mathfrak{A}'$. So by the Hahn-Banach theorem we get

$$0 = \frac{1}{2\pi i}\int_{\mathcal{C}} f(z)(z-a)^{-1}\mathrm{d}z = \frac{1}{2\pi i}\int_{\mathcal{C}_1} f(z)(z-a)^{-1})\mathrm{d}z - \frac{1}{2\pi i}\int_{\mathcal{C}_2} f(z)(z-a)^{-1}\mathrm{d}z$$

2. If $f \in \mathrm{Hol}(U)$ and $g \in \mathrm{Hol}(V)$ with $U, V \supseteq \sigma(a)$ then $f, g \in \mathrm{Hol}(U \cap V)$; so one can choose $\mathcal{C}$ to work for both f and g. The claim then follows from linearity of the integral.

3. Suppose $f, g \in \mathrm{Hol}(U)$. Choose a curve $\mathcal{C}$ as required. Let $V = \{\, \lambda : \mathrm{ind}_{\mathcal{C}}(\lambda) = 1 \,\} \supseteq \sigma(a)$; so V is open. Choose $\mathcal{C}_2$ in $V \setminus \sigma(a)$ satisfying the requirements. In particular if $\lambda \in \mathcal{C}_2$ then $\mathrm{ind}_{\mathcal{C}_1}(\lambda) = 1$ (since $\mathcal{C}_2 \subseteq V$) and if $\lambda \in \mathcal{C}_1$ then $\mathrm{ind}_{\mathcal{C}_2}(\lambda) = 0$ (since $\mathcal{C}_1 \subseteq \mathbb{C} \setminus V$). Then

$$\begin{aligned}
f(a)g(a) &= \frac{1}{2\pi i}\int_{\mathcal{C}_1} f(z)(z-a)^{-1}\mathrm{d}z \frac{1}{2\pi i}\int_{\mathcal{C}_2} g(w)(w-a)^{-1}\mathrm{d}w \\
&= \frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1}\int_{\mathcal{C}_2} f(z)g(w)R(a,z)R(a,w)\mathrm{d}z\mathrm{d}w \\
&= \frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1}\int_{\mathcal{C}_2} f(z)g(w)\frac{R(a,z)-R(a,w)}{w-z}\mathrm{d}z\mathrm{d}w \text{ (by Theorem 1.22)} \\
&= \frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1}\int_{\mathcal{C}_2} f(z)g(w)\frac{R(a,z)}{w-z}\mathrm{d}z\mathrm{d}w - \frac{1}{(2\pi i)^2}\int_{\mathcal{C}_1}\int_{\mathcal{C}_2} f(z)g(w)\frac{R(a,w)}{w-z}\mathrm{d}z\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_{\mathcal{C}_1} f(z)R(a,z)\left(\underbrace{\frac{1}{2\pi i}\int_{\mathcal{C}_2}\frac{g(w)}{w-z}\mathrm{d}w}_{=0 \text{ since } \mathrm{ind}_{\mathcal{C}_2}(\underbrace{z}_{\in\mathcal{C}_1})=0}\right)\mathrm{d}z + \frac{1}{2\pi i}\int_{\mathcal{C}_2} g(w)R(a,w)\left(\underbrace{\frac{1}{2\pi i}\int_{\mathcal{C}_1}\frac{f(z)}{z-w}\mathrm{d}z}_{=f(w) \text{ since } \mathrm{ind}_{\mathcal{C}_1}(\underbrace{w}_{\in\mathcal{C}_2})=1}\right)\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_{\mathcal{C}_2} g(w)f(w)(w-a)^{-1}\mathrm{d}w \\
&= (fg)(a)
\end{aligned}$$

TODO 8. *Typeset above better*

4. Let $\mathcal{C} = z_0 + r\exp(i\theta)$ for $0 \le \theta \le 2\pi$ and $r < R$ be sufficiently large to enclose $\sigma(a)$. Then the Taylor expansion

$$f(z) = \sum_{n \ge 0} a_n (z-z_0)^n$$

converges absolutely and uniformly on

TODO 9. *in?*

Then

$$\begin{aligned}
f(a) &= \frac{1}{2\pi i}\int_{\mathcal{C}} \sum_{n=0}^{\infty} a_n (z-z_0)^n (z-a)^{-1}\mathrm{d}z \\
&= \sum_{n=0}^{\infty} a_n \frac{1}{2\pi i}\int_{\mathcal{C}} (z-z_0)^n (z-a)^{-1}\mathrm{d}z \\
&= \sum_{n=0}^{\infty} a_n (a-z_0)^n
\end{aligned}$$

as desired. ☐ Theorem 2.3

Corollary 2.4 (Spectral mapping theorem for analytic functions)**.** *If $f \in \mathrm{Hol}(U)$ with $U \supseteq \sigma(a)$ then $\sigma(f(a)) = f(\sigma(a))$.*

Proof.

($\subseteq$) If $\lambda \notin f(\sigma(a))$ then $f(z) - \lambda \neq 0$ on $\sigma(a)$. Let $V = \{ z \in U : f(z) \neq \lambda \}$; so V is an open set containing $\sigma(a)$, and

$$g(z) = \frac{1}{\lambda - f(z)}$$

is analytic on V. But then $g(z)(\lambda - f(z)) = 1$, so $g(a)(\lambda - f(a)) = 1 = (\lambda - f(a))g(a)$; so $\lambda \notin \sigma(f(a))$.

($\supseteq$) If $\lambda \in f(\sigma(a))$ then there is $w \in \sigma(a)$ such that $f(w) = \lambda$. So $\lambda - f(z) = (z-w)g(z)$ for some $g \in \mathrm{Hol}(U)$; so $\lambda - f(a) = \underbrace{(a-w)}_{\text{not invertible}} g(a)$, and $\lambda - f(a)$ is not invertible. So $\lambda \in \sigma(f(a))$. □ Corollary 2.4

Example 2.5.

1. Let $\mathcal{H} = \ell_2$ with orthonormal basis $\{ e_n \}_{n \geq 0}$. Let $D \in \mathcal{B}(\ell_2)$ be $\mathrm{diag}(d_0, d_1, \ldots)$; i.e. $De_n = d_n e_n$. Then $\sigma(D) = \overline{\{ d_n : n \in \mathbb{N} \}}$. Suppose $f \in \mathrm{Hol}(U)$ with $U \supseteq \sigma(D)$. Find $\mathcal{C}$. Note that if $z \notin \sigma(D)$ then $zI - D = k\, \mathrm{diag}\left(\frac{1}{z-d_n} : n \in \mathbb{N} \right)$. Then

$$\begin{aligned} f(D)e_n &= \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(zI - D)^{-1} e_n \mathrm{d}z \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}} f(z) \frac{1}{z - d_n} e_n \mathrm{d}z \\ &= \frac{1}{2\pi i} \left(\int_{\mathcal{C}} f(z) \frac{1}{z - d_n} \mathrm{d}z \right) e_n \\ &= f(d_n) e_n \end{aligned}$$

   So $f(D) = \mathrm{diag}(f(d_n) : n \in \mathbb{N})$.

2. Suppose $A \in \mathcal{M}_n$. By Jordan form theorem A is similar to a direct sum of Jordan blocks

$$A \sim J_1 \oplus \cdots \oplus J_p$$

   with

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}_{k_i \times k_i}$$

   with $\sum_{i=1}^{p} k_i = n$.

   Suppose $f \in \mathrm{Hol}(U)$ with $U \supseteq \sigma(A) = \{ \lambda 1, \ldots, \lambda_p \}$. Note also that $\sigma(J_i) = \{ \lambda_i \}$. Since f is analytic near λ_i we get

$$f(z) = \sum_{m=0}^{\infty} a_m (z - \lambda_i)^m$$

on a neighborhood of λ_i. By last item of previous theorem we have

$$
\begin{aligned}
f(J_i) &= \sum_{m=0}^{\infty} a_m (J_i - \lambda_i I)^m \\
&= \sum_{n=0}^{\infty} a_n \begin{pmatrix} 0 & 1 & & \\ & \ddots & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}^n \\
&= \sum_{n=0}^{\infty} a_n \begin{pmatrix} 0 & & & 1 & & \\ & \ddots & & & \ddots & \\ & & & & & 1 \\ & & & & & \\ & & & & & \\ & & & & & 0 \end{pmatrix} \\
&= \begin{pmatrix} a_0 & a_1 & \cdots & a_{k_i-1} \\ & \ddots & \ddots & \vdots \\ & & & a_1 \\ & & & a_0 \end{pmatrix} \\
&= \begin{pmatrix} f(\lambda_i) & f'(\lambda_i) & \cdots & \frac{f^{(k_i-1)}}{(k_i-1)!} \\ & \ddots & \ddots & \vdots \\ & & & f'(\lambda_i) \\ & & & f(\lambda_i) \end{pmatrix}
\end{aligned}
$$

So

$$f(A) = f(S(\sum^{\oplus} J_i)S^{-1}) = Sf(\sum^{\oplus} J_i)S^{-1} = S(\sum^{\oplus} f(J_i)S^{-1}$$

If $\sigma(A) = \{ \lambda_1, \ldots, \lambda_p \}$ with

$$K_i = \max\{ k : A \text{ has a Jordan block of size } k \text{ with eigenvalue } \lambda_i \}$$

then $\dim(\ker(A - \lambda_i I)^s$ stops increasing at K_i. Write

$$f(z) = \underbrace{p(z)}_{\text{degree}<n} + \underbrace{\left(\prod_{i=1}^{p} (z - \lambda_i)^{K_i} \right)}_{\text{minimal polynomial of } A} g(z)$$

Then $f(A) = p(A)$.

Theorem 2.6. *Suppose $T \in \mathcal{B}(\mathfrak{X})$. Suppose $\sigma(T) = \sigma_1 \sqcup \sigma_2$ where the σ_i are disjoint compact sets. Then there are idempotents $E_1, E_2 \in \mathcal{B}(\mathfrak{X})$ such that $E_1 + E_2 = I$ and $E_i T = T E_i$. We may also demand that the $\mathcal{M}_i = \operatorname{Ran}(E_i)$ are complementary subspace (i.e. $\mathcal{M}_1 \cap \mathcal{M}_2 = \{ 0 \}$ and $\mathcal{M}_1 + \mathcal{M}_2 = \mathfrak{X}$), $T\mathcal{M}_i \subseteq \mathcal{M}_i$ (the $\mathcal{M}_i$ are invariant subspaces for T), and if $T_i = T \restriction \mathcal{M}_i \in \mathcal{B}(\mathcal{M}_i)$ then $\sigma(T_i) = \sigma_i$.*

Proof. Find open $U = U_1 \sqcup U_2$ with $U_i \supseteq \sigma_i$ and $U_1 \cap U_2 = \emptyset$. Let $f \in \operatorname{Hol}(U)$ be given by

$$z \mapsto \begin{cases} 1 & \text{if } z \in U_1 \\ 0 & \text{if } z \in U_2 \end{cases}$$

Let $E_1 = f(T)$ and $E_2 = I - E_1 = g(T)$ where $g = 1 - f$. Then $f = f^2$ and $g = g^2$, so $E_1 = E_1^2$ and $E_2 = E_2^2$; also $E_1 + E_2 = I$. Then since $f(T)T = Tf(T)$ we get $E_1T = TE_1$. Let $\mathcal{M}_i = \mathrm{Ran}(E_i) = \ker(E_{1-i})$. Then

$$E_1E_2 = (fg)(T) = 0(T) = 0$$

Also $\mathrm{Ran}(E_1) \subseteq \ker(E_2)$ and $\mathrm{Ran}(E_2) \subseteq \ker(E_1)$; furthermore if $x \in \ker(E_2)$ then $x = Ix = (E_1+E_2)x = E_1x$, so $\ker(E_2) \subseteq \mathrm{Ran}(E_1)$.

Thus the $\mathcal{M}_i$ are closed because $\ker(E_i)$ are closed. If $x \in \mathcal{M}_1 \cap \mathcal{M}_2$ then $x = E_1x = E_1(E_2x) = 0$. If $x \in \mathfrak{X}$ then $x = E_1x + E_2x \in \mathcal{M}_1 + \mathcal{M}_2$; so $\mathcal{M}_1 + \mathcal{M}_2 = \mathfrak{X}$. Also $T(E_1\mathfrak{X}) = E_1T\mathfrak{X} \subseteq E_1\mathfrak{X}$, so it's invariant.

Claim 2.7. $\sigma(T_1) = \sigma_1$.

Proof. If $\lambda \in \rho(T)$ then $I = (\lambda I - T)^{-1}(\lambda I - T)$. So

$$I_{\mathcal{M}_1} = I \restriction \mathcal{M}_1 = (\lambda I - T)^{-1}(\lambda I - T) \restriction \mathcal{M}_1 = \underbrace{((\lambda I - T)^{-1} \restriction \mathcal{M}_1)}_{\text{maps } \mathcal{M}_1 \text{ into } \mathcal{M}_1}\underbrace{(\lambda I_{\mathcal{M}_1} - T_1)}_{\text{range}\subseteq\mathcal{M}_1}$$

and likewise with right-multiplication. So $\lambda \in \rho(T_i)$.

If $\lambda \notin \sigma_1$ then $\frac{1}{\lambda - z}$ is analytic on a neighbourhood U_1 of σ_1 (and we may assume $\overline{U_1} \cap \sigma_2 = \emptyset$). Let

$$g(z) = \begin{cases} \frac{1}{\lambda - z} & \text{if } z \in U_1 \\ 0 & \text{if } z \in U_2 \end{cases}$$

Then $g(T)(\lambda I - T) = f(T)$ where

$$f(z) = \begin{cases} 1 & \text{if } z \in U_1 \\ 0 & \text{if } z \in U_2 \end{cases}$$

So $I_{\mathcal{M}_1} = g(T)(\lambda I - T) \restriction \mathcal{M}_1 = (\lambda I - T)g(T) \restriction \mathcal{M}_1$. So $\lambda \in \rho(T_1)$, and $\sigma(T_1) \subseteq \sigma_1$; similarly we get $\sigma(T_2) \subseteq \sigma_2$.

Subclaim 2.8. $\sigma(T_1 \oplus T_2) = \sigma(T_1) \cup \sigma(T_2)$.

Proof. Indeed, we have

$$\lambda I - (T_1 \oplus T_2) = (\lambda I_{\mathcal{M}_1} - T_1) \oplus (\lambda I_{\mathcal{M}_2} - T_2)$$

If $\lambda \in \rho(T_1) \cap \rho(T_2)$ then

$$(\lambda - T)^{-1} = (\lambda - T_1)^{-1} \oplus (\lambda - T_2)^{-1}$$

If $\lambda \in \sigma(T_1)$ then $\lambda I_{\mathcal{M}_1} - T_1$ either is not bounded below, in which case $\lambda - T$ is not bounded below, or has range not dense in $\mathcal{M}_1$, in which case $\overline{\mathrm{Ran}(T)} \subseteq \overline{\mathcal{M}_2 + \mathrm{Ran}(\lambda - T_1)}$ is proper. So $\sigma(T_1) \subseteq \sigma(T)$. □ Subclaim 2.8

So $\sigma(T_1) = \sigma_1$ and $\sigma(T_2) = \sigma_2$. □ Claim 2.7

□ Theorem 2.6

Suppose $\mathcal{H}$ is a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Suppose $U \supseteq \sigma(a)$ is open and $f \in \mathrm{Hol}(U)$. What is $f(T)^*$?

Well $\sigma(T^*) = \sigma(T)^*$ (complex conjugate) so $U^* \supseteq \sigma(T^*)$. Let $f^*(z) = \overline{f(z)}$ for $z \in U^*$; so $f^* \in \mathrm{Hol}(U^*)$.

TODO 10. *I think $f^*(z)$ should be $\overline{f(\overline{z})}$.*

Claim 2.9. $f(T)^* = f^*(T^*)$.

Proof. For $x, y \in \mathcal{H}$ we have

$$\begin{aligned}
\langle f(T)^* x, y\rangle &= \langle x, f(T)y\rangle \\
&= \left\langle x, \frac{1}{2\pi i}\int_{\mathcal{C}} f(z)(z-T)^{-1}\mathrm{d}z \cdot y\right\rangle \\
&= \overline{\left\langle \frac{1}{2\pi i}\int_{\mathcal{C}} f(z)(z-T)^{-1}y\mathrm{d}z, x\right\rangle} \\
&= \overline{\frac{1}{2\pi i}\int_{\mathcal{C}} f(z)\langle (z-T)^{-1}y, x\rangle \mathrm{d}z} \\
&= \frac{-1}{2\pi \overline{i}}\int_{\mathcal{C}^*} f^*(w)\overline{\langle(\overline{w}-T)^{-1}y, x\rangle}\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_{\mathcal{C}^*} f^*(w)\langle x, (\overline{w}-T)^{-1}y\rangle \mathrm{d}w \\
&= \frac{1}{2\pi i}\int_{\mathcal{C}^*} f^*(w)\langle (w-T^*)^{-1}x, y\rangle \mathrm{d}w \\
&= \langle f^*(T^*)x, y\rangle
\end{aligned}$$

Indeed, in general we have

$$\begin{aligned}
\overline{\int_{\mathcal{C}} g(z)\mathrm{d}z} &= \lim \overline{\sum g(\xi_i)(z_i - z_{i-1})} \\
&= \lim \sum \overline{g(\xi_i)}(\overline{z_i} - \overline{z_{i-1}}) \\
&= \lim \sum g^*(\overline{\xi_i})(\overline{z_i} - \overline{z_{i-1}}) \text{ (where } g^*(z) = \overline{g(\overline{z})}) \\
&= \int_{\overline{\mathcal{C}}} g^*(w)\mathrm{d}w \\
&= -\int_{\mathcal{C}^*} g^*(w)\mathrm{d}w
\end{aligned}$$

where $\mathcal{C}^* = -\overline{\mathcal{C}}$ (necessary since $\overline{\mathcal{C}}$ has winding number -1 around $\sigma(T^*)$.) ☐ Claim 2.9

Proposition 2.10 (Relative spectra)**.** *Suppose $1 \in \mathfrak{A} \subseteq \mathfrak{B}$ are two Banach algebras with the same unit. Then if $a \in \mathfrak{A}$ then $\sigma_{\mathfrak{A}}(a) \supseteq \sigma_{\mathfrak{B}}(a)$ and $\partial\sigma_{\mathfrak{A}}(a) \subseteq \partial\sigma_{\mathfrak{B}}(a)$. (Here ∂ denotes the boundary.)*

Example 2.11. Consider $A(\mathbb{D})$ with $X \subseteq \overline{\mathbb{D}}$ compact with $\mathbb{T} \subseteq X$; then we get an embedding $A(\mathbb{D}) \xhookrightarrow{\alpha_x} C(X)$ given by $f \mapsto f \restriction X$. Since $\mathbb{T} \subseteq X$ we have

$$\|\alpha_X(f)\| = \sup_{x\in X}|f(x)| = \|f\|_{A(\mathbb{D})}$$

We can thus consider $A(\mathbb{D}) \subseteq C(X)$. Consider $z \in A(\mathbb{D})$; we have $\sigma_{A(\mathbb{D})}(z) = \operatorname{Ran}(z) = \overline{\mathbb{D}}$, and $\sigma_{C(X)}(z) = \operatorname{Ran}(\alpha_X(z)) = X$.

We will need a definition before proving Proposition 2.10.

Definition 2.12. We say $a \in \mathfrak{A}$ is a *right (left, two-sided) topological divisor of zero* if there are $x_n \in \mathfrak{A}$ with $\|x_n\| = 1$ and $\|x_n a\| \to 0$.

Claim 2.13. *If $\lambda - a$ is a right or left topological divisor of zero then it isn't invertible (so $\lambda \in \sigma(a)$).*

Proof. If $(\lambda - a)^{-1}$ exists then $x(\lambda - a)(\lambda - a)^{-1} = x$; so $\|x\| \le \|x(\lambda - a)\|\|(\lambda - a)^{-1}\|$, and

$$\frac{\|x\|}{\|(\lambda - a)^{-1}\|} \le \|x(\lambda - a)\|$$

So $\lambda - a$ is not a right topological zero divisor. The case of left topological zero divisors is similar. ☐ Claim 2.13

Claim 2.14. *If $\lambda \in \partial\sigma_{\mathfrak{A}}(a)$ then $\lambda - a$ is a two-sided topological divisor of zero.*

Proof. Since $\lambda \in \partial\sigma_{\mathfrak{A}}(a)$ there are $\lambda_n \in \rho_{\mathfrak{A}}(a)$ such that $\lambda_n \to \lambda$. Then

$$(\lambda_n - a)^{-1}(\lambda - a) = (\lambda_n - a)^{-1}(\lambda_n - a + \lambda - \lambda_n) = 1 + (\lambda_n - a)^{-1}(\lambda - \lambda_n)$$

is not invertible. So by Proposition 1.11 we get that

$$\|(\lambda - \lambda_n)(\lambda_n - a)^{-1}\| \geq 1$$

and

$$\|(\lambda_n - a)^{-1}\| \geq \frac{1}{|\lambda - \lambda_n|}$$

Aside 2.15. This shows that

$$\|\mu - a)^{-1}\| \geq \frac{1}{\operatorname{dist}(\mu, \sigma(a))}$$

which is occasionally useful to know.

Let

$$x_n = \frac{(\lambda_n - a)^{-1}}{\|(\lambda_n - a)^{-1}\|}$$

Then

$$\|x_n(\lambda - a)\| = \left\| \frac{1 + (\lambda_n - a)^{-1}(\lambda - \lambda_n)}{\|(\lambda_n - a)^{-1}\|} \right\| \leq \frac{1}{\|(\lambda_n - a)^{-1}\|} + |\lambda - \lambda_n| \leq 2|\lambda - \lambda_n| \to 0$$

and $\lambda - a$ is a right topological divisor of zero. Since $(\lambda - a)x_n = x_n(\lambda - a)$ it is also a left topological divisor of zero. □ Claim 2.14

We are now ready to prove Proposition 2.10.

Proof. If $\lambda \in \rho_{\mathfrak{A}}(a)$ then we have some $(\lambda - a)^{-1} \in \mathfrak{A} \subseteq \mathfrak{B}$; so $\lambda \in \rho_{\mathfrak{B}}(a)$. So $\sigma_{\mathfrak{B}}(a) \subseteq \sigma_{\mathfrak{A}}(a)$.

If $\lambda \in \partial\sigma_{\mathfrak{A}}(a)$ then $\lambda - a$ is a right topological divisor of zero by the claim. So it is a right topological divisor of zero in $\mathfrak{B}$ as well (using the same x_n). So $\lambda \in \sigma_{\mathfrak{B}}(a)$. But there are $\lambda_n \in \rho_{\mathfrak{A}}(a) \subseteq \rho_{\mathfrak{B}}(a)$ with $\lambda_n \to \lambda$. So $\lambda \in \partial\sigma_{\mathfrak{B}}(a)$. □ Proposition 2.10

3 Commutative Banach algebras

Let $\mathfrak{A}$ be a commutative Banach algebra with unity.

Definition 3.1. A linear functional $\varphi\colon \mathfrak{A} \to \mathbb{C}$ is *multiplicative* if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathfrak{A}$ and $\varphi(1) = 1$.

Proposition 3.2. *If φ is a multiplicative linear functional on $\mathfrak{A}$ then $\|\varphi\| = 1$ (and so φ is continuous).*

Proof. Since $\varphi(1) = 1$ we have $\|\varphi\| \geq \frac{|1|}{\|1\|} = 1$. Suppose we had $\|\varphi\| > 1$. Then there is $x \in \mathfrak{A}$ with $\|x\| \leq 1$ and $|\varphi(x)| > 1$. Let $a = \frac{x}{\varphi(x)}$. So $\varphi(a) = 1$ and $\|a\| \leq \frac{1}{|\varphi(x)|} < 1$. Let

$$b = \sum_{n=1}^{\infty} a^n \in \mathfrak{A}$$

Note that $v = a + ab$; so $\varphi(b) = \varphi(a) + \varphi(a)\varphi(b) = 1 + \varphi(b)$, and $0 = 1$, a contradiction. So $\|\varphi\| = 1$. □ Proposition 3.2

If φ is a multiplicative linear functional then $\ker(\varphi)$ is a closed ideal of codimension 1; so $\ker(\varphi)$ is a maximal ideal. Conversely, suppose M is a maximal ideal; so $1 \notin M$ and $\mathfrak{A}^{-1} \cap M = \emptyset$. So $b_1(1) \cap M = \emptyset$.

The closure of an ideal is a (proper) ideal; in particular $\overline{M}$ is also an ideal. Indeed, if $m \in \overline{M}$ then there are $m_n \in M$ with $m_n \to m$; so if $a \in \mathfrak{A}$ then

$$am = \lim \underbrace{am_n}_{\in M} \in \overline{M}$$

and M is a subspace, so $\overline{M}$ is a subspace. It is proper since $\overline{M} \cap b_1(1) = M \cap b_1(1) = \emptyset$.

But $M \subseteq \overline{M}$ and M is maximal; so $M = \overline{M}$ and M is closed. So $\mathfrak{A}/M$ is a field.

Aside 3.3. If $\mathfrak{A}$ is a Banach algebra and J is a closed two-sided ideal then $\mathfrak{A}/J$ is an algebra and a Banach space. Also if $a, b \in \mathfrak{A}$ and we let $\dot{a} = a + J$ and $\dot{b} = b + J$ then

$$\|\dot{a}\dot{b}\| = \|(a+J)(b+J)\| \leq \|(a + \underbrace{j}_{\in J})(b + \underbrace{k}_{\in J})\| \leq \inf_{j,k \in J} \|a + j\| \|b + k\| = \|\dot{a}\| \|\dot{b}\|$$

TODO 11. *Another* inf *somewhere?*

So $\mathfrak{A}/J$ is a Banach algebra.

So $\mathfrak{A}/M$ is a Banach field; so by Proposition 1.30 we get an isomorphism $\psi\colon \mathfrak{A}/M \cong \mathbb{C}$. So M has codimension 1. Since ψ is an isomorphism we have $\psi(\dot{1}) = 1$. Define $\varphi_M\colon \mathfrak{A} \to \mathbb{C}$ by

$$\begin{tikzcd} \mathfrak{A} \arrow[rr, "\varphi_M"] \arrow[dr, "q"] & & \mathbb{C} \\ & \mathfrak{A}/M \arrow[ur, "\psi"] & \end{tikzcd}$$

Then φ is multiplicative.

We have thus shown most of the following:

Theorem 3.4. *There is a bijective correspondence between multiplicative linear funnctionals on $\mathfrak{A}$ and maximal ideals. Moreover, this set is non-empty.*

Proof. We have seen that the map $\varphi \mapsto \ker(\varphi)$ maps multiplicative linear functionals to maximal ideals; we have seen that this has inverse taking M to the above composition φ_M.

Claim 3.5.

TODO 12. *unlhd? lhd? trianglelefteq?*
If $I \lhd \mathfrak{A}$ is a proper ideal then there is a maximal ideal $M \supseteq I$.

Proof. We use Zorn's lemma. Consider the set $\mathcal{J}$ of proper ideals $J \lhd \mathfrak{A}$ such that $J \supseteq I$. If $\mathcal{C}$ is some totally ordered (by $\subseteq$) subset of $\mathcal{J}$ then

$$J' = \bigcup_{J \in \mathcal{C}} J$$

is an ideal. It is proper since $1 \notin J$ for all $J \in \mathcal{C}$, so $1 \notin J'$. So J' is an upper bound for $\mathcal{C}$ in $\mathcal{J}$. So by Zorn's lemma $\mathcal{J}$ contains a maximal element M, which is a maximal ideal. □ Claim 3.5

But $\{ 0 \}$ is a proper ideal. So there is a maximal ideal. □ Theorem 3.4

Definition 3.6. The collection $\mathcal{M}_{\mathfrak{A}}$ of all multiplicative linear functionals on $\mathfrak{A}$ is considered as a subset of $\mathfrak{A}'$ endowed with the weak* topology; we call this the *maximal ideal subspace* of $\mathfrak{A}$.

Definition 3.7. The *Gelfand transform* is the homomorphism $\Gamma\colon \mathfrak{A} \to C(\mathcal{M}_{\mathfrak{A}})$ given by $\Gamma(a) = \widehat{a}$ where $\widehat{a}(\varphi) = \varphi(a)$.

Theorem 3.8 (Gelfand)**.** *$\mathcal{M}_{\mathfrak{A}}$ is a compact Hausdorff space, and Γ is a contractive homomorphism into $C(\mathcal{M}_{\mathfrak{A}})$ and $\Gamma(\mathfrak{A})$ separates points in $\mathcal{M}_{\mathfrak{A}}$.*

Proof. If $a, b \in \mathfrak{A}$ then $\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \widehat{a}(\varphi)\widehat{b}(\varphi)$. So $\Gamma(ab) = \Gamma(a)\Gamma(b)$. Clearly Γ is linear.

If $\varphi_\alpha \in \mathcal{M}_{\mathfrak{A}}$ with $\varphi_\alpha \xrightarrow{w^*} \varphi$ then

$$\varphi(ab) = \lim_\alpha \varphi_\alpha(ab) = \lim_\alpha \varphi_\alpha(a)\varphi_\alpha(b) = \varphi(a)\varphi(b)$$

So $\mathcal{M}_{\mathfrak{A}}$ is a weak*-closed subset of $\overline{b_1(\mathfrak{A}')}$; by Banach-Alaoglu theorem, we have that $\overline{b_1(\mathfrak{A}')}$ is weak*-compact; so $\mathcal{M}_{\mathfrak{A}}$ is weak*-compact. Also note that

$$\widehat{a}(\varphi) = \varphi(a) = \lim \varphi_\alpha(a) = \lim \widehat{a}(\varphi_a)$$

so $\widehat{a}$ is continuous. To see that Γ is contractive, note that

$$\|\widehat{a}\| = \sup_{\varphi \in \mathcal{M}_{\mathfrak{A}}} |\widehat{a}(\varphi)| = \sup_{\varphi \in \mathcal{M}_{\mathfrak{A}}} |\varphi(a)| \leq \|a\|$$

To see that $\Gamma(\mathfrak{A}$ separates points in $\mathcal{M}_{\mathfrak{A}}$, we note that if $\varphi, \psi \in \mathcal{M}_{\mathfrak{A}}$ have $\varphi \neq \psi$ then $\exists a \in \mathfrak{A}$ such that $\widehat{a}(\varphi) = \varphi(a) \neq \psi(a) = \widehat{a}(\psi)$. □ Theorem 3.8

Theorem 3.9. *Suppose $\mathfrak{A}$ is a commutative unital Banach algebra. Then*

1. *a is invertible in $\mathfrak{A}$ if and only if $\widehat{a}$ is invertible in $C(\mathcal{M}_{\mathfrak{A}})$.*
2. *$\sigma(a) = \sigma_{C(\mathcal{M}_{\mathfrak{A}})}(\widehat{a}) = \mathrm{Ran}(\widehat{a})$.*
3. *$\|\widehat{a}\| = \mathrm{spr}(a)$.*

Proof.

1. If a is invertible in $\mathfrak{A}$ then $aa^{-1} = 1$. So $\Gamma(a)\Gamma(a^{-1}) = \Gamma(1) = 1$, and $\Gamma(a)$ is invertible. If a is not invertible then $J = a\mathfrak{A}$ is proper since $1 \notin J$ (this uses commutativity of $\mathfrak{A}$). So J is contained in some maximal ideal, which corresponds to some $\varphi \in \mathcal{M}_{\mathfrak{A}}$ with $0 = \varphi(a) = \widehat{a}(\varphi)$; so $\widehat{a}$ is not invertible.
2. Follows directly from previous item.
3. We have
$$\|\widehat{a}\| = \sup|\widehat{a}(\varphi)| = \sup\{\, |\lambda| : \lambda \in \sigma(a) = \mathrm{Ran}(\widehat{a}) \,\} = \mathrm{spr}(a)$$
as desired. □ Theorem 3.9

Definition 3.10. Suppose $\mathfrak{A}$ is a commutative Banach algebra with unity. The *radical* of $\mathfrak{A}$ is $\mathrm{rad}(\mathfrak{A}) = \ker(\Gamma) = \{\, a : \widehat{a} = 0 \,\}$. We say $\mathfrak{A}$ is *semisimple* if $\mathrm{rad}(\mathfrak{A}) = \{\, 0 \,\}$; i.e. Γ is injective.

Proposition 3.11. $\mathrm{rad}(\mathfrak{A}) = \{\, a \in \mathfrak{A} : \mathrm{spr}(a) = 0 \,\} = \{\, a : \lim\|a^n\|^{\frac{1}{n}} = 0 \,\}$ *is the set of quasi-nilpotent elements of $\mathfrak{A}$.*

Example 3.12.

1. Consider $\mathfrak{A} = C(X)$ with X compact and Hausdorff. Then for $x \in X$ we have $\varepsilon_x(f) = f(x)$ is multiplicative; so $\ker(\varepsilon_x) = \{\, f : f(x) = 0 \,\}$ is a maximal ideal. Suppose M is a maximal ideal; we can define $\ker(M) = \{\, x \in X : f(x) = 0 \text{ for all } f \in M \,\}$. If $x \in \ker(M)$ then $M \subseteq \ker(\varepsilon_x)$, and hence by maximality we have $M = \ker(\varepsilon_x)$.

   What if $\ker(M) = \emptyset$? Then for all $x \in X$ there is $f_x \in M$ such that $f_x(x) \neq 0$. Let $U_x = \{\, y \in X : f_x(y) \neq 0 \,\}$; these form an open cover of X, so by compactness there is a finite subcover $X \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$. Let
$$g = \sum_{i=1}^n f_{x_i}\overline{f_{x_i}} = \sum_{i=1}^n |f_{x_i}|^2 > 0$$
   so $g \in M$. But then g is invertible; so $M = \mathfrak{A}$ is not proper.

   Hence $\mathcal{M}_{C(X)} = X$ as a set. The topology on $\mathcal{M}_{C(X)}$ is the weak* topology induced by $(\mathfrak{A}', w^*)$. The sub-basic open sets in $\mathcal{M}_{C(X)}$ are $\{\, \varphi \in \mathcal{M}_{C(X)} : |\varphi(a) - \lambda| < r \,\}$; this corresponds via the above to $\{\, x \in X : |a(x) - \lambda| < r \,\}$, which are open in X because a is continuous. Hence the map $\gamma \colon X \to \mathcal{M}_{\mathfrak{A}}$ we (implicitly) defined above is continuous, injective, and surjective; since both X and $\mathcal{M}_{\mathfrak{A}}$ are compact and Hausdorff, we get that γ is a homeomorphism. So $\mathcal{M}_{C(X)} \approx X$.

2. Consider $\ell^1(\mathbb{Z})$ with

TODO 13. *This is a Banach algebra under convolution I guess?*

$$\delta_n(k) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{else} \end{cases}$$

Note that $\delta_n * \delta_m = \delta_{n+m}$. If $\varphi \in \mathcal{M}_{\ell^1(\mathbb{Z})}$ with $\varphi(\delta_1) = \alpha$ then $\varphi(\delta_n) = \varphi(\delta_1^n) = \varphi(\delta_1)^n = \alpha^n$.

TODO 14. *connective*

$|\alpha^n| \le \|\delta_n\|_1 = 1$ for all n; also $|\alpha| \le 1$ and $|\alpha^{-1}| \le 1$ implies $|\alpha| = 1$. We have thus determined a function $\mathcal{M}_{\ell^1(\mathbb{Z})} \to \mathbb{T}$.

Conversely if $|\alpha| = 1$ define

$$\varphi_\alpha(f) = \sum_{n\in\mathbb{Z}} a_n \alpha^n$$

where

$$f = \sum_{n\in\mathbb{Z}} a_n \delta_n$$

(so $\|f\|_1 = \sum |a_n| < \infty$). Then $\|\varphi_\alpha\| = \|(\alpha^n)_{n\in\mathbb{Z}}\|_\infty = 1$ (using the fact that $\ell_1(\mathbb{Z})' = \ell_\infty(\mathbb{Z})$). If

$$g = \sum_{n\in\mathbb{Z}} b_n \delta_n$$

then

$$(f * g)(n) = \sum_{k\in\mathbb{Z}} a_k b_{n-k}$$

This lies in $\ell^1(\mathbb{Z})$; indeed

$$\sum_k \underbrace{\sum_n |a_k b_{n-k}|}_{\text{absolutely convergent}} = \sum_k |a_k| \sum_n |b_{n-k}| = \|f\|_1 \|g\|_1$$

Also

$$\begin{aligned}
\varphi_\alpha(f * g) &= \sum_{n\in\mathbb{Z}} \alpha^n (f * g)(n) \\
&= \sum_{n\in\mathbb{Z}} \alpha^n \sum_{k\in\mathbb{Z}} a_k b_{n-k} \\
&= \sum_{k\in\mathbb{Z}} a_k \alpha^k \sum_{n\in\mathbb{Z}} \alpha^{n-k} b_{n-k} \text{ (since absolute convergence lets us rearrange the sum)} \\
&= \sum_{k\in\mathbb{Z}} a_k \alpha^k \sum_{\ell\in\mathbb{Z}} \alpha^\ell b_\ell \\
&= \varphi_\alpha(g)\varphi_\alpha(f)
\end{aligned}$$

So φ_α is multiplicative. Also φ is determined by $\varphi(\delta_1) = \alpha$. So this is a bijection $\mathcal{M}_{\ell^1(\mathbb{Z})} \to \mathbb{T}$. Also $\varphi \mapsto \varphi(\delta_1)$ is continuous by definition of the weak* topology. Thus this is a homeomorphism.

What of the Gelfand transform? Well $\Gamma\colon \ell^1(\mathbb{Z}) \to C(\mathbb{T})$ by $\Gamma(f) = \widehat{f}$ where $\widehat{f}(\alpha) = \varphi_\alpha(f)$. Write $\alpha = \exp(i\theta)$ with $0 \le \theta < 2\pi$; then

$$\widehat{f}(\exp(i\theta)) = \sum_{n=-\infty}^{\infty} a_n \exp(in\theta)$$

TODO 15. *Here $f(n) = a_n$?*

The range of Γ is the algebra $A(\mathbb{T})$ of all continuous functions on $\mathbb{T}$ whose Fourier series is absolutely convergent.

Theorem 3.13 (Wiener). *If $f \in A(\mathbb{T})$ and $\widehat{f}(\exp(i\theta)) \neq 0$ for all θ then $\frac{1}{f} \in A(\mathbb{T})$.*

Proof. We have

$$\sigma_{A(\mathbb{T})}(\widehat{f}) = \sigma_{\ell^1(\mathbb{Z})}(f) = \sigma_{C(\mathbb{T})}(\widehat{f}) = \operatorname{Ran} \widehat{f}$$

where the first equality is because the algebras are isomorphic, and the second is Gelfand's theorem. But $0 \notin \operatorname{Ran} \widehat{f}$, so $0 \notin \sigma_{A(\mathbb{T})}(\widehat{f})$, and $\widehat{f}$ is invertible in $A(\mathbb{T})$. □ Theorem 3.13

3. Consider $A(\mathbb{D})$ and $\ell^1(\mathbb{Z}^+)$ with $\mathbb{Z}^+ = \mathbb{N}_0$. Note that $A(\mathbb{D})$ is the closure of the polynomials in $C(\overline{\mathbb{D}})$. If $f \in A(\mathbb{D})$ then $f_r(z) = f(rz)$ for $0 \leq r < 1$ has Fourier series

$$f \sim \sum_{n \geq 0} a_n \exp(in\theta)$$
$$f_r \sim \sum_{n \geq 0} a_n r^n \exp(in\theta)$$

So

$$f_r(z) = \sum_{n \geq 0} a_n r^n z^n$$

converges absolutely and uniformly, and lies in the $C(\overline{\mathbb{D}})$-norm-closure of $\mathbb{C}[z]$. Also f is continuous on $\overline{\mathbb{D}}$, and hence uniformly continuous. So $f_r \to f$ uniformly. Thus f is also a limit of polynomials.

So $\{ z \}$ generates $A(\mathbb{D})$ as a unital Banach algebra. So any $\varphi \in \mathcal{M}_{A(\mathbb{D})}$ is determined by $\varphi(z) = \lambda$; note that $|\lambda| \leq \|z\| = 1$. Conversely if $\lambda \in \overline{\mathbb{D}}$ we let $\varphi_\lambda(f) = f(\lambda)$, which is clearly multiplicative. We get $\mathcal{M}_{A(\mathbb{D})} = \overline{\mathbb{D}}$.

The case $\ell^1(\mathbb{Z}_+)$ is similar, using $\varphi(\delta_1) = \lambda$; note here that $|\lambda| \leq \|\delta_1\|_1 = 1$. We get $\ell^1(\mathbb{Z}) \to C(\overline{\mathbb{D}})$ given by mapping $f = (a_n)_{n \geq 0}$ to

$$\widehat{f}(z) = \sum_{n=0}^{\infty} a_n z^n$$

for $z \in \overline{\mathbb{D}}$; this is a contractive hoommorphism. If $\lambda \in \overline{\mathbb{D}}$ then $\varphi_\lambda(f) = \widehat{f}(\lambda)$ is multiplicative.

Theorem 3.14. *Suppose $\mathfrak{A}, \mathfrak{B}$ are Banach algebras; suppose $\mathfrak{B}$ is commutative and semisimple. Then every algebra homomorphism $\theta \colon \mathfrak{A} \to \mathfrak{B}$ is (automatically) continuous.*

Proof. We are given the Gelfand map $\Gamma \colon \mathfrak{B} \to C(\mathcal{M}_{\mathfrak{B}})$ is injective. Suppose $\varphi \in \mathcal{M}_{\mathfrak{B}}$. Then $\varphi \circ \theta \colon \mathfrak{A} \to \mathbb{C}$ is multiplicative; hence $\|\varphi \circ \theta\| \leq 1$.

If $\mathfrak{A}$ is not commutative then $C = \overline{\langle ab - ba \rangle}$ is the commutator ideal, and is in the kernel of θ. We get a diagram

$$\begin{tikzcd} \mathfrak{A}/C \arrow[r, "\widetilde{\theta}"] & \mathfrak{B} \arrow[d, "\varphi"] \\ \mathfrak{A} \arrow[u, "q"] \arrow[ur, "\theta"] & \mathbb{C} \end{tikzcd}$$

with $\varphi \circ \widetilde{\theta}$ continuous (has norm ≤ 1) and $\|q\| \leq 1$. So $\|\theta\varphi\| \leq 1$.

We apply the closed graph theorem. If $a_n \in \mathfrak{A}$ with $a_n \to 0$ and $\theta(a_n) \to b$, we must show that $b = 0$. If $\varphi \in \mathcal{M}_{\mathfrak{B}}$ then

$$(\varphi \circ \theta)(\underbrace{a_n}_{\to 0}) \to 0$$

But also $\varphi(\theta(a_n)) \to \varphi(b)$; so $\varphi(b) = 0$. This holds for all φ; so $\Gamma(b) = 0$. But Γ is injective; so $b = 0$. So by the closed graph theorem we get that θ is continuous. □ Theorem 3.14

Corollary 3.15. *If $\mathfrak{A}$ is a commutative semisimple Banach algebra then*

1. *$\mathfrak{A}$ has a unique Banach algebra norm up to equivalence of norms.*
2. *Every automorphism of $\mathfrak{A}$ is continuous.*

Proof.

1. Let $\|\cdot\|$ be the norm on $\mathfrak{A}$. Suppose that $|||\cdot|||$ is a norm on $\mathfrak{A}$ which makes $\mathfrak{A}$ into a Banach algebra $(\mathfrak{A}, |||\cdot|||)$ is complete and $|||ab||| \leq |||a||| |||b|||$. Define $j\colon (\mathfrak{A}, |||\cdot|||) \to (\mathfrak{A}, |||\cdot|||)$ by $j(a) = a$. Then j is an algebra homomorphism, and is thus continuous by the theorem. So j is continuous, injective, and surjective, and is htus invertible. Thus $c\|a\| \leq |||a||| \leq C\|a\|$ for some $0 < c \leq C$.
2. Easy. □ Corollary 3.15

Corollary 3.16. *$C^\infty[0,1]$ has no norm that makes it a Banach algebra.*

Proof. Suppose $\|\cdot\|$ is a Banach algebra norm on $C^\infty[0,1]$. Let $j\colon C^\infty[0,1] \to C[0,1]$ be $j(f) = f$; note that $C[0,1]$ is commutative and semisimple. So j is continuous by theorem. Thus

$$\|f\|_\infty = \sup_{0 \leq x \leq 1} |f(x)| \leq C\|f\|$$

Claim 3.17. *The map $D\colon C^\infty[0,1] \to C^\infty[0,1]$ given by $Df = f'$ is continuous.*

Proof. D is everywhere defined, so we can use the closed graph theorem. Suppose $f_n \in C^\infty[0,1]$ has $\|f_n\| \to 0$ and $Df_n = f_n' \to g \in C^\infty[0,1]$; i.e. $\|f_n' - g\| \to 0$. Suppose $f_n \in C^\infty[0,1]$ has $\|f_n\| \to 0$ and $Df_n = f_n' \to g \in C^\infty[0,1]$; i.e. $\|f_n' - g\| \to 0$. Then $\|f_n\|_\infty \to 0$, so $\|f_n' - g\|_\infty \to 0$. If $0 \leq x < y \leq 1$ then

$$\int_x^y g(t)\mathrm{d}t = \int_x^y f_n'(t)\mathrm{d}t + \int_x^y (g - f_n')(t)\mathrm{d}t = (f_n(y) - f_n(x)) + \int_x^y (g - f_n')(t)\mathrm{d}t$$

Thus

$$\left|\int_x^y g(t)\mathrm{d}t\right| \leq |f_n(y)| + |f_n(x)| + \int_x^y \|g - f_n'\|_\infty \mathrm{d}t \leq 2\|f_n\|\infty + \|g - f_n'\|_\infty \cdot 1 \to 0$$

Thus

$$\int_x^y g(t)\mathrm{d}t = 0$$

for all x, y; thus $g = 0$. Thus D is continuous by the closed graph theorem. □ Claim 3.17

So there is c_2 such that $\|f'\| \leq c_2\|f\|$. Let $f(t) = \exp(2c_2 t)$, so $f' = 2c_2 f$. Then $2c_2\|f\| = \|f'\| \leq c_2\|f\|$; so $\|f\| = 0$ and $f = 0$, a contradiction. □ Corollary 3.16

3.1 The non-unital case

In this section, $\mathfrak{A}$ is non-unital.

TODO 16. *Are we still commutative?*

Definition 3.18. An ideal $I \lhd \mathfrak{A}$ is *modular* if $\mathfrak{A}/I$ is unital; i.e. there is $u \in \mathfrak{A}$ such that $a - au, a - ua \in I$ for all $a \in \mathfrak{A}$. An ideal is *maximal modular* if it is maximal among modular ideals.

Remark 3.19.

1. If $\mathfrak{A}$ is unital, then every proper ideal is modular.
2. If I is modular with unit u modulo I, then if $I \subseteq J \lhd \mathfrak{A}$ with $u \notin J$, then J is modular with unit u modulo J.

Theorem 3.20. *Every modular ideal is contained in a maximal modular ideal, and maximal modular ideals are closed.*

Proof. Suppose I is a modular ideal with unit u modulo I. Suppose J is a proper ideal containing I; then u is also a unit modulo J, and thus since J is proper we have $u \notin J$.

We now use Zorn's lemma. Suppose $\mathcal{C} = \{ J_\alpha \}$ is a chain of modular ideals containing I. Then $J = \bigcup \mathcal{C}$ is an ideal; since $u \notin J_\alpha$ for all α we get $u \notin J$, so J is modular by previous remark. So by Zorn's lemma we get a maximal modular ideal containing I.

Claim 3.21. *If M is modular with unit u modulo M, then $b_1(u) \cap M = \emptyset$.*

Proof. Suppose $x \in M$ with $\|x - u\| < 1$. Work in $\mathfrak{A}_+ = \mathfrak{A} \oplus \mathbb{C}e$, a unital Banach algebra containing $\mathfrak{A}$. Then $e + (x - u)$ is invertible in $\mathfrak{A}_+$, with inverse $\lambda e + y$ for some $y \in \mathfrak{A}$. Then

$$e = (e + x - u)(\lambda e + y) = \lambda e + y + \lambda x + xy - \lambda u - uy$$

Thus

$$(1 - \lambda)e = \underbrace{(y - uy)}_{\in M} + \underbrace{(\lambda x + xy)}_{\in M} - \lambda u \in \mathfrak{A}$$

So $\lambda = 1$; so $u \notin M$, a contradiction. ☐ Claim 3.21

In particular, we get $u \notin \overline{M}$, so $\overline{M}$ is also a modular ideal; hence if M is maximal then $M = \overline{M}$ is closed. ☐ Theorem 3.20

Proposition 3.22. *Suppose $\mathfrak{A}$ is a non-unital commutative Banach algebra. If φ is a multiplicative linear functional then $\|\varphi\| \leq 1$.*

Proof. Same as in the unital case for bounded above. ☐ Proposition 3.22

Remark 3.23. In the unital case we required $\varphi(1) = 1$ for φ to be a multiplicative linear functional; this no longer makes sense (since we're non-unital), so we instead require $\varphi \neq 0$.

Theorem 3.24. *There is a natural bijection $\varphi \mapsto \ker(\varphi)$ between $\mathcal{M}_{\mathfrak{A}}$ and maximal modular ideals of $\mathfrak{A}$.*

Proof. If φ is multiplicative and non-zero then $\varphi \colon \mathfrak{A} \to \mathbb{C}$ is surjective, so $\mathbb{C} \cong \mathfrak{A}/\ker(\varphi)$ is unital; so $M = \ker(\varphi)$ is modular and has codimension 1, and is thus maximal.

Conversely, suppose M is a maximal modular ideal. So M is closed; so $\mathfrak{A}/M$ is a (unital, by modularity) Banach algebra. We show that $\mathfrak{A}/M$ is a field, and is thus $\mathbb{C}$ by Mazur.

Suppose otherwise. Let $\varphi \colon \mathfrak{A} \to \mathfrak{A}/M$ be the quotient map; so there is $a \in \mathfrak{A} \setminus M$ such that $\varphi(a) \neq 0$ is not invertible. Then $J = \langle \varphi(a) \rangle = \varphi(a)\mathfrak{A}/M$ is a proper ideal; so $\varphi^{-1}(J) \trianglelefteq \mathfrak{A}$ with $M \subsetneq \varphi^{-1}(J)$. But $\mathfrak{A}/\varphi^{-1}(J) = (\mathfrak{A}/M)/J$ is unital; so $\varphi^{-1}(J)$ is modular, contradicting maximality of M.

So $\mathfrak{A}/M$ is a Banach field, and is thus $\mathbb{C}$. So $\mathfrak{A}/M \cong \mathbb{C}$, and φ defines a multiplicative linear functional. ☐ Theorem 3.24

Theorem 3.25. *Suppose $\mathfrak{A}$ is a non-unital commutative Banach algebra; let $\mathfrak{A}_+ = \mathfrak{A} \oplus \mathbb{C}e$ be the unitization. Then $\mathcal{M}_{\mathfrak{A}} = \mathcal{M}_{\mathfrak{A}_+} \setminus \{ \varphi_\infty \}$ where $\varphi_\infty(a + \lambda e) = \lambda$ is the multiplicative linear functional on $\mathfrak{A}_+$ with kernel $\mathfrak{A}$. Moreover, $\mathcal{M}_{\mathfrak{A}}$ is the locally compact Hausdorff space with topology induced as a subset of $\mathcal{M}_{\mathfrak{A}_+}$ and $\mathcal{M}_{\mathfrak{A}_+}$ is the 1-point compactification of $\mathcal{M}_{\mathfrak{A}}$.*

Definition 3.26. If X is Hausdorff and locally compact (i.e. every point $x \in X$ has a neighbourhood U such that $\overline{U}$ is compact) then the *1-point compactification* of X is the space $X_+ = X \cup \{ p \}$ where $U \subseteq X$ open is open in X_+ and neighbourhoods of p have the form $\{ p \} \cup (X \setminus K)$ where $K \subseteq X$ is compact.

Remark 3.27. X_+ is compact because if $\{ U_\alpha \}$ is an open cover, then there is α_0 with $p \in U_{\alpha_0}$; so $K = X_+ \setminus U_{\alpha_0}$ is compact in X, and the U_α cover K. So there is a finite subcover $K \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$; then $X \subseteq U_{\alpha_0} \cup \cdots \cup U_{\alpha_n}$.

X_+ is Hausdorff because if $x \in X$ then there is open $U \subseteq X$ such that $K = \overline{U}$ is compact. Then $x \in U$ and $p \in X \setminus K$ are separated by disjoint opens. (That $x, y \in X$ are separated by opens is just that X is Hausdorff.)

Proof of Theorem 3.25. If $\varphi \in \mathcal{M}_{\mathfrak{A}_+}$ then $\varphi \restriction \mathfrak{A}$ is a multiplicative lienar functional. But $\varphi_\infty \restriction \mathfrak{A} = 0$, and otherwise $\varphi \restriction \mathfrak{A} \neq 0$ (since $\mathfrak{A} \subseteq \ker(\varphi)$ implies $\varphi = \varphi_\infty$). So $\mathcal{M}_{\mathfrak{A}_+} \setminus \{ \varphi_\infty \}$ restricts to elements of $\mathcal{M}_{\mathfrak{A}}$. If $\varphi_1 \restriction \mathfrak{A} = \varphi_2 \restriction \mathfrak{A}$ then for $a + \lambda e \in \mathfrak{A}_+$ we have

$$\varphi_1(a + \lambda e) = \varphi_1(a) + \lambda = \varphi_2(a) + \lambda = \varphi_2(a + \lambda e)$$

So $\varphi_1 = \varphi_2$.

Conversely, if $\varphi \in \mathcal{M}_{\mathfrak{A}}$ we define $\widetilde{\varphi}(a + \lambda e) = \varphi(a) + \lambda$; one can check that $\varphi \mapsto \widetilde{\varphi}$ is a homomorphism.

We now verify the statement about the topology. In $\mathcal{M}_{\mathfrak{A}}$, the basic open sets have form

$$U(F, \varphi_0) = \{ \varphi \in \mathcal{M}_{\mathfrak{A}} : |\varphi(a_i) - \varphi_0(a_i)| < 1, 1 \leq i \leq n \}$$

where $\varphi_0 \in \mathcal{M}_{\mathfrak{A}}$ and $F = \{ a_1, \ldots, a_n \} \subseteq \mathfrak{A}$ is finite. In $\mathcal{M}_{\mathfrak{A}_+}$ the basic open neighbourhoods are of the form

$$V(G, \varphi_0) = \{ \varphi \in \mathcal{M}_{\mathfrak{A}_+} : |\varphi(b_i) - \varphi_0(b_i)| < 1, 1 \leq i \leq n \}$$

for $\varphi_0 \in \mathcal{M}_{\mathfrak{A}}$

TODO 17. *$\mathfrak{A}_+$?*

and $G = \{ b_1, \ldots, b_n \} \subseteq \mathcal{M}_{\mathfrak{A}_+}$ is finite. Write $b_i = a_i + \lambda_i e$, where $a_i \in \mathfrak{A}$ and $\lambda_i \in \mathbb{C}$. Then

$$|\varphi(b_i) - \varphi_0(b_i)| = |\varphi(a_i) + \lambda_i - \varphi_0(a_i) - \lambda_i|$$

So if $F = \{ a_1, \ldots, a_n \}$ then

$$V(G, \varphi_0) = V(F, \varphi_0) = \begin{cases} U(F, \varphi_0) & \text{if } \exists i_0 \text{ such that } |\varphi_0(a_{i_0})| \geq 1 \\ U(F, \varphi_0) \cup \{ \varphi_\infty \} & \text{else} \end{cases}$$

Thus the open sets of $\mathcal{M}_{\mathfrak{A}}$ have form $V \setminus \{ \varphi_\infty \}$ for V open in $\mathcal{M}_{\mathfrak{A}_+}$. Thus the topology on $\mathcal{M}_{\mathfrak{A}}$ is induced from $\mathcal{M}_{\mathfrak{A}_+}$. Since $\mathcal{M}_{\mathfrak{A}_+}$ is compact and Hausdorff, we get that $\mathcal{M}_{\mathfrak{A}}$ is locally compact and Hausdorff.

If $x \in \mathcal{M}_{\mathfrak{A}}$ then by Hausdorfness there is $U \ni x$ and $V \ni p$ open such that $U \cap V = \emptyset$. So $\overline{U} \subseteq \mathcal{M}_{\mathfrak{A}_+} \setminus V$ is compact; so $\mathcal{M}_{\mathfrak{A}}$ is locally compact and Hausdorff. Neighbourhoods of φ_∞ have the form $\{ \varphi_\infty \} \cup (\mathcal{M}_{\mathfrak{A}} \setminus K)$ where $K \subseteq \mathcal{M}_{\mathfrak{A}}$ is compact. So $\mathcal{M}_{\mathfrak{A}_+}$ is the one-point compactification of $\mathcal{M}_{\mathfrak{A}}$. $\square$ Theorem 3.25

3.1.1 $L^1(G)$

Suppose G is a locally compact abelian grape; i.e.

- G is an abelian grape
- G has a locally compact topology
- $(x, y) \mapsto xy$ is continuous $G \times G \to G$
- $x \mapsto x^{-1}$ is continuous $G \to G$.

Then $L^1(G)$ is a commutative Banach algebra under convolution. It is unital if and only if G is discrete, in which case δ_e is the unit. (Examples to keep in mind are $G = \mathbb{T}$ and $G = \mathbb{R}$.)

Such grapes have a *Haar measure*: a translation-invariant σ-finite Borel measure such that $\sigma(K) < \infty$ if K is compact. We usually normalize so that if G is compact then $m(G) = 1$ and if G is discrete then $m(e) = 1$. When integrating with respect to m we will sometimes just write $\mathrm{d}x$. (So on $\mathbb{T}$ we have $\mathrm{d}x = \frac{\mathrm{d}\theta}{2\pi}$.)

Definition 3.28. A *character* of a locally compact abelian grape G is a continuous homomorphism $\gamma \colon G \to \mathbb{T}$.

If γ, δ are characters then $(\gamma\delta)(x) = \gamma(x)\delta(x)$ is also a character; also $(\gamma^{-1})(x) = (\gamma(x))^{-1} = \overline{\gamma(x)}$ is also a character. So the set $\widehat{G}$ of all characters on G is a grape; we call this the *dual grape* of G.

Theorem 3.29. *Suppose G is a locally compact abelian grape. Then $\gamma \in \widehat{G}$ determines*

$$\varphi_\delta(f) = \int_G f(x)\overline{\gamma(x)}\mathrm{d}x$$

Then φ_δ is a multiplicative linear functional in $L^1(G)$, and every multiplicative linear functional arises in this way.

Proof. $\gamma(x)$ is continuous and $|\gamma(x)| = 1$; so $\gamma \in L^\infty(G)$. So φ_γ is a continuous linear functional on $L^1(G)$. Suppose $f, g \in L^1(G)$. Then

$$\begin{aligned}
\varphi_\gamma(f * g) &= \int_G \overline{\gamma(x)}(f * g)(x)\mathrm{d}x \\
&= \int_G \overline{\gamma(x)} \int_G f(y)g(y^{-1}x)\mathrm{d}y\mathrm{d}x \\
&= \int_G \overline{\gamma(y)}f(y) \int_G \overline{\gamma(y^{-1}x)}g(y^{-1}x)\mathrm{d}x\mathrm{d}y \text{ (using Fubini and } \gamma(x) = \gamma(y)\gamma(y^{-1}x)) \\
&= \int_G \overline{\gamma(y)}f(y) \int_G \overline{\gamma(t)}g(t)\mathrm{d}t\mathrm{d}y \text{ (by translation invariance)} \\
&= \varphi_\gamma(f)\gamma(g)
\end{aligned}$$

(Note that $\overline{\gamma(x)}f(x)g(y^{-1}x) \in L^1(G \times G)$, so Fubini's theorem holds.) So $\varphi_\gamma \in \mathcal{M}_{L^1(G)}$.

Conversely let $\varphi \in \mathcal{M}_{L^1(G)}$. Since $L^1(G)' = L^\infty(G)$ there is $\chi \in L^\infty(G)$ such that

$$\varphi(f) = \int_G f(x)\chi(x)\mathrm{d}x$$

with $\|\chi\|_\infty = \|\varphi\| \leq 1$. Also $\varphi \neq 0$ so there is $g \in L^1(G)$ such that $\varphi(g) = 1$. For $f \in L^1(G)$ we have

$$\begin{aligned}
\varphi(f) &= \varphi(f)\varphi(g) \\
&= \varphi(f * g) \\
&= \int_G \chi(x) \underbrace{\int_G f(y)g(y^{-1}x)\ y}_{(f*g)(x)} \mathrm{d}x \\
&= \int_G f(y) \int_G g(y^{-1}x)\chi(x)\mathrm{d}x\mathrm{d}y \text{ (Fubini)}
\end{aligned}$$

Let $L_y g)(x) = g(y^{-1}x$ be the (left) translation of g. A basic measure theory fact is that $y \mapsto L_y g$ is contained in L^1. (e.g. for $f \in L^1(\mathbb{R})$ if we define $f_y(x) = f(x - y)$ then $\|f - f_y\| \to 0$ as $y \to 0$.) Hence, continuing the above equations, we find

$$\varphi(f) = \int_G f(y)\varphi(L_y g)\mathrm{d}y$$

Since $y \mapsto L_y g$ is continuous, we get that φ is continuous. Define $\gamma(y) = \overline{\varphi(L_y g)}$ is a continuous map $G \to \mathbb{C}$.

A computation:

$$\begin{aligned}
(g * L_{xy}g)(t) &= \int g(s)(L_{xy}g)(s^{-1}t)\mathrm{d}s \\
&= \int g(s)g(y^{-1}x^{-1}s^{-1}t)\mathrm{d}s \\
&= \int g(x^{-1}s)g(y^{-1}x^{-1}(x^{-1}s)^{-1}t)\mathrm{d}s \\
&= \int g(x^{-1}s)g(s^{-1}y^{-1}t)\mathrm{d}s \\
&= \int (L_x g)(s)(L_y g)(s^{-1}t)\mathrm{d}s \\
&= ((L_x g) * (L_y g))(t)
\end{aligned}$$

So

$$\begin{aligned}\gamma(xy) &= \overline{\varphi(L_{xy}g)} \\ &= \overline{\varphi(g)\varphi(L_{xy}g)} \\ &= \overline{\varphi(g * L_{xy}g)} \\ &= \overline{\varphi(L_x g * L_y g)} \\ &= \overline{\varphi(L_x g)\varphi(L_y g)} \\ &= \gamma(x)\gamma(y)\end{aligned}$$

So γ is multiplicative. So

$$|\gamma(x)| = |\overline{\varphi(L_x g)}| \leq \|\varphi\|\|L_x g\|_1 \leq 1 \cdot \|g\|_1$$

So $|\gamma(x^n)| = |\gamma(x)^n| \leq \|g\|_1$ for all $n \in \mathbb{Z}$. So taking $n \geq 0$ we find $|\gamma(x)| \leq 1$, and taking $n \leq 0$ we find $|\gamma(x)| \geq 1$. So $\gamma(x) \in \mathbb{T}$, and γ is a character. □ Theorem 3.29

Corollary 3.30. *$\widehat{G}$ has a locally compact Hausdorff topology induced by this bijection with $\mathcal{M}_{L^1(G)}$, with $\gamma_\alpha \to \gamma$ if and only of $\varphi_{\gamma_\alpha} \xrightarrow{w^*} \varphi_\gamma$ in $L^1(G)' = L^\infty$, which occurs if and only if $\gamma_\alpha \xrightarrow{w^*} \gamma$ in L^∞.*

Definition 3.31. For $f \in L^1(G)$ we define

$$\widehat{f}(\gamma) = \Gamma f(\gamma) = \int f(x)\overline{\gamma(x)}\mathrm{d}x \in C_0(\widehat{G})$$

the *Fourier transform* of f.

Example 3.32.

1. In $\ell_1(\mathbb{Z})$ we have $\widehat{\mathbb{Z}} = \mathbb{T}$, done earlier.

   TODO 18. *ref*

2. Consider $L^1(\mathbb{T})$. We claim $\widehat{\mathbb{T}} = \mathbb{Z}$. Indeed, for all $n \in \mathbb{Z}$ we have $\gamma_n(t) = t^n$ a multiplicative map $\mathbb{T} \to \mathbb{T}$. Then $L^1(\mathbb{T}) \supseteq C(\mathbb{T}) \supseteq \{ f_k(t) = t^k : k \in \mathbb{Z} \}$, with $L^1(\mathbb{T}) = \overline{\operatorname{span}\{ f_k : k \in \mathbb{Z} \}}^{\|\cdot\|_1}$. Then

   $$\varphi_{\gamma_n}(f_k) = \int t^k \overline{t}^n \mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \exp(i\theta(k-n))\mathrm{d}\theta = \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases}$$

   and

   $$(f_k * f_\ell)(t) = \int s^k(s^{-1}t)^\ell \mathrm{d}s = t^\ell \int s^{k-\ell}\mathrm{d}s = \begin{cases} f_k & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases}$$

   So $f_k * f_k = f_k$ is an idempotent, and $f_k * f_\ell = 0$ with $k \neq \ell$.
   If $\varphi \in \mathcal{M}_{L^1(\mathbb{T})}$ then

   $$\varphi(f_k)^2 = \varphi(f_k * f_k) = \varphi(f_k) \in \{ 0, 1 \}$$

   and

   $$\varphi(f_k)\varphi(f_\ell) = \varphi(f_k * f_\ell) = 0$$

   if $k \neq \ell$. Then $\varphi(f_k)$ is not zero for all k implies $\varphi = 0$. So there is a unique n such that $\varphi(f_n) = 1$; so $\varphi = \varphi_n$. So $\widehat{\mathbb{T}} = \mathbb{Z}$.

3. Consider $L^1(\mathbb{R})$. We claim $\widehat{\mathbb{R}} = \mathbb{R}$. If $s \in \mathbb{R}$ we have $\varphi_s(x) = \exp(isx) \in \widehat{\mathbb{R}}$. Suppose φ is a character on $L^1(\mathbb{R})$; so φ is a continuous, multiplicative map $\mathbb{R} \to \mathbb{T}$. So $\varphi(0) = 1$, and $\operatorname{Re}(\varphi(x)) > \frac{1}{2}$ on some $[-\delta, \delta]$. So

   $$c_\delta = \int_0^\delta \varphi(x)\mathrm{d}x \neq 0$$

So

$$\varphi(t)c_\delta = \varphi(t)\int_0^\delta \varphi(x)\mathrm{d}x = \int_0^\delta \varphi(t+x)\mathrm{d}x = \int_t^{t+\delta} \varphi(x)\mathrm{d}x$$

φ is continuous, so RHS is differentiable. So

$$\begin{aligned}\varphi'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_t^{t+\delta} \varphi(x)\mathrm{d}x\right)\\ &= \varphi(t+\delta) - \varphi(t)\\ &= \varphi(t)(\varphi(\delta)-1)\end{aligned}$$

Let

$$s = \frac{\varphi(\delta)-1}{ic_\delta}$$

Then $\varphi'(t) = (is)\varphi(t)$. So $\varphi(t) = c\exp(ist)$ and $1 = \varphi(0) = c$ and $1 = |\varphi(t)| = |\exp(ist)|$ for all t; so $s \in \mathbb{R}$. So $\varphi = \varphi_s$.

So as a set we have $\widehat{\mathbb{R}} = \mathbb{R}$. The topology on $\widehat{\mathbb{R}}$ is induced by $(L^\infty(\mathbb{R}), w^*)$. If $s_\alpha \to s$ in $\mathbb{R}$ then $\exp(is_\alpha t) \to \exp(ist)$ uniformly on $[-n,n]$ for all $n \in \mathbb{N}$. So $\exp(is_\alpha t) \xrightarrow{w^*} \exp(ist)$ in L^∞. If $g \in C_{00}(\mathbb{R})$ (i.e. has compact support) then $g(t)\exp(-is_\alpha t) \to g(t)\exp(ist)$ uniformly. Thus

$$\varphi_{s_\alpha}(g) = \int g(t)\exp(-is_\alpha t)\mathrm{d}t \to \int g(t)\exp -ist\mathrm{d}t = \varphi_s(g)$$

But we can approximate $f \in L^1$ by $g \in C_{00}(\mathbb{R})$. So $\mathbb{R} \to \widehat{\mathbb{R}}$ is continuous.

Lemma 3.33. *If $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$ is uniformly continuous and $\lim_{|x|\to\infty}(f * g)(x) = 0$.*

TODO 19. *Defer until later?*

Lemma 3.34 (Riemann-Lebesgue). *If $f \in L^1(\mathbb{R})$ then*

$$\lim_{|x|\to\infty} \widehat{f}(x) = 0$$

Proof. Suffices to prove this for $g \in C_{00}(\mathbb{R})$; so g is uniformly continuous with $\mathrm{supp}(K) \subseteq [-n,n]$

TODO 20. *I assume $K = \mathrm{supp}(g)$ instead?*

and if $\varepsilon > 0$ there is $\delta > 0$ such that whenever $|x-y| < \delta$ we have $|g(x) - g(y)| < \varepsilon$. If $|x|$ is big then

$$\begin{aligned}\widehat{g}(x) &= \int g(t)\exp(-ixt)\mathrm{d}t\\ &= -\int g(t)\exp\Big(-ix\Big(t+\frac{\pi}{x}\Big)\Big)\mathrm{d}t\\ &= -\int g\Big(t-\frac{\pi}{x}\Big)\exp(-ixt)\mathrm{d}t\\ &= \frac{1}{2}\int \Big(g(t) - g\Big(t-\frac{\pi}{x}\Big)\Big)\exp(-ixt)\mathrm{d}t\end{aligned}$$

If $\left|\frac{\pi}{x}\right| < \delta$ (so $|x| > \frac{\pi}{\delta}$)) then

$$|\widehat{g}(x)| \le \frac{1}{2}\int_{n-\delta}^n \varepsilon|\exp(-ixt|\mathrm{d}t \le \frac{2n+\delta}{2}\varepsilon \to 0$$

□ Lemma 3.34

In particular if $\varphi_{s_\alpha} \xrightarrow{w^*} \varphi_s$ in L^∞ then either there is a cofinal subset s_β such that $s_\beta \to \infty$, which by Riemann-Lebesgue implies $\varphi_{s_\beta} \xrightarrow{w^*} 0$, a contradiction, or it is eventually bounded. Look at the cluster points in $\mathbb{R}$. If $s_\beta \to t$ and $s_{\beta'} \to s$ with $s \neq t$ then $\varphi_{s_\beta} \xrightarrow{w^*} \varphi_t$ and $\varphi_{s_{\beta'}} \xrightarrow{w^*} \varphi_s$, so $\varphi_{s\alpha} \not\xrightarrow{w^*} \varphi_s$; all this implies the topology on $\widehat{\mathbb{R}}$ is homeomorphic to $\mathbb{R}$.

TODO 21. *Connectives.*

Theorem 3.35. *Suppose G is a locally compact abelian grape; let $\widehat{G}$ be the dual grape with the w^* topology. Then*

1. *$(x, \gamma) \mapsto \gamma(x)$ is continuous on $G \times \widehat{G}$.*
2. *If $K \subseteq G$ is compact and $C \subseteq \widehat{G}$ is compact then*
$$N(K, r) = \{\, \gamma \in \widehat{G} : |\gamma(x) - 1| < r \text{ for all } x \in K \,\}$$
$$N(C, r) = \{\, x \in G : |\gamma(x) - 1| < r \text{ for all } \gamma \in C \,\}$$
are open in $\widehat{G}$ and G, respectively.
3. *$\{\, N(K, r)\gamma_0 : K \subseteq G \text{ compact}, r > 0, \gamma_0 \in \widehat{G} \,\}$ is a base for the topology of $\widehat{G}$.*
4. *$\widehat{G}$ is a locally compact grape (i.e. $(\gamma_1, \gamma_2) \mapsto \gamma_1\gamma_2^{-1}$ is continuous.)*

Proof.

1. Write $f_x(y) = f(x^{-1}y)$. Then
$$\begin{aligned}
\widehat{f_x}(\gamma) &= \int_G f_x(t)\overline{\gamma(t)}\mathrm{d}t \\
&= \int_G f(\underbrace{x^{-1}t}_{s})\overline{\gamma(\underbrace{t}_{xs})}\mathrm{d}t \\
&= \int f(s)\overline{\gamma(xs)}\mathrm{d}t \text{ (translation-invariance)} \\
&= \overline{\gamma(x)}\int_G f(s)\overline{\gamma(s)}\mathrm{d}t \\
&= \overline{\gamma(x)}\widehat{f}(\gamma)
\end{aligned}$$

Claim 3.36. *$(x, \gamma) \mapsto \widehat{f_x}(\gamma)$ is continuous on $G \times \widehat{G}$.*

Proof. Fix (x_0, γ_0). Translation is continuous in $L^1(G)$, so there is open $V \ni x_0$ such that $\|f_x - f_{x_0}\|_1 < \varepsilon$ for all $x \in V$. Since γ_0 is weak*-continuous there is open $W \ni \gamma_0$ such that $|\widehat{f_{x_0}}(\gamma) - \widehat{f_{x_0}}(\gamma_0)| < \varepsilon$ for all $\gamma \in W$. Then if $x \in V$ and $\gamma \in W$ we have
$$\begin{aligned}
|\widehat{f_x}(\gamma) - \widehat{f_{x_0}}(\gamma_0)| &\leq |\widehat{f_x}(\gamma) - \widehat{f_{x_0}}(\gamma)| + |\widehat{f_{x_0}}(\gamma) - f_{x_0}(\gamma_0)| \\
&< \left|\int_G (f_k(t) - f_{x_0}(t))\overline{\gamma(t)}\mathrm{d}t\right| + \varepsilon \\
&< \|f_x - f_{x_0}\|_1 + \varepsilon \\
&< 2\varepsilon
\end{aligned}$$
as desired. □ Claim 3.36

Now
$$\gamma(x) = \overline{\left(\frac{\widehat{f_x}(\gamma)}{\widehat{f}(\gamma)}\right)}$$
Pick f so that $\widehat{f}(\gamma_0) \neq 0$. So $\widehat{f}(\gamma) \neq 0$ on some neighbourhood $W \ni \gamma_0$. So $\gamma(x)$ is the quotient of continuous functions with non-zero denominator near γ_0, and is thus continuous at (x_0, γ_0).

2. Suppose $K \subseteq G$ is open and $r > 0$. Then

$$N(K, r) = \{ \gamma : |\gamma(x) - 1| < r, x \in K \}$$

Suppose $\gamma_0 \in N(K, r)$; so $|\gamma_0(x) - 1| < r$ for $x \in K$. But for each $x \in K$, continuity of $(x, \gamma) \mapsto \gamma(x)$ means there is are neighbourhoods $V_x \ni x$ and $W_x \ni \gamma_0$ such that for all $y \in V_x$ and $\gamma \in W_x$ we have $|\gamma(y) - 1| < r$. The V_x form an open cover of K; so there is a finite subcover $K \subseteq V_{x_1} \cup \cdots \cup V_{x_n}$. Let

$$W = \bigcap_{i=1}^{n} W_{x_i}$$

which is open in $\widehat{G}$ and contains γ_0. So if $\gamma \in W$ then $|\gamma(x) - 1| < r$ (since $x \in V_{x_i}$ for some i and $W \subseteq W_{x_i}$). So $W \subseteq N(K, r)$.

The second part is quite similar.

3. Without loss of generality we may assume $\gamma_0 = e$. Suppose W is open in $\widehat{G}$ with $0 \in W$. So there are $f_1, \ldots, f_n \in L^1(G)$ such that

$$0 \in \{ \gamma : |\widehat{f}_i(\gamma) - \widehat{f}_i(e)| < 1, 1 \le i \le n \} \subseteq W$$

We use the fact that $C_{00}(G)$ is dense in $L^1(G)$; we replace f_i by continuous, compactly supperted function. Let K be compact and contain

$$\bigcup_{i=1}^{n} \operatorname{supp}(f_i)$$

Let

$$r = \frac{1}{\max_i \|f_i\|_1}$$

If $\gamma \in N(K, r)$ then for $1 \le i \le n$ we have

$$\begin{aligned} |\widehat{f}_i(\gamma) - \widehat{f}_i(e)| &= \left| \int_G f_i(t)(\overline{\gamma(t)} - 1) \mathrm{d}t \right| \\ &\le \int_K |f_i(t)||\gamma(t) - 1| \mathrm{d}t \\ &< r\|f_i\|_1 \\ &\le 1 \end{aligned}$$

So $e \in N(K, r) \subseteq \{ \gamma : |\widehat{f}_i(\gamma) - \widehat{f}_i(e)| < 1, 1 \le i \le n \} \subseteq W$. So the $N(K, r)$ form a base for the topology.

4. Suppose $\gamma_1, \gamma_2 \in \widehat{G}$. Suppose $\gamma_1\gamma_2^{-1} \in N(K, r)\gamma_1\gamma_2^{-1}$. If $\gamma_1' \in N(K, \frac{r}{2})\gamma_1$ and $\gamma_2' \in N(K, \frac{r}{2})\gamma_2$ then $\gamma_1\gamma_2^{-1} \subseteq N(K, \frac{r}{2})N(K, \frac{r}{2})^{-1}\gamma_1\gamma_2^{-1}$. But

$$\begin{aligned} N(K, \frac{r}{2}) &= \{ \gamma : |\gamma(t) - 1| < \frac{r}{2}, t \in K \} \\ N(K, \frac{r}{2}) &= \{ \gamma : |\overline{\gamma(t)} - 1| < \frac{r}{2}, t \in K \} \\ N(K, \frac{r}{2}) &= \{ \gamma : |\gamma^{-1}(t) - 1| < \frac{r}{2}, t \in K \} \end{aligned}$$

So for $\gamma_1' \in N(K, \frac{r}{2}), \gamma_2' \in N(K, \frac{r}{2})^{-1}$ we have

$$|\gamma_1\gamma_2^{-1}(t) - 1| = |\gamma_1(t) - \gamma_2(t)| \le |\gamma_1(t) - 1| + |1 - \gamma_2(t)| < \frac{r}{2} + \frac{r}{2} = r$$

So

$$\gamma_1\gamma_2^{-1} \subseteq N(K, \frac{r}{2})N(K, \frac{r}{2})^{-1}\gamma_1\gamma_2^{-1} \subseteq N(K, r)\gamma_1\gamma_2^{-1}$$

and continuity follows. □ Theorem 3.35

4 Banach *-algebras

Definition 4.1. A *Banach *-algebra* is a Banach algebra $\mathfrak{A}$ with a continuous involution $a \mapsto a^*$ such that

1. $(a^*)^* = a$.
2. $(\lambda a)^* = \overline{\lambda} a^*$ and $(a+b)^* = a^* + b^*$.
3. $(ab)^* = b^* a^*$.

Example 4.2.

1. $C(X)$ and $C_0(X)$ with $f^*(x) = \overline{f(x)}$.
2. $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space, and the involution is the Hilbert space adjoint.
3. Consider $L^1(\mathbb{R})$.

   Proposition 4.3. *$L^1(\mathbb{R})$ is a Banach *-algebra with involution $f^*(x) = \overline{f(-x)}$. Moreover the Gelfand/Fourier transformation is a *-homomorphism.*

   Proof. Easy to check the *-algebra properties. Also

$$\begin{aligned}\widehat{f^*}(s) &= \int_{\mathbb{R}} f^*(x) \exp(-isx) \mathrm{d}x \\ &= \int_{\mathbb{R}} \overline{f(-x)} \exp(-isx) \mathrm{d}x \\ &= \overline{\int f(-x) \exp(isx) \mathrm{d}x} \\ &= \overline{\int f(y) \exp(-isy) \mathrm{d}y} \\ &= \overline{\widehat{f}(s)}\end{aligned}$$

   as desired. ☐ Proposition 4.3

Definition 4.4. If $\mathfrak{A}$ is a (non-unital) Banach algebra, a *bounded (norm 1) approximate identity* is a net $\{ e_\alpha \}$ such that $\sup \|e_\alpha\| < \infty$ (≤ 1) such that $a e_\alpha \to a$ and $e_\alpha a \to a$ for all $a \in \mathfrak{A}$.

Proposition 4.5. *$e_n = \frac{n}{2} \chi_{[-n^{-1}, n^{-1}]}$ form a norm 1 approximate identity for $L^1(\mathbb{R})$.*

Proof. Indeed, if $f \in L^1(\mathbb{R})$, then since translation is continuous then for any $\varepsilon > 0$ there is $\delta > 0$ such that $\|f_x - f\|_1 < \varepsilon$ if $|x| < \delta$. The if $\frac{1}{n} < \delta$ we have

$$\begin{aligned}(e_n * f - f)(t) &= \int_{\mathbb{R}} f(t-x) e_n(x) \mathrm{d}x - f(t) \\ &= \frac{2}{n} \int_{-\frac{1}{n}}^{\frac{1}{n}} f(t-x) \mathrm{d}x - \frac{2}{n} \int_{\frac{1}{n}}^{n} f(t) \mathrm{d}x \\ &= \frac{2}{n} \int_{-\frac{1}{n}}^{\frac{1}{n}} (f_x(t) - f(t)) \mathrm{d}x\end{aligned}$$

Thus

$$\begin{aligned}\|e_n * f - f\|_1 &\leq \int_{\mathbb{R}} \frac{2}{n} \int_{-\frac{1}{n}}^{\frac{1}{n}} |f_x(t) - f(t)| \mathrm{d}x \mathrm{d}t \\ &= \frac{2}{n} \int_{-\frac{1}{n}}^{\frac{1}{n}} \underbrace{\int |f_x(t) - f(t)| \mathrm{d}t}_{\|f_x - f\|_1 < \varepsilon} \mathrm{d}x \\ &< \varepsilon\end{aligned}$$

as desired. ☐ Proposition 4.5

Most of these facts hold for arbitrary locally compact grapes, but we hope to save ourselves some technicalities by working just with $\mathbb{R}$.

Lemma 4.6. *If $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$ then $f * g \in C_0(\mathbb{R})$ and is uniformly continuous.*

Proof. Note that

$$\begin{aligned}
|(f*g)(x) - (f*g)(y)| &\le \int |(f(x-t) - f(y-t))g(t)|\mathrm{d}t \\
&\le \|g\|_\infty \int |f(t-x) - f(t-y)|\mathrm{d}t \\
&= \|g\|_\infty \|f_x - f_y\|_1 \\
&= \|g\|_\infty \|f - f_{y-x}\|_1 \\
&\to 0 \text{ as } x - y \to 0
\end{aligned}$$

So $f * g$ is uniformly continuous; it remains to show that $f \in C_0(\mathbb{R})$. Suppose for contradiction that there were $\varepsilon > 0$ and $|x_n| \to \infty$ such that $|f * g(x_n)| \ge \varepsilon$. By uniform continuity there is $\delta > 0$ such that $|x - y| < \delta$ implies $|f * g(x) - f * g(y)| < \frac{\varepsilon}{2}$. Without loss of generality assume $|x_n - x_m| \ge 2\delta$ for all $n \ne m$. Then the $(x_n - \delta, x_n + \delta)$ are disjoint, and

$$\int_{x_n-\delta}^{x_n+\delta} |f * g(t)|\mathrm{d}t \ge \int_{x_n-\delta}^{x_n+\delta} \frac{\varepsilon}{2}\mathrm{d}t = \varepsilon\delta$$

So

$$\infty > \|f * g\|_1 \ge \sum_{n\ge 1} \int_{x_n-\delta}^{x_n+\delta} |f * g| \ge \sum_{n=1}^{\infty} \varepsilon\delta = \infty$$

a contradiction. So $f * g \in C_0(\mathbb{R})$. □ Lemma 4.6

Theorem 4.7. *$L^1(\mathbb{R})$ is semisimple.*

Proof. Suppose $0 \ne f \in \mathrm{rad}(L^1(\mathbb{R}))$; i.e. $\mathrm{spr}(f) \ne 0$ (by Theorem 3.9). Let

$$u_n = \frac{n}{2}\chi_{[-\frac{1}{n}, \frac{1}{n}]} \in L^\infty$$

be a norm 1 approximate identity for $L^1(\mathbb{R})$; so $f * u_n \to f$, so there is n_0 such that $f * u_{n_0} \ne 0$ and $f * u_{n_0} \in \mathrm{rad}(L^1(\mathbb{R}))$. Replace f with $f * u_n$, so without loss of generality we have $f \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\mathrm{spr}(f) = 0$. Define $f^* \in L^1(\mathbb{R})$ by $f^*(t) = \overline{f(-t)}$. So $f * f^* \in \mathrm{rad}(L^1(\mathbb{R}) \cap C_0(\mathbb{R})$; so

$$\begin{aligned}
f * f^*(0) &= \int f(t) f^*(-t)\mathrm{d}t \\
&= \int f(t)\overline{f(t)}\mathrm{d}t \\
&= \|f\|_2^2 \\
&> 0
\end{aligned}$$

Note that

$$\|f\|_2^2 = \int |f(t)||f(t)|\mathrm{d}t \le \|f\|_\infty \|f\|_1$$

is finite, since $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Define $F\colon L^1 \to \mathbb{C}$ by $F(g) = f * f^* * g(0)$. Then

$$|F(g)| = \left| \int f * f^*(t) g(-t)\mathrm{d}t \right| \le \|f * f^*\|_\infty \|g\|_1$$

so F is continuous. Define a sesquilinear form on $L^1(\mathbb{R})$ by $\psi(g, h) = F(g * h^*)$, which is then continuous by the above. Then

$$\psi(g, g) = f * f^* * g * g^*(0) = (f * g) * (f * g)^*(0) = \|f * g\|_2^2 \ge 0$$

Then

$$\begin{aligned}
\psi(h,g) &= f * f^* * h * g^*(0) \\
&= \int\int (f * f^*)(t) h(s) g^*(-s-t) \mathrm{d}s\mathrm{d}t \\
&= \int\int (f * f^*)(t) h(s) \overline{g(s+t)} \mathrm{d}s\mathrm{d}t \\
\overline{\psi(h,g)} &= \int\int \overline{(f * f^*(t))h(s)} g(s+t) \mathrm{d}s\mathrm{d}t \\
&= \int\int \underbrace{(f * f^*)^*}_{f*f^*}(-t) h^*(-s) g(s+t) \mathrm{d}s\mathrm{d}t \\
&= \psi(g,h)
\end{aligned}$$

So ψ is conjugate linear. Then by Cauchy-Schwarz we get $|\psi(g,h)| \leq \psi(g,g)^{\frac{1}{2}}\psi(h,h)^{\frac{1}{2}}$. Then

$$\begin{aligned}
\psi(u_n,u_n) &= (f * u_n) * (f * u_n)^*(0) \\
&= \|f * u_n\|_2^2 \\
&\leq \frac{n}{2}\int_{-\frac{1}{n}}^{\frac{1}{n}} \|f_s\|_2^2 \\
&= \|f\|_2^2
\end{aligned}$$

TODO 22. *f_s? f_5?*

Let $K = \|f\|_2^2$. Then

$$\begin{aligned}
|F(g)| &= \lim_{n\to\infty} |\underbrace{F(g * u_n)}_{\psi(g,u_n)}| \\
&\leq \lim_{n\to\infty} |F(g * g^*)|^{\frac{1}{2}} |\psi(u_n,u_n)|^{\frac{1}{2}} \\
&= K^{\frac{1}{2}} F(g * g^*)^{\frac{1}{2}} \\
&\leq K^1 \frac{1}{2} (K^{\frac{1}{2}} F(g * g^* * g * g^*)^{\frac{1}{2}})^{\frac{1}{2}} \\
&= K^{\frac{1}{2}} K^{\frac{1}{4}} F((g * g^*)^2)^{\frac{1}{4}} \\
&\leq K^{\frac{1}{2}} K^{\frac{1}{4}} K^{\frac{1}{8}} \cdots K^{\frac{1}{2^n}} F((g * g^*)^{2^{n-1}})^{\frac{1}{2^n}} \\
&\leq K^{1-\frac{1}{2^n}} \|f * f^*\|_\infty \|(g * g^*)^{2^{n-1}}\|^{\frac{1}{2^n}} \\
&\to K\|f * f^*\|_\infty \operatorname{spr}(g * g^*)^{\frac{1}{2}}
\end{aligned}$$

Take $g = f * f^*$. Then

$$F(f * f^*) = f * f^* * f * f^*(0) = \|f * f^*\|_2^2 > 0$$

a contradiction. So $\operatorname{rad}(L^1(\mathbb{R})) = 0$. □ Theorem 4.7

5 Non-commutative Banach algebras and their representation theory

Definition 5.1. A left ideal J of a (Banach) algebra $\mathfrak{A}$ is *modular* if there is $e \in \mathfrak{A} \setminus J$ such that $\mathfrak{A}(1-e) \subseteq J$.

Remark 5.2.

1. If $\mathfrak{A}$ is unital then every proper left ideal is modular.

2. If J is a 2-sided ideal which is left and right modular then the same e works for both. Indeed, given $e_1, e_2 \in \mathfrak{A} \setminus J$ such that $\mathfrak{A}(1-e_1) \subseteq J$ and $(1-e_2)\mathfrak{A} \subseteq J$, we have $e_2 - e_2e_1 \in J$ and $e_1 - e_2e_1 \in J$; so $e_1 - e_2 \in J$. Then
$$(1-e_1)\mathfrak{A} = (1-e_2)\mathfrak{A} + (e_2-e_1)\mathfrak{A} \subseteq J + J = J$$
as desired.

Proposition 5.3. *Suppose $\mathfrak{A}$ is a non-unital Banach algebra; let $\mathfrak{A}_+ = \mathfrak{A} + \mathbb{C}1$ be the unitization. If I is a proper ideal of $\mathfrak{A}_+$ with $I \not\subseteq \mathfrak{A}$ then $I_0 = I \cap \mathfrak{A}$ is a modular left ideal of $\mathfrak{A}$. Conversely if I_0 is a modular left ideal of $\mathfrak{A}$ with right modular unit e then $I = I_0 + \mathbb{C}(1-e)$ is a proper left ideal of $\mathfrak{A}_+$.*

Proof.

($\Longrightarrow$) Since $I_0 \subsetneqq I$ and $\mathfrak{A}$ has codimension 1 in $\mathfrak{A}_+$, we get that I_0 has codimension 1 in I. Pick $a+\lambda 1 \in I \setminus I_0$; note that $\lambda \neq 0$. So $1 + \lambda^{-1}a \in I$. Let $e = -\lambda^{-1}a$. So $\mathfrak{A}(1-e) = \mathfrak{A}(1+\lambda^{-1}a) \subseteq I \cap \mathfrak{A} = I_0$; so I_0 is modular.

($\Longleftarrow$) I_0 is proper, so I is proper (by a dimension argument). Then $\mathfrak{A}_+ I = \mathfrak{A}I_0 + \mathfrak{A}(1-e) + (\mathbb{C}1)I \subseteq I_0 + I + I = I$. So I is a left ideal. □ Proposition 5.3

Corollary 5.4. *If I is a left modular ideal of $\mathfrak{A}$ with right modular unit e then $b_1(e) \cap I = \emptyset$.*

Proof. Suppose $a \in I$ has $\|a - e\| < 1$. Then $(1-e) + a \in I + \mathbb{C}(1-e)$ is contained in a proper left ideal of $\mathfrak{A}_+$; but $(1-e)+a = 1 + a - e$ is invertible in $\mathfrak{A}_+$ by Proposition 1.11, a contradiction. (Proper ideals don't contain invertibles.) □ Corollary 5.4

Proposition 5.5. *If I is a left modular ideal with right modular unit e and $I \subseteq J$ with J a proper left ideal then J is modular with the same unit e. Hence I is contained in a maximal modular left ideal, and such ideals are closed.*

Proof. Note that $J \cap b_1(e) = \emptyset$. Indeed, otherwise by proof of the previous corollary we would have $J + \mathbb{C}(1-e) = \mathfrak{A}_+$ which is impossible since J is proper; so $J + \mathbb{C}(1-e)$ has codimension ≥ 1. Then $\mathfrak{A}(1-e) \subseteq I \subseteq J$. Maximality is by Zorn's lemma, and we note that $\overline{J}$ is still proper since it is disjoint from $b_1(e)$, so maximal implies closed. □ Proposition 5.5

Definition 5.6. If X is a vector space and $\mathcal{L}(X)$ the space of linear maps from $X \to X$, a *representation* of $\mathfrak{A}$ is a homomorphism $\pi\colon \mathfrak{A} \to \mathcal{L}(X)$. This makes X into a left $\mathfrak{A}$-module by $a \cdot x = \pi(a)x$. We say (X, π) is a *trivial module* if $X = \mathbb{C}$ and $\pi = 0$. We say X is *irreducible* if 0 and X are the only submodules and X is not trivial.

Proposition 5.7. *Suppose X is an irreducible left $\mathfrak{A}$-module.*

1. *If $0 \neq x_0 \in X$ then $\mathfrak{A}x_0 = X$.*
2. *$I_{x_0} = \{\, a : a \cdot x_0 = 0 \,\} = \ker_\pi(x_0)$ is a maximal modular left ideal with right modular unit e for any e satisfying $e \cdot x_0 = x_0$.*
3. *$\ker(\pi) = \bigcap_x \ker_\pi(x)$ is the intersection of maximal modular ideals (and is thus closed). Also $\ker(\pi) = I_{x_0} : \mathfrak{A} = \{\, a : a\mathfrak{A} \subseteq I_{x_0} \,\}$ for any $x_0 \neq 0$.*

Proof.

1. $\mathfrak{A}x_0$ is a submodule of X, so by irreducibility either $\mathfrak{A}x_0 = X$ or $\mathfrak{A}x_0 = \{\,0\,\}$. Suppose the latter; then $\mathbb{C}x_0$ is a non-zero submodule and is thus X, so X is trivial, a contradiction.
2. Pick e such that $ex_0 = x_0$ by (1). Then for $a \in \mathfrak{A}$ we have
$$a(1-e)x_0 = ax_0 - a(ex_0) = ax_0 - ax_0 = 0$$
So $\mathfrak{A}(1-e) \subseteq I_{x_0}$, and I_{x_0} is modular. Suppose $J \supsetneqq I$ is a left ideal; then $Jx_0 \neq 0$ is a submodule, so $Jx_0 = X$. So there is $f \in J$ such that $fx_0 = x_0$. So $\mathfrak{A} - \mathfrak{A}f = \mathfrak{A}(1-f) \subseteq I_{x_0} \subseteq J$; so $\mathfrak{A} \subseteq \mathfrak{A}f + J \subseteq J$, and $J = \mathfrak{A}$. So I_{x_0} is maximal.

3. First part is evident. If $a\mathfrak{A} \subseteq I_{x_0}$ and $x \in X$, we can pick $b \in \mathfrak{A}$ such that $bx_0 = x$. Then $ax = abx_0 \subseteq (a\mathfrak{A})x_0 = \{0\}$; so $a \in \ker(\pi)$. Conversely if $a \in \ker(\pi)$ then for all $b \in \mathfrak{A}$ we have $0 = a(bx_0) = (ab)x_0$, so $ab \in I_{x_0}$. So $\ker(\pi) = I_{x_0} : \mathfrak{A}$. □ Proposition 5.7

Proposition 5.8. *If I is a maximal modular left ideal in $\mathfrak{A}$ then there is a continuous representation π on a Banach space X with a vector $0 \neq x_0 \in X$ such that $I = I_{x_0}$ and $\ker(\pi) = I : \mathfrak{A}$.*

Proof. Let $X = \mathfrak{A}/I$ as a Banach space. Define $\pi(a)(b+I) = ab + I$. (Check that this is well-defined.) Then

$$\begin{aligned} \|\pi(a)\| &= \sup_{\|\dot{b}\|<1} \|\dot{ab}\| \\ &= \sup_{\|b\|<1} \inf_{i\in I} \|ab+i\| \\ &\leq \sup_{\|b\|<1} \inf_{i\in I} \|ab+ai\| \\ &\leq \sup_{\|b\|<1} \inf_{i\in I} \|a\|\|b+i\| \\ &\leq \|a\| \end{aligned}$$

So it is continuous. Let e be a right modular unit for I; let $x_0 = \dot{e}$. Then

$$I_{x_0} = \{\, a : a\dot{e} = 0 \,\} = \{\, a : ae \in I \,\} = I$$

So $\ker(\pi) = I_{x_0} : \mathfrak{A} = I : \mathfrak{A}$. □ Proposition 5.8

Definition 5.9 (Talked to Ken after the fact)**.** Suppose $\mathfrak{A}$ is a Banach algebra. A *Banach module* is an $\mathfrak{A}$-module $\mathfrak{X}$ that is also a Banach space such that for any $a \in \mathfrak{A}$ the map $\ell_a : X \to X$ given by $x \mapsto ax$ is a bounded linear operator on X and furthermore the map $\mathfrak{A} \mapsto \mathcal{C}(X)$ given by $a \mapsto \ell_a$ is continuous. A *continuous representation* is a continuous algebra homomorphism $\pi : \mathfrak{A} \to \mathcal{C}(\mathfrak{X})$ for some Banach space $\mathfrak{X}$; i.e. a representation such that each $\pi(a)$ lies in $C(\mathfrak{X})$ (rather than just $\mathcal{L}(\mathfrak{X})$) and $\pi : \mathfrak{A} \to \mathcal{C}(\mathfrak{X})$ is continuous.

Theorem 5.10. *Suppose X is an irreducible $\mathfrak{A}$-module and $x_0 \neq 0$; so by the above $I_{x_0} = \{\, a : a \cdot x_0 = 0 \,\}$ is a maximal ideal. Then $\theta : \mathfrak{A}/I_{x_0} \to X$ defined by $\theta(a+I) = a \cdot x_0$ is a well-defined module isomorphism and the norm $\|ax_0\| = \|a+I\|$ makes X into a Banach space. Moreover if X is already a Banach module then θ is a Banach space isomorphism.*

Proof. Since $I_{x_0} \cdot x_0 = 0$, we get that θ is well-defined. If $x \in X$ then there is b such that $x = bx_0$. So

$$\theta(a \underbrace{\dot{b}}_{\in \mathfrak{A}/I}) = \theta(\dot{ab}) = abx_0 = a(bx_0 = a\theta(\dot{b})$$

So θ is a morphism of modules; it is bijective since

$$\begin{aligned} \theta(\dot{a}) = \theta(\dot{b}) &\iff ax_0 = bx_0 \\ &\iff (a-b)x_0 = 0 \\ &\iff a - b \in I_{x_0} \\ &\iff a = b \end{aligned}$$

The proposed norm is just the norm on $\mathfrak{A}/I$ and $\mathfrak{A}/I$ is a Banach $\mathfrak{A}$-module.

If X already has a norm $\|\cdot\|_X$ and the action is continuous then $\|\pi\| < \infty$. Thena

$$\|\theta(\dot{a})\|_X = \|a \cdot x_0\|_X = \|(a+i)x_0\|_X$$

for all $i \in I_{x_0}$. So

$$\|\theta(\dot{a})\| \leq \inf_{i\in I} \|\pi\|\|a+i\|\|x_0\|_X = (\|\pi\|\|x_0\|)\|\dot{a}\|$$

So θ is continuous and bijective, and is thus invertible by the Banach isomorphism theorem.

Exercise 5.11. Check that θ^{-1} is also a morphism of bimodules.

So θ is an isomorphism of Banach modules. □ Theorem 5.10

TODO 23. *I think $I = I_{x_0}$ throughout.*

Definition 5.12. A 2-sided ideal $J \trianglelefteq \mathfrak{A}$ is *primitive* if it is the kernel of an irreducible representation.

Corollary 5.13. *The primitive ideals of $\mathfrak{A}$ have form $I : \mathfrak{A} = \{ a : a\mathfrak{A} \subseteq I \}$ for I a maximal modular left ideal.*

Definition 5.14. The *radical* $\mathrm{rad}(\mathfrak{A})$ is

$$\bigcap_{\pi \text{ irreducible}} \ker(\pi)$$

We say $\mathfrak{A}$ is *semisimple* if $\mathrm{rad}(\mathfrak{A}) = \{ 0 \}$. We say $\mathfrak{A}$ is *radical* if $\mathfrak{A}$ has no irreducible representations.

Example 5.15.

1. Consider $\mathfrak{A} = \mathfrak{T}_n \subseteq M_n(\mathbb{C}) = \mathcal{B}(\mathbb{C}^n)$ consisting of the upper triangular $n \times n$ matrices; we use the norm

   $$\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$$

   What are the left ideals of $\mathfrak{T}_n$? Suppose we have such I, and $A \in I$. Supose $a_{i_0,j_0} \neq 0$. Recall the matrix units $E_{ij} = e_i e_j^*$, so $E_{ij}x = \langle x, e_j \rangle e_i$; note $E_{ij} \in \mathfrak{T}_n$ if $i \leq j$. Then

   $$E_{ii_0} A e_{j_0} = a_{i_0 j_0} e_i$$

   But

   $$Ae_{i_0} = \sum_j a_{i_0,j} e_j$$

   So

   TODO 24. *Some conclusion about upward closed sets of indices within a column.*

   For $i \leq j \leq n$ we have $J_i = \{ T \in \mathfrak{T}_n : t_{jj} = 0 \}$ is a maximal 2-sided ideal of codimension 1, and is thus maximal as a left ideal. Then we have $\pi \colon \mathfrak{T}_n / J_j \to \mathbb{C}$ given by $T \mapsto t_{jj}$; then π_j is irreducible and $\ker(\pi_j) = J_j$. Suppose I is a left ideal but $I \not\subseteq J_j$ for all j. Then there is $A_j \in I$ such that $a_{jj} \neq 0$; so $E_{jj} A_j \in I$. But $\mathrm{Ran}(E_{jj} A_j) = \mathbb{C} e_j \frac{1}{a_{jj}} E_{jj} A_j$ is the set of matrices with 0 outside the j^{th} column and 1 in the (j,j)-entry (and upper triangular).

   $$\begin{pmatrix} 0 & \cdots & & & \\ & & & & \\ & & 1 & x & x \\ & & & & \\ & & & & \\ & & & & 0 \end{pmatrix}$$

   So

   $$I \ni A = \sum \frac{1}{a_j j} E_{jj} A_j$$

   and A is upper triangular with 1's on the diagonal. So A is invertible, and $I = \mathfrak{T}_n$. So the J_j are the maximal left ideals. So

   $$\mathrm{rad}(\mathfrak{T}_n) = \bigcap_{j=1}^{n} \ker(\pi_j) = \mathfrak{T}_n^0$$

   the strictly upper triangular matrices.

2. Consider $\mathfrak{A} = M_n$; the only ideals are $\{ 0 \}$ and M_n.

   Claim 5.16. *The maximal left ideals have form $I_x = \{ A \in M_n : Ax = 0 \}$ for $x \neq 0$.*

Proof. Clearly $\mathrm{id}\colon M_n \hookrightarrow \mathcal{B}(\mathbb{C}^n)$ is irreducible. So the I_x are maximal modular left ideals. Conversely suppose I is a left ideal but for all $x \neq 0$ there is $A_x \in I$ such that $A_x x \neq 0$. Let $e_1, \dots, e_n$ be the standard basis; let $A_{e_1} e_1 = u \neq 0$. Let $B = \|u\|^{-2} e_1 u^*$; so $BA_{e_1} e_1 = e_1$. Then B, and hence $C_1 = BA_{e_1}$, have rank 1; so $C_1 = e_1 v_1^*$ for some v_1 with $\langle v_1, e_1 \rangle \neq 0$.

Take $x \perp v_1$; then $A_x x \neq 0$; find rank-1 B_2 (again can take $\|x\|^{-2} e_2 x^*$)

TODO 25. *Really?*

and let $C_2 = B_2 A_x$; so $C_2 = e_2 v_2^*$ for some v_2 with $\langle v_2, x \rangle \neq 0$. So $\{ v_1, v_2 \}$ is linearly independent.

Now take $x \perp \{ v_1, v_2 \}$, etc. We build $e_j v_j^* \in I$ such that $\{ v_1, \dots, v_n \}$ are linearly independent and

$$\sum e_i v_j^*$$

is invertible. So $I = M_n$. □ Claim 5.16

The representation on M_n / I_x is just the identity representation because $\mathrm{id}\colon M_n \to \mathcal{B}(\mathbb{C}^n)$. Fix $x \neq 0$, and get $I_x = \{ A : Ax = 0 \}$ maximal modular. So id is isomorphic to a representation on M_n / I_x. So id is the unique (up to equivalence) irreducible representation of M_n.

Theorem 5.17. *Suppose $\mathfrak{A}$ is a Banach algebra, and consider $1 \in \mathfrak{A}_+$ if $\mathfrak{A}$ is not unital. Then the following are equivalent:*

(1) $a \in \mathrm{rad}(\mathfrak{A})$.

(2l) *a is in the intersection of all maximal modular left ideals of $\mathfrak{A}$.*

(2r) *a is in the intersection of all maximal modular right ideals of $\mathfrak{A}$.*

(3) *$\sigma(ab) = \{ 0 \}$ for all $b \in \mathfrak{A}$.*

(3') *$\sigma(ba) = \{ 0 \}$ for all $b \in \mathfrak{A}$.*

(4l) *$ab - \lambda$ is left-invertible for all $\lambda \neq 0$ and $b \in \mathfrak{A}$.*

(4l') *$ba - \lambda$ is left-invertible for all $\lambda \neq 0$ and $b \in \mathfrak{A}$.*

(4l) *$ab - \lambda$ is right-invertible for all $\lambda \neq 0$ and $b \in \mathfrak{A}$.*

(4l') *$ba - \lambda$ is right-invertible for all $\lambda \neq 0$ and $b \in \mathfrak{A}$.*

TODO 26. *Mathmode for description labels?*

Lemma 5.18. *If $\lambda \neq 0$ and $ab - \lambda$ is left (right) invertible then so is $ba - \lambda$.*

Proof. Let $u \in \mathfrak{A}_+$ satisfy $u(ab - \lambda) = 1$. Then $bua(ba - \lambda) = bu(ab - \lambda)a = ba$. Then

$$\left(\frac{bua - 1}{\lambda} \right)(ba - \lambda) = \frac{ba - (ba - \lambda)}{\lambda} = 1$$

as desired. □ Lemma 5.18

Hence

- 4l is equivalent to 4l'.
- 4r is equivalent to 4r'.
- 3 is equivalent to 3'.

Proof of Theorem 5.17.

(1) $\iff$ (2l) Done. Indeed we have

$$\operatorname{rad}(\mathfrak{A}) = \bigcap_{\pi \text{ irreducible}} \ker(\pi) = \bigcap \{ \ker(\pi_I) : I \text{ maximal left modular} \}$$

(3) $\implies$ (4l,4r) Immediate.

(1) $\implies$ (4l) Suppose there is $\lambda \neq 0$ and b such taht $ab - \lambda$ is not left invertible. Then $J = \mathfrak{A}(1 - \lambda^{-1}ab) = \mathfrak{A}(ab - \lambda)$ is a proper ideal and has $\lambda^{-1}ab$ as a right modular unit; so J is left modular, and is contained in some I maximal left modular. Then we have $\pi\colon \mathfrak{A} \to \mathcal{L}(\mathfrak{A}/I)$ with

$$\pi(a)\dot{b} = \dot{ab} = \lambda \dot{1} \neq 0$$

so $a \notin \ker \pi$, and $a \notin \operatorname{rad}(\mathfrak{A})$.

(4l') $\implies$ (2l) Suppose there is a maximal modular left ideal I with $a \notin I$. So $\dot{a} \neq 0$ in $\mathfrak{A}/I$, which is an irreducible module. So there is $b \in \mathfrak{A}$ such that $b\dot{a} = \dot{1}$. So $ba - 1 \in I$ is contained in a proper left ideal; so $ba - 1$ is not left invertible.

(1) $\implies$ (3) Suppose $a \in \operatorname{rad}(\mathfrak{A})$, $b \in \mathfrak{A}$, and $\lambda \neq 0$. Since 1 implies 4l we get that $ab - \lambda$ has left inverse u; so $1 = u(ab - \lambda) = uab - \lambda u$, and $uab \in \operatorname{rad}(\mathfrak{A})$ (since $a \in \operatorname{rad}(\mathfrak{A})$). So $\lambda u = uab - 1$ is left-invertible again since 1 implies 4l. So there is v such that $v(\lambda u) = 1$; so u is left- and right-invertible, and is thus invertible. So $ab - \lambda = u^{-1}$ is invertible.

(2l) $\iff$ (2r) Use the fact that 3 is left-right blind. $\square$ Theorem 5.17

Definition 5.19. If X is a non-trivial Banach $\mathfrak{A}$-module we say X is *topologically irreducible* if the only closed submodules are $\{0\}$ and X.

Example 5.20. There are topologically irreducible Banach modules that aren't algebraically irreducible. (i.e. what we called irreducible before.) Consider $\mathbb{F}_2^+$ the free monoid on $\{x, y\}$; this is the set of words $i_1 \cdots i_k$ with $k \geq 0$ and each $i_j \in \{x, y\}$. We define $v \cdot w$ to be their concatenation: if $v = i_1 \cdots i_k$ and $w = j_1 \cdots j_\ell$ then $v \cdot w = i_1 \cdots i_k j_1 \cdots j_\ell$. Let $\mathfrak{A} = \ell_1(\mathbb{F}_2^+)$ be the set of

$$\sum_{v \in \mathbb{F}_2^+} \lambda_v v$$

subject to

$$\left\| \sum \lambda_v v \right\| = \sum |\lambda_v| < \infty$$

We define $v \cdot w = vw$, so $(\lambda v)(\mu w) = (\lambda\mu)vw$. Define $\pi\colon \ell_1(\mathbb{F}_2^+) \to \mathcal{B}(\ell_2)$ by $\pi(x) = S$ the unilateral shift and $\pi(y) = S^*$. So

$$\pi(x^{k_1} y^{\ell_1} \cdots x^{k_m} y^{\ell_m} = S^{k_1}(S^*)^{\ell_1} \cdots S^{k_m}(S^*)^{\ell_m}$$

If ε is the empty word

$$\begin{aligned} \pi(\varepsilon) &= I \\ &= \pi(yx) \\ &= S^* S \\ \pi(xy) &= SS^* \\ \pi(\varepsilon - xy) &= I - SS^* \\ &= e_0 e_0^* \end{aligned}$$

So

$$\pi(x^n(\varepsilon - xy)y^j) = S^i e_0 e_0^* (S^*)^j = (S^i e_0)(S^j e_0^*) = e_i e_j^*$$

So $\overline{\operatorname{Ran} \ell_1(\mathbb{F}_2^+)} \supseteq \overline{\operatorname{span}\{ E_{ij} \}} = \mathcal{K}$ is the space of compact operators, which acts transitively. So it's topologically irreducible. But $X = \ell_1(\mathbb{F}_2^+)e_0 \subseteq \ell_1 \subsetneqq \ell_2$; so it's not algebraically irreducible.

Theorem 5.21 (Schur's lemma)**.** *Suppose $\mathfrak{A}$ is a Banach algebra and X an irreducible $\mathfrak{A}$-module. Let $\mathcal{D} = \{ T \in \mathcal{L}(X) : Ta = aT \text{ for all } a \in \mathfrak{A} \}$. Then $\mathcal{D} = \mathbb{C}I$.*

Proof. Note that $\mathcal{D}$ is an algebra (it's a subspace, and closed under multiplication). We claim that $\mathcal{D} \subseteq \mathbb{C}I$.

Suppose $T \in \mathcal{D} \setminus \{ 0 \}$; so $TX \neq \{ 0 \}$ is a submodule and $a(Tx) = T(ax) \in TX$. So $TX = X$; so $\ker(T) \neq X$ is a submodule. If $x \in \ker(T)$ then $T(ax) = a(Tx) = 0$; so $\ker(T) = \{ 0 \}$, and T is invertible. But now

$$aT^{-1} = T^{-1}(Ta)T^{-1} = T^{-1}ATT^{-1} = T^{-1}a$$

so $T^{-1} \in \mathcal{D}$, and $\mathcal{D}$ is a division algebra.

Now, X is irreducible, so without loss of generality we take $X = \mathfrak{A}/I_{x_0}$ for any $0 \neq x_0 \in X$.

TODO 27. *ref*

(Recall $I_{x_0} = \{ a \in \mathfrak{A} : ax_0 = 0 \}$.) In particular the $\mathfrak{A}$-action is continuous on X. So if $T \in \mathcal{D}$ then

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\| = \sup_{\|a+I_{x_0}\|\leq 1} \|T(a+\underbrace{i}_{\in I_{x_0}})x_0\| \leq \sup_{\|a+I_{x_0}\|\leq 1} \|(x+i)Tx_0\| \leq \sup_{\|a+I_{x_0}\|\leq 1} \inf_{i\in I_{x_0}} \|a\|\|Tx_0\| = \|Tx_0\| < \infty$$

So $\mathcal{D} \subseteq \mathcal{B}(X)$. Also $\mathcal{D}$ is closed: if $T_n \in \mathcal{D}$ and $(T_n)_n \to T$ then

$$aT = \lim_{n\to\infty} aT_n = \lim_{n\to\infty} T_n a = Ta$$

So $\mathcal{D}$ is a Banach division ring containing $\mathbb{C}I$; so $\mathcal{D} = \mathbb{C}I$ by Mazur's theorem. ☐ Theorem 5.21

Definition 5.22. Suppose $\mathfrak{A}$ is a Banach algebra and X a $\mathfrak{A}$-module. We say $\mathfrak{A}$ is

- *transitive* if $\mathfrak{A}x_0 = X$ for all $x_0 \neq 0$
- *k-transitive* if whenever $x_1, \ldots, x_k$ linearly independent in X and $y_1, \ldots, y_k \in X$ there is $a \in \mathfrak{A}$ such that $ax_i = y_i$ for $1 \leq i \leq k$
- *strictly transitive* if it is k-transitive for all $k \geq 1$.

Theorem 5.23 (Jacobson density theorem)**.** *If X is an irreducible $\mathfrak{A}$-module then $\mathfrak{A}$ is strictly transitive in $\mathfrak{A}$.*

Standing assumption: X is an irreducible $\mathfrak{A}$-module.

Lemma 5.24. *Suppose $x_1, x_2 \in X$ are linearly independent. Then there is $a \in \mathfrak{A}$ such that $ax_1 = 0$ and $ax_2 \neq 0$.*

Proof. Suppose not; suppose $ax_1 = 0$ implies $ax_2 = 0$. Define $T \colon X \to X$ linear by $T(ax_1) = ax_2$ for all $a \in \mathfrak{A}$; this is defined on $X = \mathfrak{A}x_1$ and if $ax_1 = bx_1$ then $(a-b)x_1 = 0$ implies $(a-b)x_2 = 0$ and $ax_2 = bx_2$. So T is well-defined and linear. If $b \in \mathfrak{A}$ and $x = ax_1$ then

$$T(bx) = T(bax_1) = b(ax_2) = bT(ax_1) = bTx$$

So $Tb = bT$ and $T \in \mathcal{D} = \mathbb{C}I$. So $x_2 \in \mathbb{C}x_1$, a contradiction. ☐ Lemma 5.24

Lemma 5.25. *Suppose $n \geq 3$ and $x_1, \ldots, x_n$ are linearly independent in X. Then there is $a \in \mathfrak{A}$ such that $ax_1 = ax_2 = \cdots = ax_{n-1} = 0 \neq ax_n$.*

Proof. Proceed by induction on n. Our induction hypothesis: if $\mathfrak{B}$ is any Banach algebra and Y an irreducible $\mathfrak{B}$-module and $y_1, \ldots, y_{n-1} \in Y$ linearly independent then there is $b \in \mathfrak{B}$ such that $by_1 = \cdots = by_{n-2}0 \neq by_{n-1}$.

Lemma 5.24 gives the base case $n = 2$.

For the induction step, let $M = \operatorname{span}\{ x_1, \ldots, x_{n-2} \}$. Let

$$\mathfrak{B} = \bigcap_{i=1}^{n-2} \underbrace{I_{x_i}}_{\text{closed left ideal}} = \{ a : aM = 0 \}$$

Let $Y = X/M$. Then if $b \in \mathfrak{B}$ we have $b(x+M) = bx \in bx + M$; so Y is a $\mathfrak{B}$-module with $b(\dot{x}) = \dot{bx}$.

Claim 5.26. *Y is an irreducible $\mathfrak{B}$-module.*

Proof. Suppose $0 \neq y_1 \in Y$ and $y_2 \in Y$; say $y_1 = x + M$ and $y_2 = x' + M$. Then $x \notin M$ and $x_1, \ldots, x_{n-2}$ are linearly independent and span M; so $x_1, \ldots, x_{n-2}, x$ is linearly independent. So by induction hypothesis (for $\mathfrak{A}$ acting on X) there is $a \in \mathfrak{A}$ such that $ax_1 = \cdots = ax_{n-2} = 0 \neq ax$. Then $a \in \mathfrak{B}$ and $ax \neq 0$, so there is $c \in \mathfrak{A}$ such that $cax = x'$. Then $ca \in \mathfrak{B}$ and

$$(ca)y_1(ca)\dot{x} = c\dot{a}x = \dot{x'} = y_2$$

TODO 28. *Dot cax*

So $\mathfrak{B}$ is transitive in Y. So Y is irreducible. □ Claim 5.26

Now $x_1, \ldots, x_n$ are linearly independent and $\dot{x_{n-1}}, \dot{x_n}$ are linearly independent in $Y = X/M$. Since Y is an irreducible $\mathfrak{B}$ Lemma 5.24 yields that there is $b \in \mathfrak{B}$ such that $b\dot{x_{n-1}} = \dot{0}$ and $b\dot{x_n} \neq \dot{0}$. So $bx_1 = bx_2 = \cdots = bx_{n-2} = 0$ and $bx_{n-1} \in M$ but $bx_n \notin M$. So either $bx_{n-1} = 0$ or $\{ bx_{n-1}, bx_n \}$ is linearly independent. By Lemma 5.24 there is $c \in \mathfrak{A}$ such that $cbx_{n-1} = 0$ and $cbx_n \neq 0$. So if $a = cb$ then $ax_1 = \cdots = ax_{n-1} = 0 \neq ax_n$, as desired. □ Lemma 5.25

Proof of Theorem 5.23. Suppose $x_1, \ldots, x_n$ are linearly independent in X and $y_1, \ldots, y_n \in X$. Then by Lemma 5.25 there is $a_j \in \mathfrak{A}$ such that

$$a_j x_i = \begin{cases} 0 \text{if } i \neq j \\ z_j \neq 0 & \text{if } i = j \end{cases}$$

By transitivity there is $b_j \in \mathfrak{A}$ such that $b_j z_j = y_j$. Let

$$a = \sum_{j=1}^{n} b_j a_j \in \mathfrak{A}$$

Then $ax_j = y_j$ for $1 \leq j \leq n$. So $\mathfrak{A}$ is n-transitive for $n \geq 1$. □ Theorem 5.23

5.1 Automatic continuity

Theorem 5.27 (B. Johnson)**.** *If X is a Banach space and $\pi \colon \mathfrak{A} \to \mathcal{B}(X)$ makes X an irreducible $\mathfrak{A}$ module then π is continuous.*

Proof. First note that $\ker(\pi)$ is primitive, and is thus closed. We have the following commuting diagram:

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "q"] \arrow[dr, "\pi"] & \mathfrak{A}/\ker(\pi) \arrow[d, "\dot{\pi}"] \\ & \mathcal{B}(X) \end{tikzcd}$$

Then X is also an irreducible $\mathfrak{A}/\ker(\pi)$-module. If $\dot{\pi}$ is continuous then $\pi = \dot{\pi} \circ q$ is continuous. So without loss of generality we may assume π is injective.

If $\dim(X) < \infty$ then $\dim(\mathcal{B}(X)) = (\dim(X))^2 < \infty$; since π is injective we get $\dim(\mathfrak{A}) < \infty$, and linearity of π implies continuity.

Suppose then that $\dim(X) = \infty$. For $x \in X$ define a linear map $T_x \colon \mathfrak{A} \to X$ by $T_x a = ax$. Let $Y = \{ x \in X : T_x \text{ continuous} \}$; so $Y \subseteq X$ is a subspace. Also if $b \in \mathfrak{A}$ then

$$\|T_{bx} a\| = \|abx\| = \|T_x(ab)\| \leq \|T_x\| \|ab\| \leq (\|T_x\| \|b\|) \|a\|$$

So $x \in Y$ implies $bx \in Y$, and Y is an $\mathfrak{A}$-submodule of X. So Y is $\{ 0 \}$ or X.

Case 1. Suppose $Y = X$ and $x \in X$. Then

$$\sup_{\|a\|\leq 1} \|\pi(a)x\| = \sup_{\|a\|\leq 1} \|ax\| = \|Tx\| < \infty$$

Hence by the uniform boundedness principle we have

$$\|\pi\| = \sup_{\|a\|\leq 1} \|\pi(a)\| < \infty$$

and π is continuous.

Case 2. Suppose $Y = \{0\}$. Since $\dim(X) = \infty$ there are linearly independent unit vectors $x_1, x_2, x_3, \ldots$. By the Jacobson density theorem there is $a_n \in \mathfrak{A}$ such that $a_n x_i = 0$ for $1 \leq i < n$ and $a_n x_n \neq 0$. Let

$$L_n = \bigcap_{i=1}^{n-1} I_{x_i}$$

so $a_n \in L_n$ and $a_n \notin L_{n+1}$. Then $a_n x_n \neq 0$ so $T_{a_n x_n}$ is unbounded. Pick $b_n \in \mathfrak{A}$ with $\|b_n\| < \frac{2^{-n}}{\|a_n\|}$ such that

$$\|b_n a_n x_n\| = \|T_{a_n x_n} b_n\| > n + \left\| \left(\sum_{i=1}^{n-1} b_i a_i \right) x_n \right\|$$

Let

$$b = \sum_{i=1}^{\infty} b_i a_i$$

This converges since $\|b_n a_n\| < 2^{-n}$. Then

$$b = \sum_{i=1}^{n} b_i a_i + \sum_{i>n} b_i a_i$$

But for $i > n$ we have $a_i \in L_n$ and then L_n are closed left ideals; so $b_i a_i \in L_n$ for $i > n$, and

$$\sum_{i=n+1}^{\infty} b_i a_i \in L_n$$

and hence

$$\left(\sum_{i=n+1}^{\infty} b_i a_i \right) x_n = 0$$

But now

$$\|\pi(b)\| \geq \|b x_n\| = \left\| \sum_{i=1}^{n-1} b_i a_i x_n + b_n a_n x_n + \underbrace{\left(\sum_{i=n+1}^{\infty} b_i a_i \right) x_n}_{=0} \right\| \geq \|b_n a_n x_n\| - \left\| \left(\sum_{i=1}^{n-1} b_i a_i \right) x_n \right\| > n$$

a contradiction. So this case cannot hold, and we land in the first case. $\square$ Theorem 5.27

Definition 5.28. Suppose X, Y are Banach spaces and $T\colon X \to Y$ is linear. The *separating space* is

$$\mathfrak{S}(T) = \{ y \in Y : \text{ there are } x_n \in X \text{ such that } x_n \to 0, Tx_n \to y \}$$

Remark 5.29. By the closed graph theorem T is continuous if and only if $\mathfrak{S}(T) = \{0\}$.

Theorem 5.30 (Johnson)**.** *Suppose $\mathfrak{A}k, \mathfrak{B}$ are Banach algebras and $\theta\colon \mathfrak{A} \to \mathfrak{B}$ is a surjective homomorphism. Then $\mathfrak{S}(\theta) \subseteq \operatorname{rad}(\mathfrak{B})$.*

Proof. Suppose (X, π) is an irreducible Banach module for $\mathfrak{B}$. Then $\pi \circ \theta$ is an irreducible representation, making X an irreducible $\mathfrak{A}$-module. (Indeed, if $x_1 \neq 0$ in X and $x_2 \in X$ then there is $b \in \mathfrak{B}$ such that $bx_1 = x_2$; but there is $a \in \mathfrak{A}$ such that $\theta(a) = b$, and hence $(\pi \circ \theta)(a)x_1 = x_2$.) So $\pi \circ \theta\colon \mathfrak{A} \to \mathcal{B}(X)$ is an irreducible representation; so by Johnson's theorem we have $\pi \circ \theta$ is continuous.

If $b \in \mathfrak{S}(\theta)$ then

$$\pi(b) = \lim_{n\to\infty} \pi(\theta(a_n)) = \lim_{n\to\infty} \underbrace{(\pi \circ \theta)}_{\text{continuous}} \underbrace{(a_n)}_{\to 0} = 0$$

So

$$b \in \bigcap_{\pi \text{ irreducible}} \ker(\pi) = \operatorname{rad}(\mathfrak{B})$$

as desired. ☐ Theorem 5.30

Corollary 5.31 (Johnson)**.** *Every surjective homomorphism from a Banach algebra $\mathfrak{A}$ to a semisimple Banach algebra $\mathfrak{B}$ is continuous.*

Proof. Given such $\theta\colon \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ we have $\mathfrak{S}(\theta) \subseteq \operatorname{rad}(\mathfrak{B}) = \{0\}$. So by the closed graph theorem θ is continuous. ☐ Corollary 5.31

Corollary 5.32. *Every automorphism of a semisimple Banach algebra is continuous.*

Corollary 5.33 (Uniqueness of norm)**.** *If $\mathfrak{B}$ is a semisimple Banach algebra then all Banach algebra norms are equivalent. i.e. if $\|\cdot\|$ and $\|\!|\cdot|\!\|$ are two Banach algebra norms and $\|\cdot\|$ makes $\mathfrak{B}$ semisimple then there is $0 < c_1 \leq c_2 < \infty$ such that $c_1\|b\| \leq \|\!|b|\!\| \leq c_2\|b\|$ for all $b \in \mathfrak{B}$.*

Proof. $\mathrm{id}\colon (\mathfrak{B}, \|\!|\cdot|\!\|) \to (\mathfrak{B}, \|\cdot\|)$ is a homomorphism and is thus continuous and bijective; so θ is invertible. ☐ Corollary 5.33

Fact 5.34. *Even in the commutative case, this last corollary fails if we drop the assumption of semisimplicity.*

6 C*-algebras

Definition 6.1. A C*-algebra is a Banach *-algebra $\mathfrak{A}$ such that $\|a^*a\| = \|a\|^2$ for all $a \in \mathfrak{A}$.

Remark 6.2. $\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$, so $\|a\| \leq \|a^*\| \leq \|a^{**}\| = \|a\|$; so $\|a^*\| = \|a\|$.

Example 6.3.

(1) Consider $\mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ a Hilbert space. If $T \in \mathcal{B}(\mathcal{H})$ then

$$\begin{aligned}
\|T\|^2 &= \|T^*\|\|T\| \\
&\geq \|T^*T\| \\
&= \sup\{\, |\langle T^*Tx, y\rangle| : x, y \in \mathcal{H}, |x| = |y| = 1 \,\} \\
&\geq \sup_{\|x\|=1} |\langle T^*Tx, x\rangle| \\
&= \sup_{\|x\|=1} |\langle Tx, Tx\rangle| \\
&= \sup_{\|x\|=1} \|Tx\|^2 \\
&= \|T\|^2
\end{aligned}$$

So $\|T^*T\| = \|T^2\|$.

(1') If $\mathfrak{A}$ is a closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ (i.e. if $A \in \mathfrak{A}$ then $A^* \in \mathfrak{A}$) then $\mathfrak{A}$ is a *concrete C*-algebra*.

(1") If $T \in \mathcal{B}(\mathcal{H})$ we define $C^*(T) = \overline{\mathrm{alg}\{\, I, T, T^* \,\}}^{\|\cdot\|}$. (Here alg means "the algebra generated by".)

(2) If X is locally compact and Hausdorff then $C_0(X)$ is a C*-algebra with $f^* = \overline{f}$ for $f \in C_0(X)$. Then

$$\|\overline{f}f\| = \sup_{x \in X} |\overline{f(x)}f(x)| = \sup |f(x)|^2 = \|f\|^2$$

Definition 6.4. We say $a \in \mathfrak{A}$ is

- *self-adjoint* if $a = a^*$
- *normal* if $aa^* = a^*a$
- *unitary* if $a^*a = aa^* = 1$
- *positive* if $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$.

Proposition 6.5. *If $\mathfrak{A}$ is a C*-algebra without unit then $\mathfrak{A}^+ = \mathfrak{A} + \mathbb{C}1$ has a C*-algebra norm.*

Proof. Setting $(a + \lambda 1)^* = a^* + \overline{\lambda 1}$ makes $\mathfrak{A}^+$ a Banach *-algebra. Let $\mathfrak{A}^+$ act on $\mathfrak{A}$ by left multiplication: $a + \lambda \mapsto L_a + \lambda I \in \mathcal{B}(\mathfrak{A})$. This yields a Banach *-algebra norm

$$|\!|\!| a + \lambda |\!|\!| = \|L_a + \lambda I\|_{\mathcal{B}(\mathfrak{A})}$$

Then

$$|\!|\!| a |\!|\!| = \sup_{\substack{\|b\| \le 1 \\ b \in \mathfrak{A}}} \|ab\| \le \sup_{\|b\| \le 1} \|a\|\|b\| = \|a\|$$

and

$$|\!|\!| a |\!|\!| \ge \left\| a \frac{a^*}{\|a^*\|} \right\| = \frac{\|aa^*\|}{\|a^*\|} = \frac{\|a^*\|^2}{\|a^*\|} = \|a\|$$

So $|\!|\!| a |\!|\!| = \|a\|$. But

$$\begin{aligned}
|\!|\!| a + \lambda |\!|\!|^2 &= \sup_{\|b\| \le 1} \|ab + \lambda b\|^2 \\
&= \sup_{\|b\| \le 1} \|(b^*a^* + \overline{\lambda} b^*)(ab + \lambda b)\| \\
&= \sup_{\|b\| \le 1} \|b^*(a^*a + \lambda a^* + \overline{\lambda} a + |\lambda|^2)b\| \\
&\le \sup_{\|b\| \le 1} \|(a^*a + \lambda a^* + \overline{\lambda} a + |\lambda|^2)b\| \\
&= |\!|\!| a^*a + \lambda a^* + \overline{\lambda} a + |\lambda|^2 |\!|\!| \\
&= |\!|\!| (a + \lambda)^*(a + \lambda) |\!|\!| \\
&\le |\!|\!| (a + \lambda)^* |\!|\!| \, |\!|\!| a + \lambda |\!|\!| \\
&= |\!|\!| a + \lambda |\!|\!|^2
\end{aligned}$$

So $|\!|\!| (a + \lambda)^*(a + \lambda) |\!|\!| = |\!|\!| a + \lambda |\!|\!|^2$. ☐ Proposition 6.5

Theorem 6.6. *If $\mathfrak{A}$ is an abelian C*-algebra then the Gelfand transform $\Gamma \colon \mathfrak{A} \to C_0(\mathcal{M}_{\mathfrak{A}})$ is an isometric *-isomorphism.*

TODO 29. *extra word? onto? continuous?*

Proof. First suppose $\mathfrak{A}$ is unital. Then $\mathcal{M}_{\mathfrak{A}}$ is compact and $\Gamma\colon \mathfrak{A} \to \mathcal{C}(\mathcal{M}_{\mathfrak{A}})$ is a (unital) homomorphism with $\operatorname{Ran}(\Gamma)$ separates points. Let $a = a^* \in \mathfrak{A}$ and let $u_t = \exp(ita)$ for $t \in \mathbb{R}$. Then

$$u_t^* = \left(\sum_{n\geq 0} \frac{(ita)^n}{n!}\right)^* = \sum_{n\geq 0} \frac{(-ita)^n}{n!} = u_{-t}$$

Then $u_t^* u_t = \exp(-ita)\exp(ita) = \exp(0) = 1$, and similarly $u_t u_t^* = 1$. If $\varphi \in \mathcal{M}_{\mathfrak{A}}$ then $\varphi(u_t) = \varphi(\exp(ita)) = \exp(it\varphi(a))$; so $|\exp(it\varphi(a))| \leq \|u_t\| = 1$ for all $t \in \mathbb{R}$. So $\varphi(a) \in \mathbb{R}$; i.e. $\Gamma(a)$ is real-valued and thus self-adjoint. If $a \in \mathfrak{A}$ is arbitrary we let $x = \frac{a+a^*}{2}$ be the "real part" of a and $y = \frac{a-a^*}{2i}$ the "imaginary part". Then $x = x^*$ and $y = y^*$ and $a = x + iy$. Then

$$\Gamma(a^*) = \Gamma((x+iy)^*) = \Gamma(x-iy) = \underbrace{\Gamma(x)}_{\in\mathbb{R}} - i\underbrace{\Gamma(y)}_{\in\mathbb{R}} = \overline{\Gamma(x) + i\Gamma(y)} = \overline{\Gamma(x+iy)} = \Gamma(a)^*$$

So Γ preserves $*$.

Suppose $a = a^*$. Then $\|a^2\| = \|a^*a\| = \|a\|^2$. Since a^* is self-adjoint we have $\|a^4\| = \|(a^2)^2\| = \|a^2\|^2 = \|a\|^4$; continuing thus we get $\|a^{2^n}\| = \|a\|^{2^n}$. So

$$\|a\| = \lim_n \|a^{2^n}\|^{2^{-n}} = \operatorname{spr}(a) = \|\Gamma(a)\| = \sup_{\varphi\in\mathcal{M}_{\mathfrak{A}}} |\varphi(a)|$$

Note that $\varphi(a)$ runs over $\sigma(a)$ since $\operatorname{Ran}(\Gamma(a)) = \sigma(a)$.

If $a \in \mathfrak{A}$ is arbitrary then $\|a\|^2 = \|a^*a\| = \|\Gamma(a^*a)\| = \|\Gamma(a)\|^2$; so Γ is isometric.

So $\Gamma(\mathfrak{A})$ is a norm-closed, self-adjoint subalgebra of $\mathcal{C}(\mathcal{M}_{\mathfrak{A}})$ which separates points. By Stone-Weierstrass theorem we get $\Gamma(\mathfrak{A}) = \mathcal{C}(\mathcal{M}_{\mathfrak{A}})$.

Suppose now that $\mathfrak{A}$ not unital.

TODO 30. *caselist*

Then $\mathfrak{A}$ lies in the unitization $\mathfrak{A}^+$ and $\mathcal{M}_{\mathfrak{A}^+} = \mathcal{M}_{\mathfrak{A}} \cup \{\varphi_\infty\}$ is the one-point compactification of the locally compact space $\mathcal{M}_{\mathfrak{A}}$ (where $\varphi_\infty(a+\lambda) = \lambda$). Then by above we have $\Gamma\colon \mathfrak{A}^+ \to \mathcal{C}(\mathcal{M}_{\mathfrak{A}^+}$ is an isometric $*$-isomorphism. But $\Gamma(\mathfrak{A}) = \{ f : f(\varphi_\infty) = 0 \}$ has codimension 1. Since $\mathfrak{A}$ has codimension 1 in $\mathfrak{A}^+$ we have $\Gamma(\mathfrak{A})$ has codimension 1 in $\Gamma(\mathfrak{A}) = \mathcal{C}(\mathcal{M}_{\mathfrak{A}})$.

TODO 31. *pluses?*

So Γ maps $\mathfrak{A}$ onto $\mathcal{C}_0(\mathcal{M}_{\mathfrak{A}}) = \{ f \in \mathcal{C}(\mathcal{M}_{\mathfrak{A}^+}) : f(\varphi_\infty) = 0 \}$. ☐ Theorem 6.6

Corollary 6.7. *Suppose $\mathfrak{A}$ is a unital C*-algebra (not necessarily abelian) and $n \in \mathfrak{A}$ is normal. Then if $C^*(n) = \overline{\operatorname{alg}\{1, n, n^*\}}^{\|\cdot\|}$ then there is a homeomorphism $\sigma(n)$ to $\mathcal{M}_{C^*(n)}$ that sends $\lambda \in \sigma(n)$ to φ_λ where $\varphi_\lambda(n) = \lambda$. Thus $C^*(n)$ is $*$-isomorphic to $\mathcal{C}(\sigma(n))$.*

Proof. $C^*(n)$ is a unital abelian C*-algebra. Let $X = \mathcal{M}_{C^*(n)}$. If $\varphi \in X$ then $\varphi(n) = \lambda \in \sigma(n)$. But then $\varphi(n^*) = \overline{\lambda}$ so $\varphi(p(n,n^*)) = p(\lambda, \overline{\lambda})$ where $p \in \mathbb{C}[x,y]$. But such $p(n,n^*)$ are dense in $C^*(n)$; so since φ is continuous we have that φ is determined by λ. So the map $X \to \sigma(n)$ given by $\varphi \mapsto \varphi(n)$ is bijective and continuous and is thus a homeomorphism. So

$$\begin{array}{ccccc} C^*(n) & \cong & \mathcal{C}(X) & \cong & \mathcal{C}(\sigma(n)) \\ n & \mapsto & \widehat{n} & \mapsto & \widetilde{n} \end{array}$$

where $\widetilde{n}(\lambda) = \lambda$ (so $\widetilde{n} = \operatorname{id}_{\sigma(n)}$). ☐ Corollary 6.7

Corollary 6.8. *The C*-algebra $C^*(n,n^*)$ is isomorphic to $C_0(\sigma(n) \setminus \{0\})$.*

Corollary 6.9 (Continuous functional calculus for normal elements)**.** *Suppose $\mathfrak{A}$ is a unital C*-algebra and $n \in \mathfrak{A}$ is normal. Then there is a $*$-isomorphism $\Gamma^{-1}\colon \mathcal{C}(\sigma(n)) \to C^*(n)$. So for $f \in \mathcal{C}(\sigma(n))$ we define $f(n) = \Gamma^{-1}(f)$.*

Note that $\Gamma^{-1}(\mathrm{id}_{\sigma(n)}) = n$ and $\Gamma^{-1}(\overline{z}) = n^*$. Also $\Gamma^{-1}(p(z, \overline{z})) = p(n, n^*)$. This extends to all continuous functions.

Corollary 6.10. *If n is normal and $f \in \mathcal{C}(\sigma(n))$ then $\sigma(f(n)) = f(\sigma(n))$*

Corollary 6.11.

1. *If n is normal then $\|n\| = \mathrm{spr}(n)$.*
2. *If $a = a^*$ then $\sigma(a) \subseteq \mathbb{R}$.*
3. *If u is unitary then $\sigma(u) \subseteq \mathbb{T}$.*

Proof.

1. $\|n\| = \|\Gamma(n)\| = \mathrm{spr}(n)$.
2. If $a = a^*$ then $\Gamma(a)$ is real-valued, so $\sigma(a) = \mathrm{Ran}(\Gamma(a)) \subseteq \mathbb{R}$.
3. If $uu^* = u^*u = 1$ then $|\Gamma(u)|^2 = 1$ so $\varphi(u) \in \mathbb{T}$ for all φ, and thus $\sigma(u) \subseteq \mathbb{T}$. □ Corollary 6.11

6.1 Operators on a Hilbert space

If $T \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$ then

$$\begin{aligned}\langle Tx, y\rangle &= \frac{1}{4}(\langle Tx + y, x + y\rangle + i\langle T(x + iy), x + iy\rangle - \langle T(x - y), x - y\rangle - i\langle T(x - iy), x - iy\rangle) \\ &= \frac{1}{4}\sum_{k=0}^{3} i^k \langle T(x + i^k y), x + i^k y\rangle\end{aligned}$$

This is the *polarization identity.*

TODO 32. *missing parens on $Tx + y$?*

Proposition 6.12. *If $U \in \mathcal{B}(\mathcal{H})$ then the following are equivalent:*

1. *U is unitary.*
2. *$\langle Ux, Uy\rangle = \langle x, y\rangle$ for all $x, y \in \mathcal{H}$ and $U\mathcal{H} = \mathcal{H}$.*
3. *U is isometric (i.e. $\|Ux\| = \|x\|$ for all x) and surjective.*

Proof.

(1) $\Longrightarrow$ (2) $\langle Ux, Uy\rangle = \langle U^*Ux, y\rangle = \langle x, y\rangle$. Since U is invertible we get that U is surjective.

(2) $\Longrightarrow$ (3) Take $x = y$.

(3) $\Longrightarrow$ (1) $\|x\|^2 = \langle Ux, Ux\rangle = \langle U^*Ux, x\rangle$ for all x. The polar identity yields $\langle Ix, y\rangle = \langle U^*Ux, y\rangle = \langle Ux, Uy\rangle$ for all x, y. So $I = U^*U$. So U is bijective and thus invertible; so $U^* = U^{-1}$ and U is unitary. □ Proposition 6.12

Proposition 6.13. *If $N \in \mathcal{B}(\mathcal{H})$ is normal then $\|N^*x\| = \|Nx\|$ for all $x \in \mathcal{H}$. Hence $\ker(N^*) = \ker(N)$.*

Proof. We have

$$\|N^*x\|^2 = \langle N^*x, N^*x\rangle = \langle NN^*x, x\rangle = \langle N^*Nx, x\rangle = \langle Nx, Nx\rangle = \|Nx\|^2$$

as desired. □ Proposition 6.13

Corollary 6.14. *If N is normal and Fredholm then* $\operatorname{ind}(N) = 0$.

Proof. T is Fredholm if $\operatorname{Ran}(T)$ is closed, $\operatorname{nul}(T) = \dim(\ker(T)) < \infty$, and $\operatorname{nul}(T^*) = \dim(\mathcal{H}/T\mathcal{H}) < \infty$. Then $\operatorname{ind}(T) = \operatorname{nul}(T) - \operatorname{nul}(T^*)$. ☐ Corollary 6.14

Proposition 6.15. *Suppose $A \in \mathcal{B}(\mathcal{H})$. Then $A = A^*$ if and only if $\langle Ax, x\rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$.*

Proof.

($\implies$) We have

$$\overline{\langle Ax, x\rangle} = \langle x, Ax\rangle = \langle A^*x, x\rangle = \langle Ax, x\rangle$$

So $\langle Ax, x\rangle \in \mathbb{R}$.

($\impliedby$) We have

$$\begin{aligned}
\langle A^*y, x\rangle &= \langle y, Ax\rangle \\
&= \overline{\langle Ax, y\rangle} \\
&= \overline{\frac{1}{4}\sum_{k=0}^{3} i^k \underbrace{\langle A(x + i^k y), x + i^k y\rangle}_{\in\mathbb{R}}} \\
&= \frac{1}{4}\sum_{k=0}^{3} (-i)^k \langle A(x + i^k y), x + i^k y\rangle \\
&= \frac{1}{4}\sum_{k=0}^{3} (-i)^k \langle i^k A(y + (-i)^k x), i^k(y + (-i)^k x)\rangle \\
&= \frac{1}{4}\sum_{k=0}^{3} (-i)^k \langle A(y + (-i)^k x), (y + (-i)^k x)\rangle \\
&= \langle Ay, x\rangle
\end{aligned}$$

as desired.

☐ Proposition 6.15

Corollary 6.16. *If $A \in \mathcal{B}(\mathcal{H})$ then $\sigma(A^*A) \subseteq [0, \infty)$. So A^*A is positive.*

Proof. Note $(A^*A)^* = A^*A$ is self-adjoint. If $r > 0$ then

$$\begin{aligned}
\langle (A^*A + rI)x, x\rangle &= \langle A^*Ax, x\rangle + \langle rx, x\rangle \\
&= \|Ax\|^2 + r\|x\|^2 \\
&\geq r\|x\|^2
\end{aligned}$$

So $A^*A + rI$ is bounded below, and thus has closed range and thus is surjective and is thus invertible. So $-r \notin \sigma(A^*A) \subseteq \mathbb{R}$. Also $\ker(A^*A + rI) = \{0\}$ so $A^*A + rI$ has dense range, and thus $\operatorname{Ran}(A^*A + rI)^\perp = \ker((A^*A + rI)^*) = \{0\}$. So $\sigma(A^*A) \subseteq [0, \infty)$

TODO 33. *Tidy*

☐ Corollary 6.16

6.2 Positive elements

Proposition 6.17. *If $a \in \mathfrak{A}$ and $a \geq 0$ then there is a unique $b \in \mathfrak{A}$ with $b \geq 0$ such that $b^2 = a$.*

Proof. Let $f(x) = x^{\frac{1}{2}}$, which is continuous on $\sigma(a) \subseteq [0, \|a\|]$. Let $b = f(a)$. Note that $f(x) = \lim p_n(x)$ with $p_n \in \mathbb{C}[x]$ and $p_n(0) = 0$. So $p_n(a) \in \mathfrak{A}$ even if $\mathfrak{A}$ is not unital. So $f(a) \in \mathfrak{A}$. Then $b^2 = f^2(a) = \operatorname{id}(a) = a$.

TODO 34. *I guess we're implicitly using the fact that $(f \circ g)(a) = f(g(a))$.*

For uniqueness, suppose $c \geq 0$ with $c^2 = a$. Then $x = \mathrm{id}(x) = f(x^2)$. In $C^*(c)$ we have

$$c = \mathrm{id}(c) = f(x^2(c)) = f(c^2) = f(a) = b$$

as desired. ☐ Proposition 6.17

Proposition 6.18. *If $a = a^*$ then there is $a_+, a_- \in \mathfrak{A}$ such that $a_+ \geq 0$, $a_- \geq 0$, $a_+a_- = 0$, and $a = a_+ - a_-$.*

Proof. Let $f \in \mathcal{C}(\sigma(a))$ be

$$x \mapsto \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Let $a_+ = f(a)$ and $a_- = a_+ - a = g(a)$ where

$$g(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ -x & \text{else} \end{cases}$$

so $f - g = \mathrm{id}$. Then $f \geq 0$ so $a_+ \geq 0$; likewise $g \geq 0$ so $a_- \geq 0$. Also $a_+ - a_- = (f-g)(a) = a$ and $a_+a_- = (fg)(a) = 0$ since $(fg)(x) = 0$. ☐ Proposition 6.18

Lemma 6.19. *If $a = a^* \in \mathfrak{A}$ then the following are equivalent:*

1. *$a \geq 0$.*
2. *$a = b^2$ for some $b \geq 0$.*
3. *For all $c \geq \|a\|$ we have $\|c1 - a\| \leq c$. (Work in $\mathfrak{A}_+$ if $\mathfrak{A}$ is not unital.)*
4. *There exists $c \geq \|a\|$ such that $\|c1 - a\| \leq c$.*

Proof.

(1) $\implies$ (2) Done.

(2) $\implies$ (3) If $f(x) = c - x^2$ we have

$$\|c1 - a\| = \|f(b)\| = \sup_{\lambda \in \sigma(b)} |f(\lambda)| \leq \sup_{\lambda \in [0, \|a\|^{\frac{1}{2}}]} |c - x^2| = c$$

since $\sigma(b) \subseteq [0, \|b\|]$ and $\|b\|^2 = \|b^2\| = \|a\|$.

(3) $\implies$ (4) Clear.

(4) $\implies$ (1) We have $\sigma(a) \subseteq \mathbb{R} \cap \overline{b_c(c)} = [0, 2c] \subseteq \mathbb{R}^+$ (since $\|c - a\| \leq c$). So $a \geq 0$. ☐ Lemma 6.19

Corollary 6.20. *If $a, b \in \mathfrak{A}$ with $a \geq 0$ and $b \geq 0$ then $a + b \geq 0$.*

Proof. There is $r \geq \|a\|$ such that $r1 - a \leq r$, and there is $s \geq \|b\|$ such that $\|s1 - b\| \leq s$. But then

$$\|(r+s)1 - (a+b)\| \leq \|r1 - a\| + \|s1 - b\| \leq r + s$$

So $a + b \geq 0$. ☐ Corollary 6.20

Theorem 6.21. *If $a \in \mathfrak{A}$ then $a^*a \geq 0$.*

Proof. Write $a^*a = b_+ - b_-$ where $b_+ \geq 0$, $b_- \geq 0$, and $b_+b_- = 0$. Pick $c \geq 0$ such that $c^2 = b_-$; let $t = ac$. Then $c = f(b_-)$ where $f(x) = \sqrt{x} = \lim p_n(x)$ where $p_n \in \mathbb{C}[x]$ and $p_n(0) = 0$. Then

$$cb_+ = \lim p_n(b_-)b_+ = \lim\Big(\frac{p_n}{x}\Big)(b_-)b_-b_+ = 0$$

Now

$$t^*t = c(a^*a)c = c(b_+ - b_-)c = -cb_-c = -c^4 = -b_-^2 \leq 0$$

So $\sigma(t^*t) \subseteq (-\infty, 0]$. Write $t = x + iy$ with $x = \mathrm{Re}(t)$ and $y = \mathrm{Im}(t)$ self-adjoint. Then

$$\begin{aligned} t^*t &= (x - iy)(x + iy) = x^2 + y^2 + i(xy - yx) \\ tt^* &= (x + iy)(x - iy) = x^2 + y^2 - i(xy - yx) \end{aligned}$$

So $t^*t + tt^* = 2x^2 + 2y^2 \geq 0$ by corollary.

TODO 35. *ref*

So $tt^* = (t^*t + tt^*) - t^*t = 2x^2 + 2y^2 + b_-^2 \geq 0$. So $\sigma(tt^*) \subseteq [0, \infty)$.
But $\sigma(t^*t) \cup \{0\} = \sigma(tt^*) \cup \{0\}$

TODO 36. *ref*

So $\sigma(t^*t) = \{0\}$. Then $\|t\|^2 = \|t^*t\| = \mathrm{spr}(t^*t) = 0$, and $t = 0$. So $b_-^2 = 0$, and $b_- = 0$. Thus $a^*a = b_+ \geq 0$. ☐ Theorem 6.21

Definition 6.22. If $a = a^*$ and $b = b^*$ we say $a \leq b$ if $b - a \geq 0$.

Corollary 6.23. *If $a \leq b$ in $\mathfrak{A}$ and $x \in \mathfrak{A}$ then $x^*ax \leq x^*bx$.*

Proof. Since $0 \leq b - a$ there is $c \geq 0$ with $c^2 = b - a$; then $x^*bx - x^*ax = x^*(b - a)x = x^*ccx = (cx)^*(cx) \geq 0$. ☐ Corollary 6.23

Corollary 6.24. *If $0 \leq a \leq b$ and a, b invertible then $b^{-1} \leq a^{-1}$.*

Proof. Since $b \geq 0$ we get from spectral mapping theorem that $b^{-1} \geq 0$, and hence $b^{-\frac{1}{2}} = \sqrt{b^{-1}}$ is well-defined.

TODO 37. *ref?*

Then previous corollary gives

$$0 \leq b^{-\frac{1}{2}}(b - a)b^{-\frac{1}{2}} = 1 - (b^{-\frac{1}{2}}a^{\frac{1}{2}})(a^{\frac{1}{2}}b^{-\frac{1}{2}})$$

So $(b^{-\frac{1}{2}}a^{\frac{1}{2}})(a^{\frac{1}{2}}b^{-\frac{1}{2}}) \leq 1$. So $\|a^{\frac{1}{2}}b^{-\frac{1}{2}}\|^2 = \|(b^{-\frac{1}{2}}a^{\frac{1}{2}})(a^{\frac{1}{2}}b^{-\frac{1}{2}})\| \leq 1$.
Aside 6.25. If $\|x\| \leq 1$ then $0 \leq x^*x \leq 1$. Since $x^*x \geq 0$ and $\|x^*x\| = \|x\|^2 \leq 1$ then $\sigma(x^*x) \subseteq [0, 1]$; so $x^*x \leq 1$. (Indeed, $1 - x^*x = g(x^*x)$ where $g(t) = 1 - t$ for $t \in [0, 1]$; so $g \geq 0$.)

TODO 38. *Better environment*

Thus $a^{\frac{1}{2}}b^{-1}a^{\frac{1}{2}} = (a^{\frac{1}{2}}b^{-\frac{1}{2}})(b^{-\frac{1}{2}}a^{\frac{1}{2}}) \leq 1$. Thus $b^{-1} = a^{-\frac{1}{2}}(a^{\frac{1}{2}}ba^{\frac{1}{2}})a^{-\frac{1}{2}} \leq a^{-\frac{1}{2}}1a^{-\frac{1}{2}} = a^{-1}$. ☐ Corollary 6.24

Definition 6.26. An *approximate identity* for a C*-algebra $\mathfrak{A}$ is a net e_λ where $0 \leq e_\lambda \leq 1$ and

$$\lim_\lambda \|a - ae_\lambda\| = 0 = \lim_\lambda \|a - e_\lambda a\|$$

for all $a \in \mathfrak{A}$.

Theorem 6.27. *Suppose $\mathfrak{A}$ is a C*-algebra. Then there is a bounded approximate identity for $\mathfrak{A}$.*

Proof. Let $\Lambda = \{ e \in \mathfrak{A} : e \geq 0, \|e\| < 1 \}$.

Claim 6.28. *Λ is directed by $\leq$.*

Proof. Suppose $a, b \in \Lambda$. We want to find $c \in A$ such that $a \leq c$ and $b \leq c$. Let $f\colon [0,1) \to \mathbb{R}^+$ be $f(t) = \frac{t}{1-t}$; let $g\colon \mathbb{R}^+ \to [0,1)$ be $g(t) = \frac{t}{1+t} = 1 - \frac{1}{1+t}$. Then

$$g(f(t)) = 1 - \frac{1}{1+f(t)} = 1 - \frac{1}{1+\frac{t}{1-t}} = 1 - \frac{1-t}{1-t+t} = t$$

Let $y = f(a) + f(b) \geq 0$; let $c = g(y) \geq 0$. Then $\sigma(c) = g(\sigma(y)) \subseteq [0,1)$; so $\|c\| < 1$, and $c \in \Lambda$. Since $y \geq f(a)$ we get $1 + y \geq 1 + f(a)$. Also note that if $x \geq 0$ then $1 + x \geq 0$, and $\sigma(1+x) \subseteq [1, \infty)$; so $1 + x$ is invertible. Applying this to y and $f(a)$ we get $(1+y)^{-1} \leq (a + f(a))^{-1}$. Then

$$c = g(y) = 1 - (1+y)^{-1} \geq 1 - (1+f(a))^{-1} = g(f(a)) = a$$

Similarly we get $c \geq b$. So Λ is directed. $\square$ Claim 6.28

If $0 \leq a \leq b \in \Lambda$ and $x \in \mathfrak{A}$ then

$$\|x - bx\|^2 = \|(x^* - x^*b)(x - bx)\| = \|x^*(1-b)^2 x\|$$

Aside 6.29. $0 \leq a \leq b$ does not imply that $a^2 \leq b^2$. Indeed, if

$$a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$b = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

then $a \leq b$ but

$$b^2 - a^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$$

has determinant -1.

Now, $0 \leq 1 - b \leq 1$ so since $x^2 \leq x$ on $[0,1]$ we have $(1-b)^2 \leq 1 - b$; so $x^*(1-b)^2 x \leq x^*(1-b)x$. Thus

$$\begin{aligned} \|x - bx\|^2 &= \|x^*(1-b)^2 x\| \\ &\leq \|x^*(1-b)x\| \\ &\leq \|x^*(1-a)x\| \text{ (since } 1 - a \geq 1 - b) \\ &\leq \|x\|^* \|x - ax\| \end{aligned}$$

Now suppose that $x \geq 0$. Let $a_n = g(nx) = \frac{nx}{1+nx}$; let

$$h(t) = t\left(1 - \frac{nt}{1+nt}\right)t = \frac{t^2}{1+nt} \leq \frac{t}{n}$$

Then

$$\|x(1-a_n)x\| = \|h(x)\| \leq \sup_{t \in [0, \|x\|]} |h(t)| \leq \frac{\|x\|}{n}$$

If $\varepsilon > 0$ choose n such that $\frac{\|x\|}{n} < \varepsilon^2$. Then for all $b \in \Lambda$ with $b \geq a_n$ we have

$$\|x - bx\|^2 \leq \|x(1-a_n)x\| \leq \frac{\|x\|}{n} < \varepsilon^2$$

so $\|x - bx\| < \varepsilon$. So

$$\lim_{b \in \Lambda} bx = x$$

Also

$$\lim_{b \in \Lambda} xb = \left(\lim_{b \in \Lambda} bx\right)^* = x^* = x$$

For general $x \in A$ we have

$$\|x - xb\|^2 = \|x(1-b)\|^2 = \|(1-b)x^*x(1-b)\| \leq \underbrace{\|1-b\|}_{\leq 1}\|(x^*x) - (x^*x)b\| \to 0$$

as desired. $\square$ Theorem 6.27

Corollary 6.30. *If $\mathfrak{A}$ is a separable C*-algebra then $\mathfrak{A}$ has an approximate identity $\{ e_n : n \geq 1 \}$ with $0 \leq e_n \leq e_{n+1} < 1$.*

Proof. Exercise. $\square$ Corollary 6.30

6.3 Ideals and quotients

Definition 6.31. An *ideal* of a C*-algebra is a closed two-sided ideal.

Lemma 6.32. *Suppose $\mathfrak{J} \triangleleft \mathfrak{A}$ is an ideal of $\mathfrak{A}$. Then $\mathfrak{J}$ is self-adjoint.*

Proof. Let $\mathfrak{B} = \mathfrak{J} \cap \mathfrak{J}^*$; so $\mathfrak{B}$ is a C*-algebra. (Indeed, it is closed and self-adjoint, and if $a, b \in \mathfrak{B}$ then $ab \in \mathfrak{J}$ and $ab \in \mathfrak{J}^*$ since $\mathfrak{J}, \mathfrak{J}^*$ are ideals.) Let $\{ e_\lambda \}$ be an approximate identity for $\mathfrak{B}$. Then $\mathfrak{B} \supseteq \mathfrak{J}\mathfrak{J}^*$ since $\mathfrak{J}\mathfrak{J}^* \subseteq \mathfrak{J}\mathfrak{A} \subseteq \mathfrak{J}$ and $\mathfrak{J}\mathfrak{J}^* \subseteq \mathfrak{A}\mathfrak{J}^* = (\mathfrak{J}\mathfrak{A})^* = \mathfrak{J}^*$.

Suppose $a \in \mathfrak{J}$ and e_λ is in our approximate identity. Then

$$\begin{aligned}
\|a^* - a^*e_\lambda\|^2 &= \|(a - e_\lambda a)(a^* - a^*e_\lambda)\| \\
&= \|(aa^* - aa^*e_\lambda) - e_\lambda(aa^* - aa^*e_\lambda)\| \\
&= \|(1 - e_\lambda)(aa^* - aa^*e_\lambda)\| \\
&\leq \|aa^* - \underbrace{aa^*}_{\in \mathfrak{B}} e_\lambda\| \\
&\to 0
\end{aligned}$$

So $a^*e_\lambda \to a^*$, and $a^*e_\lambda \in \mathfrak{J}$ since $e_\lambda \in \mathfrak{B} \subseteq \mathfrak{J}$. So since $\mathfrak{J}$ is closed we get $a^* \in \mathfrak{J}$. So $\mathfrak{J} = \mathfrak{J}^*$. $\square$ Lemma 6.32

Aside 6.33. If $0 \leq a \leq b$ then $\|a\| \leq \|b\|$. Indeed, we have $\sigma(b) \subseteq [0, \|b\|]$ so $b \leq \|b\|1$ and $a \leq \|b\|1$. So if $r > \|b\|$ then $r - a \geq (r - \|b\|)1$. So $\sigma(r - a) \subseteq [r - \|b\|, \infty)$, and $\sigma(a) \subseteq (-\infty, \|b\|) \cap \mathbb{R}^+ = [0, \|b\|]$. So $\|a\| = \operatorname{spr}(a) \leq \|b\|$.

There's probably an easier proof of the above; he came up with this on the spot when asked.

Lemma 6.34. *Suppose $\mathfrak{A}$ is a C*-algebra; suppose $x, a \in \mathfrak{A}$ with $x^*x \leq a$. Then there is $b \in \mathfrak{A}$ such that $x = ba^{\frac{1}{4}}$ and $\|b\| \leq \|a\|^{\frac{1}{4}}$.*

Proof. Let $b_n = x\big(a + \frac{1}{n}\big)^{-\frac{1}{2}}a^{\frac{1}{4}}$. (Note that $a \geq 0$ so $a + \frac{1}{n} \geq \frac{1}{n}$ is invertible in $\mathfrak{A}_+$. Then

$$\left(a + \frac{1}{n}\right)^{-\frac{1}{2}} a^{\frac{1}{4}} = f(a)$$

where

$$f(x) = \frac{x^{\frac{1}{4}}}{\sqrt{x + \frac{1}{n}}} \in \mathcal{C}_0[0, \|a\|]$$

So $f(a) \in \mathfrak{A}$ even whe $\mathfrak{A}$ is not unital.) Let

$$d_{nm} = \left(a + \frac{1}{n}\right)^{-\frac{1}{2}} - \left(a + \frac{1}{m}\right)^{-\frac{1}{2}}$$

for $n, m \geq 1$. Then

$$\begin{aligned}
\|b_n - b_m\|^2 &= \|x d_{nm} a^{\frac{1}{4}}\|^2 \\
&= \|a^{\frac{1}{4}} d_{nm} x^* x d_{nm} a^{\frac{1}{4}}\| \\
&\leq \|a^{\frac{1}{4}} d_{nm} a d_{nm} a^{\frac{1}{4}}\| \\
&= \|d_{nm} a^{\frac{3}{4}}\|^2 \\
&= \left\| \left(a + \frac{1}{n}\right)^{-\frac{1}{2}} a^{\frac{3}{4}} - \left(a + \frac{1}{m}\right)^{-\frac{1}{2}} a^{\frac{3}{4}} \right\|^2 \\
&= \|f_n(a) - f_m(a)\|^2 \\
&\to 0
\end{aligned}$$

as $n, m \to \infty$, where

$$f_n(x) = \frac{x^{\frac{3}{4}}}{\sqrt{x + \frac{1}{n}}} \in \mathcal{C}_0[0, \|a\|]$$

So $0 \leq f_n \leq f_{n+1} \leq x^{\frac{1}{4}}$, and $f_n \to x^{\frac{1}{4}}$ uniformly on $[0, \|a\|]$. Thus $f_n(a) \to a^{\frac{1}{4}}$ in $\mathfrak{A}$, and $(f_n(a))_n$ is a Cauchy sequence. So $(b_n)_n$ is Cauchy, and there is a limit

$$b = \lim_{n \to \infty} b_n \in \mathfrak{A}$$

Then

$$\begin{aligned}
\|x - b a^{\frac{1}{4}}\|^2 &= \lim_{n \to \infty} \|x - b_n a^{\frac{1}{4}}\|^2 \\
&= \lim_{n \to \infty} \left\| x - x \left(a + \frac{1}{n}\right)^{-\frac{1}{2}} a^{\frac{1}{2}} \right\|^2 \\
&= \lim_{n \to \infty} \left\| x \left(1 - \left(a + \frac{1}{n}\right)^{-\frac{1}{2}} a^{\frac{1}{2}}\right)^2 \right\| \\
&= \lim_{n \to \infty} \left\| \left(1 - \left(a + \frac{1}{n}\right)^{-\frac{1}{2}} a^{\frac{1}{2}}\right) x^* x \left(1 - \left(a + \frac{1}{n}\right)^{-\frac{1}{2}} a^{\frac{1}{2}}\right) \right\| \\
&\leq \lim_{n \to \infty} \left\| \left(1 - \left(a + \frac{1}{n}\right)^{-\frac{1}{2}} a^{\frac{1}{2}}\right) a \left(1 - \left(a + \frac{1}{n}\right)^{-\frac{1}{2}} a^{\frac{1}{2}}\right) \right\| \\
&= \lim_{n \to \infty} h_n(a) \\
&= 0
\end{aligned}$$

where

$$h_n(x) = x \left(1 - \sqrt{\frac{x}{x + \frac{1}{n}}}\right)^2 \to 0$$

uniformly on $[0, \|a\|]$. So $x = ba^{\frac{1}{4}}$. Also

$$\begin{aligned}
\|b_n\|^2 &= \|b_n^* b_n\| \\
&= \left\| a^{\frac{1}{4}} \left(a + \frac{1}{n} \right)^{-\frac{1}{2}} x^* x \left(a + \frac{1}{n} \right)^{-\frac{1}{2}} a^{\frac{1}{4}} \right\| \\
&\leq \left\| a^{\frac{1}{4}} \left(a + \frac{1}{n} \right)^{-\frac{1}{2}} a \left(a + \frac{1}{n} \right)^{-\frac{1}{2}} a^{\frac{1}{4}} \right\| \\
&= g_n(a) \\
&\leq \|g_n\|_{[0,\|a\|]} \\
&\leq \|a^{\frac{1}{2}}\| \\
&= \|a\|^{\frac{1}{2}}
\end{aligned}$$

where

$$g_n(x) = \frac{x^{\frac{3}{2}}}{x + \frac{1}{n}} \xrightarrow{\leq} \sqrt{x}$$

uniformly on $[0, \|a\|]$. $\square$ Lemma 6.34

Definition 6.35. A C*-subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$ is *hereditary* if whenever $b \in \mathfrak{B}$ with $b \geq 0$ and $a \in \mathfrak{A}$ with $0 \leq a \leq b$ we must have $a \in \mathfrak{B}$.

Corollary 6.36. *Ideals are hereditary subalgebras of $\mathfrak{A}$. Indeed, if $\mathfrak{J} \triangleleft \mathfrak{A}$ and $x^* x \leq a \in \mathfrak{J}$ then $x \in \mathfrak{J}$.*

Proof. Write $x = ba^{\frac{1}{4}}$ with $a \in \mathfrak{J}$; so $a^{\frac{1}{4}} \in \mathfrak{J}$ and $x \in \mathfrak{J}$. Then $0 \leq b \leq a$ implies $b^{\frac{1}{2}} \in \mathfrak{J}$, and thus $b \in \mathfrak{J}$. So $\mathfrak{J}$ is hereditary. $\square$ Corollary 6.36

Theorem 6.37. *If $\mathfrak{A}$ is a C*-algebra and $\mathfrak{J} \trianglelefteq \mathfrak{A}$ then $\mathfrak{A}/\mathfrak{J}$ is a C*-algebra.*

Proof. $\mathfrak{J} = \mathfrak{J}^*$, so $\mathfrak{A}/\mathfrak{J}$ is a *-algebra: if $\dot{a} = a + \mathfrak{J}$ then $(\dot{a})^* = \dot{a^*} = a^* + \mathfrak{J}$. This is a Banach algebra with the quotient norm. Let $\{ e_\lambda \}$ be an approximate identity for $\mathfrak{J}$.

Claim 6.38. $\|\dot{a}\| = \lim_\lambda \|a - ae_\lambda\|$.

Proof. $ae_\lambda \in \mathfrak{J}$, so $\|\dot{a}\| \leq \|a - ae_\lambda\|$. For all $\varepsilon > 0$ there is $b \in \mathfrak{J}$ such that $\|a - b\| < \|\dot{a}\| + \varepsilon$. Then since $0 \leq e_\lambda \leq 1$ we have

$$\begin{aligned}
\lim_\lambda \|a - ae_\lambda\| &\leq \lim_\lambda \|(a - b)(1 - e_\lambda)\| + \|b - be_\lambda\| \\
&\leq \lim_\lambda (\|\dot{a}\| + \varepsilon)(1) + \underbrace{\lim_\lambda \|b - be_\lambda\|}_{=0} \\
&= \|\dot{a}\| + \varepsilon
\end{aligned}$$

But $\varepsilon > 0$ was arbitrary. So

$$\lim_\lambda \|a - ae_\lambda\| = \|\dot{a}\|$$

as claimed. $\square$ Claim 6.38

Then

$$\begin{aligned}
\|\dot{a^*}\dot{a}\| &= \|\dot{(a^* a)}\| \\
&= \lim_\lambda \|\underbrace{a^* a - a^* a e_\lambda}_{a^* a(1 - e_\lambda)}\| \\
&\geq \lim_\lambda \|(1 - e_\lambda) a^* a (1 - e_\lambda)\| \\
&= \lim_\lambda \|a(1 - e_\lambda)\|^2 \\
&= \|\dot{a}\|^2
\end{aligned}$$

Then

$$\|\dot a\|^2 \le \|(\dot a)^*\dot a\| \le \|(\dot a)^*\|\|\dot a\| = \|\dot a\|^2$$

where for the last equality note that $\mathfrak{J}$ is self-adjoint, so $\operatorname{dist}(a^*, \mathfrak{J}) = \operatorname{dist}(a, \mathfrak{J})$. Thus $\|(\dot a)^*\dot a\| = \|\dot a\|^2$, and the C*-identity holds. So $\mathfrak{A}/\mathfrak{J}$ is a C*-algebra. □ Theorem 6.37

Theorem 6.39. *Suppose $\pi\colon \mathfrak{A} \to \mathfrak{B}$ is a non-zero *-homomorphism between C*-algebras. Then $\|\pi\| = 1$. So $\mathfrak{J} = \ker(\pi)$ is a closed two-sided ideal. Let $\widetilde{\pi}$ be the induced map on the quotient; so the following diagram commutes:*

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "q"] \arrow[dr, "\pi"'] & \mathfrak{A}/\mathfrak{J} \arrow[d, "\widetilde{\pi}"] \\ & \mathfrak{B} \end{tikzcd}$$

*THen $\widetilde{\pi}$ is an isometric *-monomorphism (i.e. injective *-homomorphism), and $\pi(\mathfrak{A})$ is a C*-subalgebra of $\mathfrak{B}$.*

Proof. If $a = a^*$ then $\sigma_{\mathfrak{B}}(\pi(a)) \subseteq \sigma_{\mathfrak{A}}(a)$: indeed, if $\lambda \notin \sigma(a)$ then $(a-\lambda)^{-1} \in \mathfrak{A}$, and $\pi((a-\lambda)^{-1}) = (\pi(a)-\lambda)^{-1}$. (If $\mathfrak{A}$ is not unital, define $\pi_+\colon \mathfrak{A}_+ \to \mathfrak{B}_+$ by $\pi_+(1) = 1$; now we can sensibly talk about spectra.) Then

$$\|\pi(a)\| = \operatorname{spr}(\pi(a)) \le \operatorname{spr}(a) = \|a\|$$

For general a we have

$$\|\pi(a)\|^2 = \|\pi(a^*a)\| \le \|a^*a\| = \|a\|^2$$

So $\|\pi\| \le 1$ and π is continuous. So $\mathfrak{J}$ is closed, and $\mathfrak{A}/\mathfrak{J}$ is a C*-algebra; so $\widetilde{\pi}(\dot a) = \pi(a)$ is well-defined and injective.

Claim 6.40. *$\widetilde{\pi}$ is isometric.*

Proof. If not, then there is $\dot a \in \mathfrak{A}/\mathfrak{J}$ such that

$$r = \|\widetilde{\pi}(\dot a)\|^2 = \|\widetilde{\pi}((\dot a)^*\dot a)\| < s = \|\dot a\|^2 = \|(\dot a)^*\dot a\|$$

so $s \in \sigma((\dot a)^*\dot a)$.

TODO 39. *How'd this happen?*

Let

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le r \\ \frac{x-r}{s-r} & \text{if } r \le x \le s \end{cases}$$

Then

$$\|f((\dot a)^*\dot a)\| = \sup_{x \in \sigma((\dot a)^*\dot a)} |f(x)| = 1$$

so

$$\|\widetilde{\pi}(f((\dot a)^*a))\| = \|f(\pi((\dot a)^*\dot a))\| = \|0\| = 0$$

as $\sigma((\dot a)^*\dot a) \subseteq [0,1]$.

TODO 40. *?*

So $\widetilde{\pi}$ is not injective, a contradiction. So$\widetilde{\pi}$ is isometric. □ Claim 6.40

So in particular $\pi(\mathfrak{A}) = \widetilde{\pi}(\mathfrak{A}/\mathfrak{J})$ is closed, and is thus a C*-subalgebra of $\mathfrak{B}$. □ Theorem 6.39

Corollary 6.41. *If $\mathfrak{J} \triangleleft \mathfrak{A}$ and $\mathfrak{B}$ a C*-subalgebra of $\mathfrak{A}$ then $\mathfrak{B}+\mathfrak{J}$ is a C*-subalgebra, and $\mathfrak{B}/\mathfrak{B}\cap\mathfrak{J} \cong \mathfrak{B}+\mathfrak{J}/\mathfrak{J}$.*

Proof. Let $q\colon \mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$ be the quotient mapping; so q is a *-homomorphism. So $q \restriction \mathfrak{B}\colon \mathfrak{B} \to \mathfrak{A}/\mathfrak{J}$ is a *-homomorphism. Then using the above theorem there is an isometric *-homomorphism such that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{B} \arrow[rr, "q\restriction\mathfrak{B}"] \arrow[d] & & \mathfrak{A}/\mathfrak{J} \\
\mathfrak{B}/\ker(q\restriction\mathfrak{B}) = \mathfrak{B}/\mathfrak{B}\cap\mathfrak{J} \arrow[urr, dashed] & &
\end{tikzcd}$$

So $q(B) = \mathfrak{B}+\mathfrak{J}/\mathfrak{J}$ is closed; so $\mathfrak{B}+\mathfrak{J} = q^{-1}(q(\mathfrak{B}))$ is a closed self-adjoint subalgebra, and is thus a C*-algebra. ☐ Corollary 6.41

Corollary 6.42. *If $a \in \mathfrak{A} \subseteq \mathfrak{B}$ with $\mathfrak{A}, \mathfrak{B}$ unital C*-algebras then $\sigma_{\mathfrak{A}}(a) = \sigma_{\mathfrak{B}}(a)$. i.e. C*-algebras are inverse-closed: if $a \in \mathfrak{A}$ and $a^{-1} \in \mathfrak{B}$ then $a^{-1} \in \mathfrak{A}$.*

Proof. We know $\sigma_{\mathfrak{B}}(a) \subseteq \sigma_{\mathfrak{A}}(a)$; it remains to show that if there is $b \in \mathfrak{B}$ such that $ab = ba = 1$ then $b \in \mathfrak{A}$.

Case 1. Suppose $a = a^*$. Then $\mathfrak{C} = C^*(a, a^{-1})$ is abelian and contained in $\mathfrak{B}$. Then $0 \notin \sigma_{\mathcal{C}}(a)$; so there is $f \in \mathfrak{C}([-\|a\|, \|a\|])$ such that

$$f(x) = \begin{cases} x^{-1} & \text{if } x \in \sigma_{\mathfrak{C}}(a) \\ 0 & \text{if } x = 0 \end{cases}$$

Then $a^{-1} = f(a)$ in $\mathfrak{C}$. This also makes sense in $C^*(a)$ since f is a limit of polynomials p_n with $p_n(0) = 0$. So $f(a) \in C^*(a)$; so a is invertible in $C^*(a) \subseteq \mathfrak{A}$.

Case 2. For the general case, suppose $a \in \mathfrak{A}$ and $a^{-1} \in \mathfrak{B}$. Then $(a^*a)^{-1} = a^{-1}(a^{-1})^*$ is invertible in $\mathfrak{B}$. But $a^*a \geq 0$ so by the previous case we have $(a^*a)^{-1} \in \mathfrak{A}$. Then $a^{-1} = (a^*a)^{-1}a^* \in \mathfrak{A}$. ☐ Corollary 6.42

7 Concrete C*-algebras

TODO 41. *Section title?*

7.1 Review of weak and strong operator topologies

Suppose $\mathcal{H}$ is a Hilbert space. We can endow $\mathcal{B}(\mathcal{H})$ with the *weak operator topology* by declaring $T_\alpha \xrightarrow{\text{WOT}} T$ if $\langle T_\alpha x, y\rangle \to \langle Tx, y\rangle$ for all $x, y \in \mathcal{H}$; this is the weakest topology such that $T \mapsto \langle Tx, y\rangle$ is continuous for all $x, y \in \mathcal{H}$. The basic open neighbourhoods around 0 are given by

$$\mathcal{O}(0, x_1, \ldots, x_n, y_1, \ldots, y_n) = \{\, T \in \mathcal{B}(\mathcal{H}) : |\langle Tx_i, y_i\rangle| < 1 \text{ for } 1 \leq i \leq n \,\}$$

We can also endow $\mathcal{B}(\mathcal{H})$ with the *strong operator topology* by declaring $T_\alpha \xrightarrow{\text{SOT}}$ if $T_\alpha x \to Tx$ for all $x \in \mathcal{H}$; this is the weakest topology such that $T \mapsto Tx$ is continuous for all $x \in \mathcal{H}$. It is determined by seminorms $p_x(T) = \|Tx\|$; or

$$p(T) = \left(\sum_{i=1}^{n} \|Tx_i\|^2\right)^{\frac{1}{2}}$$

for $x_1, \ldots, x_n \in \mathcal{H}$. The basic open neighbourhoods aroud 0 are given by

$$\mathcal{O}(x_1, \ldots, x_n) = \left\{ T : \sum_{i=1}^{n} \|Tx_i\|^2 < 1 \right\}$$

We also have the strong* topology SOT* given by $T_\alpha \xrightarrow{\text{SOT}^*} T$ if and only if $T_\alpha \xrightarrow{\text{SOT}} T$ and $T_\alpha^* \xrightarrow{\text{SOT}} T^*$. The basic open neighbourhoods around 0 are

$$\mathcal{O}(x_1, \ldots, x_n) = \left\{ T \sum_{i=1}^{n} \|Tx_i\|^2 < 1, \sum_{i=1}^{n} \|T^*x_i\| = 1 \right\}$$

TODO 42. *I think the second sum should be norms squared? Also in the next proof*

Example 7.1. If S is the unilateral shift then $S^n \xrightarrow{\text{WOT}} 0$ and $(S^*)^n \xrightarrow{\text{SOT}} 0$ but $S^n \not\xrightarrow{\text{SOT}} 0$ since the S^n are isometries, so $\|S^n x\| = 1 \not\to 0$.

Lemma 7.2. *Suppose $\varphi\colon \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is linear. Then the following are equivalent:*

1. *There exist $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathcal{H}$ such that*

$$\varphi(T) = \sum_{i=1}^{n} \langle Tx_i, y_i \rangle$$

2. *φ is WOT-continuous.*
3. *φ is SOT-continuous.*
4. *φ is SOT*-continuous.*

Proof.

(1) $\implies$ (2) Easy.

(2) $\implies$ (3) Easy.

(3) $\implies$ (4) Easy.

(4) $\implies$ (1) We have $\varphi^{-1}(\mathbb{D})$ is a SOT*-open neighbourhood of 0. So there is $x_1, \ldots, x_n \in \mathcal{H}$ such that

$$\varphi^{-1}(\mathbb{D}) \supseteq \left\{ T : \sum \|Tx_i\|^2 < 1, \sum \|T^* x_i\| < 1 \right\} \supseteq \{ T : Tx_i = 0, T^* x_i = 0 \} \subseteq \ker(\varphi)$$

Then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{B}(\mathcal{H}) \arrow[rr, "\varphi"] \arrow[dr, "\rho"'] & & \mathbb{C} \\
& \mathcal{H}^{(n)} \oplus (\mathcal{H}^*)^{(n)} \arrow[ur, "\psi"'] &
\end{tikzcd}$$

where $T \mapsto (Tx_1, \ldots, Tx_n, T^*x_1, \ldots, T^*x_n) \mapsto \varphi(T)$ and the latter map is continuous. We extend the map $\rho(\mathcal{B}(\mathcal{H})) \to \mathbb{C}$ to ψ on $\mathcal{H}^{(2n)}$ by Hahn-Bnach. Then there are $w_i \in \mathcal{H}^*, z_i \in \mathcal{H}$ such that

$$\psi(u_1, \ldots, u_n, v_1, \ldots, v_n) = \sum \langle u_i, w_i \rangle + \sum \langle v_i, z_i \rangle$$

Then

$$\varphi(T) = \sum_{i=1}^{n} \langle Tx_i, w_i \rangle + \sum_{i=1}^{n} \langle z_i, T^* x_i \rangle = \sum_{i=1}^{n} \langle Tz_i, x_i \rangle$$

as desired.

Improved version:

TODO 43. *Delete the first version?*

Note that $\varphi^{-1}(\mathbb{D})$ is a basic SOT*-open neighbourhood of 0 and

$$\varphi^{-1}(\mathbb{D}) \supseteq \left\{ T : \sum_{i=1}^{n} \|Tx_i\|^2 < 1 \text{ and } \sum_{j=1}^{m} \|T^* y_j\|^2 < 1 \right\} \supseteq \{ T : Tx_i = 0 = T^* y_j, 1 \le i \le n, 1 \le j \le m \}$$

and this last is a closed subspace. Then we want $\psi\colon \mathcal{H}^{(n)} \oplus (\mathcal{H}^*)^{(m)} \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{B}(\mathcal{H}) \arrow[rr, "\varphi"] \arrow[dr, "\rho"'] & & \mathbb{C} \\ & \mathcal{H}^{(n)} \oplus (\mathcal{H}^*)^{(m)} \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

where $\rho(T) = (Tx_1, \dots, Tx_n, T^*y_1, \dots, T^*y_m)$. Then T^*y_j represents the linear functional in $\mathcal{H}$ given by $x \mapsto \langle x, T^*y_j\rangle = \langle Tx, y_j\rangle$; the map $T \mapsto T^*y_j \in \mathcal{H}^*$ is *linear*.

Define $\psi((Tx_1, \dots, Tx_n, T^*y_1, \dots, T^*y_m)) = \varphi(T)$. Then $\ker(\rho) \subseteq \ker(\varphi)$, so ψ is well-defined. If

$$\sum \|Tx_i\|^2 < 1$$
$$\sum \|T^*y_j\|^2 < 1$$

then $\psi((Tx_1, \dots, Tx_n, T^*y_1, \dots, T^*y_m)) \in \mathbb{D}$, so $|\psi((Tx_1, \dots, Tx_n, T^*y_1, \dots, T^*y_m))| < 1$. So $\|\psi\| \leq 1$. We can thus by Hahn-Banach extend to a linear functional on $\mathcal{H}^{(n)} \oplus (\mathcal{H}^*)^{(m)}$ of norm ≤ 1. But $(\mathcal{H}^{(n)} \oplus (\mathcal{H}^*)^{(m)})^* = (\mathcal{H}^*)^{(n)} \oplus \mathcal{H}^{(m)}$; so there are $u_1, \dots, u_n \in \mathcal{H}^*$ and $v_1, \dots, v_m \in \mathcal{H}$ such that

$$\psi((x_1, \dots, x_n, y_1, \dots, y_m)) = \sum_{i=1}^n \langle x_i, u_i\rangle + \sum_{j=1}^m \langle v_j, y_j\rangle$$

Then

$$\varphi(T) = \psi((Tx_1, \dots, Tx_n, T^*y_1, \dots, T^*y_m)) = \sum \langle Tx_i, u_i\rangle + \sum \langle v_j, T^*y_j\rangle = \sum \langle Tx_i, u_i\rangle + \sum \langle Tv_j, y_j\rangle$$

as desired. $\square$ Lemma 7.2

Corollary 7.3. *$\mathcal{B}(\mathcal{H})$ with topologies WOT, SOT, and SOT* have the same closed convex sets.*

Proof. They have the same continuous functionals, and thus the same closed half spaces $H = \{\, T : \mathrm{Re}(\varphi(T)) \leq r \,\}$. By the geometric Hahn-Banach theorem, every closed convex set in a locally convex topological vector space is the intersection of the closed half spaces containing it. $\square$ Corollary 7.3

Definition 7.4. A *von Neumann algebra* is a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$ which is WOT-closed.

Definition 7.5. If $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$, we define the *commutant* fo $\mathcal{S}$ to be $\mathcal{S}' = \{\, T \in \mathcal{B}(\mathcal{H}) : ST = TS \text{ for all } S \in \mathcal{S} \,\}$.

Remark 7.6. $\mathcal{S}'$ is always a WOT-closed unital algebra. Indeed, $\mathcal{S}$ is clearly a subspace. It is closed under multiplication, as if $T_1, T_2 \in \mathcal{S}'$ then $T_1T_2S = T_1ST_2 = ST_1T_2$. If $T_\alpha \in \mathcal{S}'$ with $T_\alpha \xrightarrow{\text{WOT}} T$ then

$$ST = \lim_\alpha ST_\alpha = \lim_\alpha T_\alpha S = TS$$

and so $\mathcal{S}'$ is WOT-closed. If $\mathcal{S} = \mathcal{S}^*$ then $\mathcal{S}'$ is self-adjoint, and is thus a von Neumann algebra.

Theorem 7.7 (Double commutant theorem)**.** *If $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ is a C*-algebra which is non-degenerate (i.e. $\overline{\mathfrak{A}\mathcal{H}} = \mathcal{H}$) then $\overline{\mathfrak{A}}^{\text{SOT}} = \overline{\mathfrak{A}}^{\text{WOT}} = \mathfrak{A}''$ (where $\mathfrak{A}'' = (\mathfrak{A}')'$).*

Proof. We know $\overline{\mathfrak{A}}^{\text{SOT}} = \overline{\mathfrak{A}}^{\text{WOT}}$ by previous corollary. We know $\overline{\mathfrak{A}}^{\text{SOT}} \subseteq \mathfrak{A}''$ since $\mathfrak{A} \subseteq \mathfrak{A}''$ and $\mathfrak{A}''$ is WOT-closed.

Suppose $T \in \mathfrak{A}''$ and $x_1, \dots, x_n \in \mathcal{H}$; we wish to find $A \in \mathfrak{A}$ such that

$$A \in \left\{ B \in \mathcal{B}(\mathcal{H}) : \sum_{i=1}^n \|(T - B)x_i\|^2 < 1 \right\}$$

where this last is a SOT neighbourhood of T.

Case 1. Suppose $n = 1$. Then $M = \overline{\mathfrak{A}x_1}$ is a closed subspace of $\mathcal{H}$, and $\mathfrak{A}M = \overline{\mathfrak{A}\overline{\mathfrak{A}x_1}} = \overline{\mathfrak{A}^2 x_1} \subseteq M$. Let P be the orthogonal projection onto M. Then if $A \in \mathfrak{A}$ we have $AP = PAP$; so for $A \in \mathfrak{A}$ we have $PA^* = PA^*P$; so $PA = PAP = AP$ for all $A \in \mathfrak{A}$, and $P \in \mathfrak{A}'$. So $TP = PT$ and $Tx_1 = TPx_1 = PTx_1 \in M$. So there is $A \in \mathfrak{A}$ such that $\|Tx_1 - Ax_1\| < 1$ (or $< \varepsilon$ for any $\varepsilon > 0$).

Aside 7.8. Why is $x_1 \in M$? Let $(e_\lambda)_\lambda$ be an approximate identity for $\mathfrak{A}$. Since $\overline{\mathfrak{A}\mathcal{H}} = \mathcal{H}$ there is $x \in \mathcal{H}$ and $A \in \mathfrak{A}$ such that $Ax \approx x_1$; then

$$\underbrace{e_\lambda x_1}_{\in \mathfrak{A}x_1} \approx e_\lambda Ax \to Ax$$

Case 2. Suppose $n > 1$. Let $\mathcal{H}^{(n)} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_n$. Let

$$\mathfrak{A}^{(n)} = \left\{ A^{(n)} = \begin{pmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{pmatrix} \in \mathcal{M}_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^{(n)}) \right\}$$

Suppose $T \in \mathcal{B}(\mathcal{H}^{(n)})$ and let P_j be the orthogonal projection onto $\mathcal{H}_j = 0 \oplus \cdots \oplus \mathcal{H} \oplus 0 \oplus \cdots \oplus 0$ with $\mathcal{H}$ in the j^{th} spot. We let $T_{ij} = P_i T P_j \restriction H_j \in \mathcal{B}(\mathcal{H})$; then

$$T = \Big(\sum P_i\Big) T \Big(\sum P_j\Big) = \sum_{i,j} T_{ij}$$

Claim 7.9. $(\mathfrak{A}^{(n)})' = \mathcal{M}_n(\mathfrak{A}')$.

Proof. Suppose $T \in \mathcal{B}(\mathcal{H}^{(n)})$ commutes with $\mathfrak{A}^{(n)}$. Then if $T = (T_{ij})_{ij}$ and

$$A^{(n)} = \begin{pmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{pmatrix}$$

we have $TA^{(n)} = (T_{ij}A)_{ij} = (AT_{ij})_{ij} = A^{(n)}T$. So $T \in (\mathfrak{A}^{(n)})'$ if and only if $T_{ij} \in \mathfrak{A}'$ for all i, j, which occurs if and only if $T \in \mathcal{M}_n(\mathfrak{A}')$. ☐ Claim 7.9

Claim 7.10. $\mathcal{M}_n(\mathfrak{A}')' = (\mathfrak{A}'')^{(n)}$.

Proof. Suppose $A = (A_{ij})_{ij} \in \mathcal{M}_n(\mathfrak{A}')'$. Let $E_{ij} \in \mathcal{M}_n(\mathfrak{A}')$ have an I in the (i, j) position and a zero elsewhere. Then

$$E_{ii}A = \begin{pmatrix} & 0 & & \\ A_{i1} & A_{i2} & \cdots & A_{in} \\ & 0 & & \end{pmatrix} = AE_{ii} = \begin{pmatrix} & A_{i1} & \\ & A_{i2} & \\ 0 & \vdots & 0 \\ & A_{in} & \end{pmatrix}$$

So $A_{ij} = 0$ if $i \neq j$. Doing a similar trick with E_{ij} we conclude that $A_{ii} = A_{jj}$ if $i \neq j$. So $A = A^{(n)}$ for some $A \in \mathcal{B}(\mathcal{H})$.

Note

$$\begin{pmatrix} T & 0 & \\ 0 & 0 & \\ & & \ddots \end{pmatrix} \in \mathcal{M}_n(\mathfrak{A}')$$

if $T \in \mathfrak{A}'$. Then $A^{(n)}T = TA^{(n)}$, so examining top-left entries we get $AT = TA$. ☐ Claim 7.10

Suppose $T \in \mathfrak{A}''$ and $x_1, \dots, x_n \in \mathcal{H}$. We have a SOT neighbourhood of T given by

$$\left\{ B \in \mathcal{B}(\mathcal{H}) : \sum \|(T-B)x_i\|^2 < 1 \right\}$$

Let

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{H}^{(n)}$$

Let $M = \overline{\mathfrak{A}^{(n)}x}$ and P be the orthogonal projection to M. Then $PA^{(n)} = A^{(n)}P$ for all $A \in \mathfrak{A}$; so $P \in (\mathfrak{A}^{(n)})' = \mathcal{M}_n(\mathfrak{A}')$. But $T^{(n)} \in (\mathfrak{A}'')^{(n)} = \mathcal{M}(\mathfrak{A}')'$; so $T^{(n)}P = PT^{(n)}$. So

$$T^{(n)}x = \begin{pmatrix} Tx_1 \\ \vdots \\ Tx_n \end{pmatrix} = T^{(n)}Px = PT^{(n)}x \in M = \overline{\mathfrak{A}^{(n)}x}$$

So there is $A \in \mathfrak{A}$ such that

$$1 > \|T^{(n)}x - A^{(n)}x\|^2 = \left\| \begin{pmatrix} Tx_1 - Ax_1 \\ \vdots \\ Tx_n - Ax_n \end{pmatrix} \right\|^2 = \sum_{i=1}^n \|(T-A)x_i\|^2$$

as desired. ☐ Theorem 7.7

Lemma 7.11. *Let $f(x) = \frac{2t}{1+t^2}$. If $A_\alpha = A_\alpha^*$ and $A_\alpha \xrightarrow{\text{SOT}} S$ then $f(A_\alpha) \xrightarrow{\text{SOT}} f(S)$.*

Proof. Note f maps $[-1,1]$ injectively onto itself, and $f(\mathbb{R}) \subseteq [-1,1]$.

Suppose $x \in \mathcal{H}$. Then

$$\begin{aligned} f(A_\alpha)x - f(S)x &= (2(I+A_\alpha^2)^{-1}A_\alpha - 2S(1+S^2)^{-1})x \\ &= 2(1+A_\alpha^2)^{-1}(A_\alpha - S)(\underbrace{(I+S^2)^{-1}x}_{u}) + \underbrace{2(1+A_\alpha^2)^{-1}A_\alpha}_{f(A_\alpha)}(S-A_\alpha)\underbrace{S(I+S^2)^{-1}x}_{v} \\ &= 2(1+A_\alpha^2)^{-1}(A_\alpha - S)u + f(A_\alpha)(S-A_\alpha)v \end{aligned}$$

Now, $A_\alpha - S)u \to 0$, and since

$$\|2(1+A_\alpha^2)^{-1}\| \le \left\| \frac{2}{1+x^2} \right\|_{\mathbb{R}} = 2$$

we get $2(1+A_\alpha^2)^{-1}(A_\alpha - S)u \to 0$, and hence $(S-A_\alpha)v \to 0$. Then since $\|f(A_\alpha)\| \le \|f\|_\infty = 1$ we have $f(A_\alpha)(S-A_\alpha)v \to 0$. ☐ Lemma 7.11

Theorem 7.12 (Kaplansky's density theorem)**.** *Suppose $\mathfrak{A}$ is a non-degenerate C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. Then $\overline{b_1(\mathfrak{A}_{\text{sa}})}^{\text{SOT}} = \overline{b_1(\mathfrak{A}''_{\text{sa}})}^{\|\cdot\|}$ and $\overline{b_1(\mathfrak{A})}^{\text{SOT}} = \overline{b_1(\mathfrak{A}'')}^{\|\cdot\|}$.*

Proof. Suppose $S \in \overline{b_1(\mathfrak{A}''_{\text{sa}})}$. Let $T = g(S)$ (where g is the inverse function of $f \restriction [-1,1] \colon [-1,1] \to [-1,1]$). Then $T = T^* \in \overline{b_1(\mathfrak{A}''_{\text{sa}})}$. By the double commutant theorem there are $A_\alpha \in \mathfrak{A}$ such that $A_\alpha \xrightarrow{\text{WOT}} T$, and thus $A_\alpha^* \xrightarrow{\text{WOT}} T^* = T$. So $\frac{A_\alpha + A_\alpha^*}{2} \xrightarrow{\text{WOT}} T$. So $T \in \overline{\mathfrak{A}_{\text{sa}}}^{\text{WOT}} = \overline{\mathfrak{A}_{\text{sa}}}^{\text{SOT}}$. So there is $A_\alpha = A_\alpha^* \in \mathfrak{A}_{\text{sa}}$ such that $A_\alpha \xrightarrow{\text{SOT}} T$. Thus by lemma we have $f(A_\alpha) \xrightarrow{\text{SOT}} f(T) = f(g(S)) = S$; also $\|f(A_\alpha)\| \le \|f\|_{\mathbb{R}} = 1$.

If $T \in \overline{b_1(\mathfrak{A}'')}$ then

$$\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix} \in \overline{b_1(\mathcal{M}_2(\mathfrak{A}''))} = \mathcal{M}_2(\mathfrak{A})''$$

So there is

$$A_\alpha = \begin{pmatrix} A_{\alpha,11} & A_{\alpha,12} \\ A_{\alpha,21} & A_{\alpha,22} \end{pmatrix} \in b_1(\mathcal{M}_2(\mathfrak{A}))_{\text{sa}}$$

Thus

$$A_\alpha \xrightarrow{\text{SOT}} \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$$

So $\|A_{\alpha,12}\| \leq \|A_\alpha\| \leq 1$, and $A_{\alpha,12} \xrightarrow{\text{SOT}} T$. □ Theorem 7.12

Definition 7.13. We say $U \in \mathcal{B}(\mathcal{H})$ is a *partial isometry* if $U \restriction (\ker(U))^\perp$ is isometric.

Proposition 7.14. *Suppose $U \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

1. *U is a partial isometry.*
2. *U^*U and UU^* are projections.*

   TODO 44. *or?*

3. *$U = UU^*U$.*

Proof.

(1) $\Longrightarrow$ (2) Suppose U is a partial isometry. Then $\mathcal{H} = (\ker(U)) \oplus (\ker(U))^\perp$. Then $U \restriction (\ker(U))^\perp$ is an isometry onto $\operatorname{Ran}(U)$ (closed $U\mathcal{H} = U(\ker(U))^\perp$).

TODO 45. *words*

But $\ker(U^*) = (\operatorname{Ran}(U))^\perp$, and $U^* \restriction \operatorname{Ran}(U)$ is an isometry onto $(\ker(U))^\perp$ such that $U^*U = P^\perp_{\ker(U)}$. Likewise UU^* is the projection onto $P^\perp_{\ker(U^*)} = P_{\operatorname{Ran}(U)}$ (since U^* is also a partial isometry).

(2) $\Longrightarrow$ (1) U^*U a projection means that $U \restriction (\ker(U))^\perp = S \in \mathcal{B}((\ker(U))^\perp, \mathcal{H})$ and $S^*S = I_{(\ker(U))^\perp}$. So S is an isometry. So $U \restriction (\ker(U))^\perp$ is an isometry; so U is a partial isomorphism.

(2) $\Longrightarrow$ (3) $U = UP^\perp_{\ker(U)} = UU^*U$.

(3) $\Longrightarrow$ (2) $U^*U = U^*(UU^*U) = (U^*U)^2$ so U^*U is a projection. Similarly UU^* is a projection.

□ Proposition 7.14

Theorem 7.15 (Polar decomposition)**.** *Suppose $T \in \mathcal{B}(\mathcal{H})$. Then $|T| = (T^*T)^{\frac{1}{2}} \in C^*(T)$ and there is a partial isometry $U \in W^*(T)$ (the von Neumann algebra generated by T, which is $C^*(T)''$) such that $T = U|T|$.*

Proof. We have $T^*T \in C^*(T)$ and $T^*T \geq 0$, so if $f(x) = x^{\frac{1}{2}} \in \mathcal{C}[0, \|T\|^2]$ then $|T| = f(T^*T) \in C^*(T)$.

If $x \in \mathcal{H})$ then

$$\||T|x\|^2 = \langle |T|x, |T|x\rangle = \langle |T|^2x, x\rangle = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2$$

so $\||T|x\| = \|Tx\|$ for all $x \in \mathcal{H}$. Define $U \in \mathcal{B}(\mathcal{H})$ as follows. If $x \in \ker(T) = \ker(|T|)$ we set $Ux = 0$. If $x \in \operatorname{Ran}(|T|)$, say $x = |T|y$ define $Ux = Ty$; so $\|Ux\| = \|Ty\| = \||T|y\| = \|x\|$. So U is isometric on $\operatorname{Ran}(|T|)$. By continuity, we extend U to an isometry on $\overline{\operatorname{Ran}(|T|)}$; but $\overline{\operatorname{Ran}(|T|)} = (\ker(|T|))^\perp$. So U is a partial isometry and $U\overline{\operatorname{Ran}(T)} = \overline{\operatorname{Ran}(T)}$.

If $x \in \ker(T)$ then $U|T|x = 0 = Tx$. If $x = |T|y \in \operatorname{Ran}(|T|)$ then $U|T|y = Ty$ (by definition); this extends by continuity to $\operatorname{Ran}(|T|)^\perp$

TODO 46. $\overline{\operatorname{Ran}(|T|)}$*?* $\ker(|T|)^\perp$*?*

To show that $U \in W^*(T) = C^*(T)''$ it suffices to show that $UX = XU$ for $X \in C^*(T)'$. So Suppose $X \in C^*(T)'$.

Note that $X\ker(T) \subseteq \ker(T)$; indeed, if $Tx = 0$ then $T(Xx) = X(Tx) = 0$. So $Ux = 0$, so $XUx = 0$ and $U(Xx) = 0$. So $XU = UX$ on $\ker(U) = \ker(T)$. Suppose $x = |T|y \in \operatorname{Ran}(|T|)$. Then

$$UXx = UX|T|y = U|T|Xy = TXy = XTy = XU|T|y = XUx$$

So $UX - XU = 0$ in $\ker(U) \oplus (\ker(U))^\perp = \mathcal{H}$. So $UX = XU$. So $U \in C^*(T)'' = W^*(T)$. □ Theorem 7.15

Remark 7.16.

1. If T is invertible then $U = T|T|^{-1} \in C^*(T)$.
2. If $f \in C_0((0, \|T\|])$ then $Uf(|T|) \in C^*(T)$. (See assignment 3.)

7.2 Projections in von Neumann algebras

Lemma 7.17. *Suppose $(A_\lambda)_{\lambda\in\Lambda}$ is an increasing net of self-adjoint operators in $\mathcal{B}(\mathcal{H})$ bounded above by M. Then in SOT we have a limit $A = \lim_\lambda A_\lambda$ and A is the least upper bound of the A_λ.*

Proof. For $x \in \mathcal{H}$ we have $\langle A_\lambda x, x\rangle \le M\|x\|^2$; so $\langle A_\lambda x, x\rangle$ is an increasing net of real numbers that is bounded above. So

$$\Omega(x) = \lim_\lambda \langle A_\lambda x, x\rangle$$

exists. Define

$$\begin{aligned}\langle Ax, y\rangle &= \frac{1}{4}(\Omega(x+y) - \Omega(x-y) + i\Omega(x+iy) - i\Omega(x+iy))\\ &= \lim_\lambda \frac{1}{4}(\langle A_\lambda(x+y), x+y\rangle - \langle A_\lambda(x-y), x-y\rangle + i\langle A_\lambda(x+iy), x+iy\rangle - i\langle A_\lambda(x-iy), x-iy\rangle)\\ &= \lim_\lambda \langle A_\lambda x, y\rangle\end{aligned}$$

So if A is the WOT limit of the A_λ then $A \in \mathcal{B}(\mathcal{H})$.

If $B \ge A_\lambda$ for all λ then

$$\langle Bx, x\rangle \ge \sup_\lambda \langle A_\lambda x, x\rangle = \lim \langle A_\lambda x, x\rangle = \langle Ax, x\rangle$$

So $\langle (B-A)x, x\rangle \ge 0$ for all x; so $B \ge A$. Thus A is the least upper bound of the A_λ.

If $B \ge 0$ then $[x, y] = \langle Bx, y\rangle$ is a sesquilinear form, and thus satisfies the Cauchy-Schwarz inequality; i.e. $[x, y] \le [x, x]^{\frac{1}{2}}[y, y]^{\frac{1}{2}}$. So

$$\|Bx\|^2 = \langle Bx, Bx\rangle = [x, Bx] \le [x, x]^{\frac{1}{2}}[Bx, Bx]^{\frac{1}{2}} = \langle Bx, x\rangle^{\frac{1}{2}}\langle B^3x, x\rangle^{\frac{1}{2}}$$

Since $A - A_\lambda \ge 0$ we have $\langle (A - A_\lambda)x, x\rangle \to 0$. Fix λ_0. For $\lambda \ge \lambda_0$ we have $A - A_\lambda \le A - A_{\lambda_0}$; so $\|A - A_\lambda\| \le \|A - A_{\lambda_0}\|$. Thus

$$\begin{aligned}\|(A - A_\lambda)x\|^2 &\le \langle (A - A_\lambda)x, x\rangle^{\frac{1}{2}}\langle (A - A_\lambda)^3x, x\rangle^{\frac{1}{2}}\\ &\le \langle (A - A_\lambda)x, x\rangle^{\frac{1}{2}}\|A - A_\lambda\|^{\frac{3}{2}}\|x\|\\ &\le \langle (\underbrace{A - A_\lambda}_{\to 0})x, x\rangle^{\frac{1}{2}}\underbrace{\|A - A_{\lambda_0}\|^{\frac{3}{2}}\|x\|}_{\text{constant}}\end{aligned}$$

so $A_\lambda x \to Ax$ for all x. So $A_\lambda \xrightarrow{\text{SOT}} A$. ☐ Lemma 7.17

Corollary 7.18. *If $(P_\lambda)_\lambda$ is an increasing net of projections then the SOT-limit P of P_λ is the projection onto*

$$\overline{\bigcup_{\lambda\in\Lambda} \operatorname{Ran}(P_\lambda)}$$

Proof. $P_\lambda \le I$, so we have a bounded, increasing net. So the SOT-limit P of P_λ exists. Let $M_\lambda = \operatorname{Ran}(P_\lambda)$ and

$$M = \overline{\bigcup_{\lambda\in\Lambda} M_\lambda}$$

If $x \perp M$ then $P_\lambda x = 0$ for all λ; so $Px = 0$. If $x \in M_{\lambda_0}$ then $x = P_\lambda x$ for all $\lambda \ge \lambda_0$; so $Px = x$. Thus $Px = x$ for all

$$x \in \bigcup_{\lambda\in\Lambda} M_\lambda$$

so by continuity of P we get $Px = x$ for all $x \in M$. So $P = P_M$. ☐ Corollary 7.18

Suppose $A = A^* \in \mathcal{B}(\mathcal{H})$; translate and scale A so that $\sigma(A) \subseteq [0,1]$. We want projections in $W^*(A)$. Suppose $\mathcal{O} \subseteq [0,1]$ is open; consider $\{ f(A) : f \in \mathcal{C}[0,1], 0 \le f \le \chi_{\mathcal{O}} \} \subseteq C^*(A)$. This is a directed set, since if $f, g \le \chi_{\mathcal{O}}$ in $\mathcal{C}[0,1]$ then $f \vee g \in \mathcal{C}[0,1]$ with $f, g \le f \vee g \le \chi_{\mathcal{O}}$. So $f(A), g(A) \le (f \vee g)(A) \in C^*(A) \cong \mathcal{C}(\sigma(A))$. By lemma (since all are bounded by I) we get

$$P_{\mathcal{O}} = \sup\{ f(A) : f \in \mathcal{C}[0,1], 0 \le f \le \chi_{\mathcal{O}} \}$$

exists as a SOT-limit, and is thus in $W^*(A)$.

Claim 7.19. $P_{\mathcal{O}} = P_{\mathcal{O}}^2$.

Proof. Note $P_{\mathcal{O}} \le I$. If $f \in \mathcal{C}[0,1]$ with $0 \le f \le \chi_{\mathcal{O}}$ then $0 \le f^{\frac{1}{2}} \le \chi_{\mathcal{O}}$.

Note by the double commutant theorem that since $P_{\mathcal{O}} \in W^*(A)$ we get $P_{\mathcal{O}}$ commutes with $C^*(A)$ (since $C^*(A)$ is abelian). But $P_{\mathcal{O}} \ge f^{\frac{1}{2}}(A)$; so since they commute we have $P_{\mathcal{O}}^2 \ge f(A)$,

TODO 47. *?*

so $P_{\mathcal{O}}^2 \ge P_{\mathcal{O}}$. But $0 \le P_{\mathcal{O}} \le I$; so $P_{\mathcal{O}}^2 \le P_{\mathcal{O}} \le P_{\mathcal{O}}^2$. So $P_{\mathcal{O}} = P_{\mathcal{O}}^2$ is a projection. □ Claim 7.19

Suppose $n \ge 1$; divide $[0,1]$ into 2^n equal segments. Let $P_{j,n} = P_{(j2^{-n},2)}$; let

$$A_n = 2^{-n} \sum_{j=1}^{2^n} P_{j,n} \in W^*(A) = \sup\left\{ f(A) : f \in \mathcal{C}[0,1], f \le 2^{-n} \sum_{j=1}^{2^n} \chi_{(j2^{-n},2)} \right\} \le A$$

Then $A_n \ge A - 2^{-n} I$, so $A = \lim_n A_n$ in norm.

Corollary 7.20. $A \in \overline{\mathrm{Conv}(\mathrm{Proj}(W^*(A)))}^{\|\cdot\|}$. *Thus if* $\mathfrak{A}$ *is a von Neumann algebra then* $\overline{\mathrm{Conv}(\mathrm{Proj}(\mathfrak{A}))}^{\|\cdot\|} = \overline{b_1(\mathfrak{A}_{\ge 0})}$.

Proof. We showed the first part above. For the second, note that if $A \in \mathfrak{A}$ with $0 \le A \le I$ then $A \in \overline{\mathrm{Conv}(\mathrm{Proj}(W^*(A)))}^{\|\cdot\|} \subseteq \overline{\mathrm{Conv}(\mathrm{Proj}(\mathfrak{A}))}^{\|\cdot\|}$. □ Corollary 7.20

Note that the projections are the extreme points of $b_1(\mathfrak{A}_+)$, and the symmetries are the extreme points of $b_1(\mathfrak{A}_{\mathrm{sa}})$.

Corollary 7.21. $\mathrm{Conv}(\mathrm{Sym}(\mathfrak{A})) = b_1(\mathfrak{A}_{\mathrm{sa}})$.

(The symmetries are self-adjoint unitaries, and we have for $P - P^\perp = 2P - I$, P projections that $A \mapsto 2A - I$ maps $b_1(\mathfrak{A}_+)$ bijectively to $b_1(\mathfrak{A}_{\mathrm{sa}})$.)

8 Representations of C*-algebras

Definition 8.1. A *representation* π of a C*-algebra $\mathfrak{A}$ is a *-homomorphism to $\mathcal{B}(\mathcal{H})$. It is *non-degenerate* if $\overline{\mathfrak{A}\mathcal{H}} = \mathcal{H}$. We say π is *topologically irreducible* if $\pi(\mathfrak{A})$ has no closed invariant subspaces; we say π is *algebraically irreducible* if $\pi(\mathfrak{A})$ has no proper submodules (i.e. if $x \neq 0$ then $\pi(\mathfrak{A})x = \mathcal{H}$)).

Lemma 8.2. *π is topologically irreducible if and only if $\pi(\mathfrak{A})' = \mathbb{C}I$.*

Proof.

($\Longleftarrow$) Suppose M is a closed subspace with $\pi(\mathfrak{A})M = M$; so $\mathcal{H} = M \oplus M^\perp$ with

$$\pi(\mathfrak{A}) \subseteq \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

But $\pi(\mathfrak{A}) = \pi(\mathfrak{A})^*$; so

$$\pi(\mathfrak{A}) \subseteq \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} = \left\{ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \right\}' = \{ P_M \}'$$

So $P_M \in \pi(\mathfrak{A})'$.

($\Longrightarrow$) Suppose $\pi(\mathfrak{A})' \neq \mathbb{C}I$; then there is a projection $P = P^2$ with $P \notin \{0, I\}$ and $P \in \pi(\mathfrak{A})'$. Then $M = \operatorname{Ran}(P)$ is invariant, and π is not topologically irreducible. □ Lemma 8.2

Lemma 8.3. *Suppose π is a topologically irreducible representation of $\mathfrak{A}$. Suppose M is a subspace with $\dim(M) < \infty$. Let $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. Then there is $a \in \mathfrak{A}$ with $\|a\| \leq \|T\|$ such that $\|(T - \pi(a)) \restriction M\| < \varepsilon$.*

Proof. Let $\dim(M) = n$ and $\{e_1, \ldots, e_n\}$ an orthonormal basis for M; without loss of generality assume $\|T\| = 1$. Then since $\pi(\mathfrak{A})' = \mathbb{C}I$ we get $\pi(\mathfrak{A})'' = (\mathbb{C}I)' = \mathcal{B}(\mathcal{H})$. By Kaplansky's density theorem we have $\overline{b_1(\mathcal{B}(\mathcal{H}))} = \overline{b_1(\pi(\mathfrak{A}))}^{\mathrm{SOT}}$. Pick $a \in \mathfrak{A}$ such that $\|\pi(a)\| < 1$ and $\|Te_i - \pi(a)e_i\| < \frac{\varepsilon}{n}$ for $1 \leq i \leq n$. Then

$$\|(T - \pi(a)) \restriction M\| \leq \|(T - \pi(a))P_M\| \leq \sum_{i=1}^{n} \|(T - \pi(a))P_{\mathbb{C}e_i}\| < n \cdot \frac{\varepsilon}{n} = \varepsilon$$

Then we have

$$\begin{aligned} \mathfrak{A} \xrightarrow{q} \mathfrak{A}/\ker(\pi) &\xrightarrow{\tilde{\pi}} \mathcal{B}(\mathcal{H}) \\ a \mapsto \ \|\dot{a}\| < 1 \ &\mapsto \|\pi(a)\| < 1 \end{aligned}$$

Choose $a_1 \in a + \ker(\pi)$ such that $\|a_1\| < \|\dot{a}\| + \delta < 1$. We then use a_1. □ Lemma 8.3

Theorem 8.4 (Kadison's transitivity theorem)**.** *Suppose $\pi\colon \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is topologically irreducible and $\dim(M) < \infty$; suppose $T \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$. Then there is $a \in \mathfrak{A}$ with $\|a\| < \|T\| + \varepsilon$ such that $\pi(a) \restriction M = T \restriction M$.*

Proof. Use the lemma to find $a_0 \in \mathfrak{A}$ with $\|a_0\| \leq \|T\|$ such that $\|(T - \pi(a)) \restriction M\| < \frac{\varepsilon}{4}$, and let $T_1 = (T - \pi(a))P_M$. Find $a_1 \in \mathfrak{A}$ with $\|a_1\| \leq \|T_1\| < \frac{\varepsilon}{4}$ such that $\|(T_1 - \pi(a_1)) \restriction M\| < \frac{\varepsilon}{8}$; then let $T_2 = T - \pi(a_0) - \pi(a_1)$. Recursively find $a_n \in \mathfrak{A}$ such that $\|a_n\| < \frac{\varepsilon}{2^{n+1}}$ such that

$$\|T - \pi(a_0 + a_1 + \cdots + a_n)\| < \frac{\varepsilon}{2^{n+2}}$$

Let $a = \sum_{n \geq 0} a_n$; so

$$\|a\| \leq \|a_0\| + \sum \frac{\varepsilon}{2^{n+1}} < \|T\| + \frac{\varepsilon}{2} + \varepsilon 2$$

and

$$(T - \pi(a)) \restriction M = \lim_n \left(T - \pi\left(\sum_{i=0}^{n} a_i\right)\right) \restriction M = 0$$

as desired. □ Theorem 8.4

Corollary 8.5. *If π is topologically irreducible then π is algebraically irreducible.*

Proof. Suppose $x, y \in \mathcal{H}$ with $x \neq 0$. Let $T = y\frac{x^*}{\|x\|^2}$, so $Tx = y$. Then there is a such that $\pi(a)x = y$; so the action is transitive. □ Corollary 8.5

8.1 GNS construction

This is Gelfand-Naimark-Segal.

Definition 8.6. A linear functional f on a C*-algebra $\mathfrak{A}$ is called *positive* if $a \geq 0$ implies $f(a) \geq 0$. A positive linear functional of norm 1 is called a *state*.

Example 8.7. If $\pi\colon \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a non-degenerate representation and $x \in \mathcal{H}$ with $\|x\| = 1$ then $f(a) = \langle \pi(a)x, x\rangle$ is a state.

Proof. If $a \geq 0$ then $\pi(a) \geq 0$, so $\langle \pi(a)x, x\rangle \geq 0$. Also $\|f\| \leq \|\pi\|\|x\|^2 = 1$. If $1 \in \mathfrak{A}$ then $f(1) = \langle \pi(1)x, x\rangle = \langle Ix, x\rangle = 1$; so $\|f\| = 1$. If $\mathfrak{A}$ is not unital, we will see that if $(e_\lambda)_\lambda$ is an approximate identity then $\pi(e_\lambda) \xrightarrow{\mathrm{SOT}} I$; so $\|f\| \geq \sup|f(e_\lambda)| = 1$. □

Remark 8.8. If f is a positive linear functional then $[a,b] = f(b^*a)$ (for $a, b \in \mathfrak{A}$) is a sesquilinear form on $\mathfrak{A}$; it is linear in a and conjugate-linear in b, and $[a,a] = f(a^*a) \geq 0$. So Cauchy-Schwarz inequality holds, and

$$|f(b^*a)| = |[a,b]| \leq [a,a]^{\frac{1}{2}}[b,b]^{\frac{1}{2}} = f(a^*a)^{\frac{1}{2}} f(b^*b)^{\frac{1}{2}}$$

Lemma 8.9. *Suppose f is a positive linear functional on $\mathfrak{A}$. If $1 \in \mathfrak{A}$ then $\|f\| = f(1)$. If $(e_\lambda)_\lambda$ is an approximate identity then $\|f\| = \sup f(e_\lambda) < \infty$. In particular, positive linear functionals are continuous.*

Proof.

Case 1. Suppose $\mathfrak{A}$ is unital. If $0 \leq a \leq 1$ then $0 \leq f(a) \leq f(1)$. If $a \in \mathfrak{A}$ with $\|a\| \leq 1$ then $0 \leq a^*a \leq 1$, so $f(a^*a) \leq f(1)$. Then

$$|f(a)| = |f(1^*a)| \leq f(a^*a)^{\frac{1}{2}} f(1^*1)^{\frac{1}{2}} \leq f(1)$$

So $\|f\| \leq f(1) \leq \|f\|$.

Case 2. Suppose $\mathfrak{A}$ is non-unital.

Claim 8.10. *$f \restriction \mathfrak{A}_{\geq 0}$ is continuous.*

Proof. If not there are $a_n \geq 0$ with $\|a_n\| < 2^{-n}$ and $f(a_n) > 1$; then

$$a = \sum_{n \geq 1} a_n \in \mathfrak{A}_{\geq 0}$$

and

$$f(a) \geq f\left(\sum_{n=1}^{N} a_n\right) = \sum_{n=1}^{N} f(a_n) > N$$

a contradiction. □ Claim 8.10

Aside 8.11. In this section we may use $\mathfrak{A}_+$ to mean $\mathfrak{A}_{\geq 0}$.

So f is continuous, and there is c such that $f(a) \leq C\|a\|$ for all $a \geq 0$. Now if $a \in \mathfrak{A}$ then

$$a = \mathrm{Re}(a) + i\,\mathrm{Im}(a) = b_+ - b_- + i(c_+ - c_-)$$

with

$$\begin{aligned} \|b_\pm\| &\leq \|\mathrm{Re}(a)\| \leq \|a\| \\ \|c_\pm\| &\leq \|\mathrm{Im}(a)\| \leq \|a\| \end{aligned}$$

Then

$$|f(a)| \leq f(b_+) + f(b_-) + f(c_+) + f(c_-) \leq 4C\|a\|$$

Thus $M = \sup_\lambda f(e_\lambda) < \infty$ and $M = \lim_\lambda f(e_\lambda)$ since the e_λ is an increasing net. Note also that $0 \leq e_\lambda \leq 1$, so $0 \leq e_\lambda^2 \leq e_\lambda$, and $f(e_\lambda^2) \leq f(e_\lambda) \leq M$.

Now, by continuity we have

$$\begin{aligned} |f(a)|^2 &= \lim_\lambda |f(e_\lambda a)|^2 \\ &\leq \lim_\lambda |f(a^*a)||f(e_\lambda^2)| \text{ (Cauchy-Schwarz)} \\ &\leq \lim_\lambda \|f\|\|a\|^2 M \\ &= \|f\|\|a\|^2 M \end{aligned}$$

So

$$\|f\|^2 = \sup_{\|a\| \leq 1} |f(a)|^2 \leq \sup \|f\| \cdot 1 \cdot M = \|f\| \cdot M$$

So $\|f\| \leq M = \sup_\lambda f(e_\lambda) \leq \|f\|$. □ Lemma 8.9

Theorem 8.12 (GNS)**.** *Suppose f is a state on a C^*-algebra $\mathfrak{A}$. Then there is a representation $\pi_f\colon \mathfrak{A} \to \mathcal{B}(\mathcal{H}_f)$ and a unit vector $\xi_f \in \mathcal{H}_f$ such that*

1. *$f(a) = \langle \pi(a)\xi_f, \xi_f\rangle$*
2. *ξ_f is a cyclic vector; i.e. $\overline{\pi(\mathfrak{A})\xi_f} = \mathcal{H}_f$.*

Proof. Let $N = \{\, a \in \mathfrak{A} : f(a^*a) = 0 \,\}$. If $a \in N$ and $b \in \mathfrak{A}$ then by Cauchy-Schwarz we have

$$|[a,b]| = |f(b^*a)| \leq f(a^*a)^{\frac{1}{2}} f(b^*b)^{\frac{1}{2}} = 0$$

So $N = \{\, a : [a,b] = 0 \text{ for all } b \in \mathfrak{A} \,\}$; so N is a subspace. If $a \in N$ and $b \in \mathfrak{A}$ then

$$f((ba)^*(ba)) = f(a^*b^*ba) \leq \|b\|^2 f(a^*a) = 0$$

so N is a left ideal. Since f is continuous, we get that N is closed. So $\mathfrak{A}/N$ is a Banach space, with elements $\dot{a} = a + N$. We define an inner product by $\langle \dot{a}, \dot{b}\rangle = f(b^*a) = [a,b]$. Given representatives $a, a+n$ and $b, b+m$ with $n, m \in N$ we have

$$f((b+m)^*(a+n)) = f(b^*a + m^*a + b^*n + m^*n)$$

Since $b^*n, m^*n \in N$ we have $f(b^*n + m^*n) = 0$. Since $f \geq 0$ if $a = x + iy$ (so $a^* = x - iy$) then

$$f(a) = \underbrace{f(x)}_{\in\mathbb{R}} + i \underbrace{f(y)}_{\in\mathbb{R}}$$

and $f(a^*) = f(x) - if(y) = \overline{f(a)}$. So $f(m^*a) = \overline{f(a^*m)} = 0$ since $m \in N$ implies $a^*m \in N$. Thus $f(b+m)^*(a+n)) = f(b^*a)$.

So $\langle \dot{a}, \dot{b}\rangle$ is well-defined. Also if $0 = \langle \dot{a}, \dot{a}\rangle = f(a^*a)$ then $a \in N$ and $\dot{a} = \dot{0}$; so this is a positive definite inner product. We have an inner product norm $\|\dot{a}\|_2 = \langle \dot{a}, \dot{a}\rangle^{\frac{1}{2}}$. The completion of $(\mathfrak{A}/N, \|\cdot\|_2)$ is a Hilbert space. Define $\pi_0\colon \mathfrak{A} \to \mathcal{L}(\mathfrak{A}/N)$ by $\pi_0(a)\dot{b} = (\dot{ab})$. Then $a\dot{b} = a(b+N) = ab + aN \subseteq ab + N$, so this is independent of the choice of b. Also π_0 is a homomorphism of algebras; furthermore

$$\begin{aligned}
\langle \pi_0(a^*)\dot{b}, \dot{c}\rangle &= \langle (\dot{a^*b}), \dot{c}\rangle \\
&= f(c^*a^*b) \\
&= f((ac)^*b) \\
&= \langle \dot{b}, (\dot{ac})\rangle \\
&= \langle \dot{b}, \pi_0(a)\dot{c}\rangle \\
&= \langle \pi_0(a)^*\dot{b}, \dot{c}\rangle
\end{aligned}$$

So $\pi_0(a^*) = \pi_0(a)^*$, and π_0 is a *-homomorphism. Also

$$\begin{aligned}
\|\pi_0(a)\| &= \sup_{\|\dot{b}\|\leq 1} \|\pi_0(a)\dot{b}\| \\
&= \sup_{f(b^*b)\leq 1} f((ab)^*ab)^{\frac{1}{2}} \\
&= \sup_{f(b^*b)\leq 1} f(b^*a^*ab)^{\frac{1}{2}} \\
&\leq \sup_{f(b^*b)\leq 1} (\|a\|^2 f(b^*b))^{\frac{1}{2}} \\
&= \|a\|
\end{aligned}$$

so $\|\pi_0\| \leq 1$ and π_0 is continuous. We can extend π_0 to a continuous linear operator $\pi_f(a)$ in $\mathcal{B}(\mathcal{H}_f)$; then $\pi_f\colon \mathfrak{A} \to \mathcal{B}(\mathcal{H}_f)$ is a *-representation on $\mathcal{H}_f$.

Case 1. Suppose $1 \in \mathfrak{A}$; we then let $\xi_f = \dot{1}$. Then $\|\xi_f\|^2 = f(1^*1) = f(1) = \|f\| = 1$. Then

$$\langle \pi(a)\xi_f, \xi_f\rangle = \langle \pi(a)\dot{1}, \dot{1}\rangle = f(1^*a1) = f(a)$$

Also $\pi(\mathfrak{A})\dot{1} = \{\, \dot{a} : a \in \mathfrak{A} \,\} = \mathcal{H}_f$; so ξ_f is cyclic.

Case 2. Suppose $\mathfrak{A}$ is not unital; let $(e_\lambda)_\lambda$ be an approximate identity.

Claim 8.13. *$(\dot{e_\lambda})_\lambda$ is Cauchy.*

Proof. Note that $1 = \|f\| = \lim_\lambda f(e_\lambda)$. If $\varepsilon > 0$ and

$$f(e_\lambda) > 1 - \varepsilon$$
$$f(e_\mu) > 1 - \varepsilon$$

then there is ν with $e_\nu \geq e_\lambda$ and $e_\nu \geq e_\mu$, and so $f(e_\nu) \geq f(e_\lambda) > 1 - \varepsilon$. So $\|e_\nu e_\lambda - e_\lambda\| < \varepsilon$ and $\|e_\nu e_\mu - e_\mu\| < \varepsilon$. Then

$$\|\dot{e_\nu} - \dot{e_\mu}\|^2 = f((e_\nu - e_\mu)^2) = f(e_\nu^2 + e_\mu^2 - e_\nu e_\mu - e_\mu e_\nu)$$

But

$$|f(e_\nu e_\mu)| = |f(e_\mu) + f(e_\nu e_\mu - e_\mu)| \geq 1 - \varepsilon - \|e_\nu e_\mu - e_\mu\| > 1 - 2\varepsilon$$

and also $|f(e_\mu e_\nu)| = |\overline{f(e_\nu e_\mu)}| > 1 - 2\varepsilon$. Thus

$$\|\dot{e_\nu} - \dot{e_\mu}\|^2 \leq f(e_\nu^2) + f(e_\mu)^2 - 2(1 - 2\varepsilon) \leq 2 - 2 + 4\varepsilon = 4\varepsilon$$

TODO 48. *something about this being because e_ν, e_μ being norm 1 and positive?*

so $\|\dot{e_\nu} - \dot{e_\mu}\| \leq 2\sqrt{\varepsilon}$. Then

$$\|\dot{e_\mu} - \dot{e_\lambda}\| \leq \|\dot{e_\mu} - \dot{e_\nu}\| + \|\dot{e_\nu} - \dot{e_\lambda}\| < 2\sqrt{\varepsilon} + 2\sqrt{\varepsilon} = 4\sqrt{\varepsilon}$$

and $(e_\lambda)_\lambda$ is Cauchy. □ Claim 8.13

Let $\xi_f = \lim_\lambda \dot{e_\lambda}$. Then

$$\langle \pi(a)\xi_f, \xi_f \rangle = \lim_\lambda \langle \pi(a)\dot{e_\lambda}, \dot{e_\lambda} \rangle = \lim_\lambda f(e_\lambda a e_\lambda) = f(a)$$

and

$$\begin{aligned} \|\dot{a} - \pi(a)\xi_f\|^2 &= \lim_\lambda \|\dot{a} - \pi(a)\dot{e_\lambda}\|^2 \\ &= \lim_\lambda \|\dot{a} - (a\dot{e}_\lambda)\|^2 \\ &= 0 \end{aligned}$$

TODO 49. *something about how the penultimate is equal to $\|\dot{a} - (e_\lambda \dot{a})\|$ and in turn to $\|\dot{a} - \pi(e_\lambda)\dot{a}\|$?*

Thus $\pi(e_\lambda)\dot{a} \to \dot{a}$; so $\pi(e_\lambda) \xrightarrow{\text{SOT}} I$. So $\overline{\pi(\mathfrak{A})\xi_f} = \overline{\mathfrak{A}/N} = \mathcal{H}_f$ is then cyclic. Also

$$1 = \lim_\lambda f(e_\lambda) = \lim_\lambda \langle \pi(e_\lambda)\xi_f, \xi_f \rangle = \langle \xi_f, \xi_f \rangle$$

□ Theorem 8.12

Corollary 8.14. *If $\mathfrak{A}$ is not unital and f is a state then f extends uniquely to a state on $\mathfrak{A}^+$ by setting $f(1) = 1$.*

Proof. Suppose g is a Hahn-Banach extension of f to $\mathfrak{A}^+$; so $\|g\| = 1 \geq |g(1)|$. Let $g(1) = \alpha$. Then $1 = \lim_\lambda f(e_\lambda)$ and $0 \leq e_\lambda \leq 1$, so $-1 \leq 1 - 2e_\lambda \leq 1$, and

$$1 \geq |g(1 - 2e_\lambda)| = |\alpha - 2f(e_\lambda)| \to |\alpha - 2|$$

Then since $|\alpha| \leq 1$ and $|\alpha - 2| \leq 1$ we get $\alpha \in \overline{\mathbb{D}} \cap (2 + \overline{\mathbb{D}}) = \{ 1 \}$. So $g(1) = 1$,a nd g is unique.

Also

$$g(\alpha + \lambda 1) = \langle (\pi(a) + \lambda I)\xi_f, \xi_f \rangle = \widetilde{\pi}(a + \lambda 1)$$

where $\widetilde{\pi} \colon \mathfrak{A}^+ \to \mathcal{B}(\mathcal{H}_f)$ is $\widetilde{\pi}(a) = \pi(a)$ and $\widetilde{\pi}(1) = I$; so $\widetilde{\pi}$ is *-linear, and $g \geq 0$. □ Corollary 8.14

Lemma 8.15. *Suppose f is a linear functional on $\mathfrak{A}$.*

1. *If $1 \in \mathfrak{A}$ and $f(1) = 1 = \|f\|$, then f is a state.*
2. *If $(e_\lambda)_\lambda$ is an approximate identity and $1 = \|f\| = \lim_\lambda f(e_\lambda)$ then f is a state.*

Proof.

1. If $a = a^*$ write $f(a) = x + iy$ for $x, y \in \mathbb{R}$. Then
$$|f(a + it1)|^2 = |(x + iy) + it|^2 = x^2 + (y + t)^2 \leq \|a + it1\|^2$$
But $a+it1$ is normal and $\sigma(a+it1) = \sigma(a)+it \subseteq [-\|a\|, \|a\|]+it$. So $\|a+it\| = \operatorname{spr}(a+it) = \sqrt{\|a\|^2 + t^2}$. Thus
$$\|a\|^2 + t^2 \geq x^2 + (y + t)^2 = x^2 + y^2 + 2yt + t^2$$
So $x^2 + y^2 + 2yt \leq \|a\|^2$ for all $t \in \mathbb{R}$. So $y = 0$, and $f(a) \in \mathbb{R}$.
If $a = a^*$ with $0 \leq a \leq 1$ then $-1 \leq 2a - 1 \leq 1$. So $-1 \leq 2f(a) - 1 \leq 1$ since $\|f\| = 1$ and $f(a) \in \mathbb{R}$. Thus $0 \leq f(a) \leq 1$. So $f \geq 0$, and f is a state.
2. Extend f by Hahn-Banach to a norm 1 functional on $\mathfrak{A}^+$. Then $\lim_\lambda f(e_\lambda) = 1$, so by the same proof as the previous corollary we get $g(1) = 1$. So by the unital case we get that g is a state. So f is a state. ☐ Lemma 8.15

Definition 8.16. The *state space* of $\mathfrak{A}$ is $S(\mathfrak{A}) = \{ f \in A^* : f \geq 0, \|f\| = 1 \}$; the *quasi-state space* of $\mathfrak{A}$ is $Q(\mathfrak{A}) = \{ f \in \mathfrak{A}^* : f \geq 0, \|f\| \leq 1 \}$.

Remark 8.17. If $\mathfrak{A}$ is unital then $S(\mathfrak{A})$ is weak*-compact: indeed,
$$S(\mathfrak{A}) = \{ f \in \mathfrak{A}^* : 1 = f(1) = \|f\| \} = \underbrace{\overline{b_1(\mathfrak{A}^*)}}_{\text{weak*-compact}} \cap \underbrace{\{ f \in \mathfrak{A}^* : f(1) = 1 \}}_{\text{weak*-closed}}$$
If $\mathfrak{A}$ is not unital then generally $S(\mathfrak{A})$ is not weak*-compact. But $Q(\mathfrak{A})$ is always weak*-compact: indeed,
$$Q(\mathfrak{A}) = \overline{b_1(\mathfrak{A}^*)} \cap \bigcap_{a \geq 0} \underbrace{\{ f \in \mathfrak{A}^* : f(a) \geq 0 \}}_{weak*-closed}$$

Example 8.18. Consider $\mathfrak{A} = \mathcal{C}_0((0, 1])^* = \mathcal{M}((0, 1])$, the space of complex regular Borel measures; then $S(\mathfrak{A})$ is the space of probability measures. Let $\mu_n = n \cdot (m \upharpoonright (0, n^{-1}])$ (where m is the Lebesgue measure); then $\mu_n \xrightarrow{w^*} 0$ (i.e. δ_0). So $S(\mathfrak{A})$ isn't weak*-closed.

Definition 8.19. A state f is *pure* if $g \in \mathfrak{A}^*$ and $0 \leq g \leq f$ implies there is $t \in [0, 1]$ with $g = tf$.

Proposition 8.20. *$f \in S(\mathfrak{A})$ is pure if and only if it is extreme.*

Aside 8.21. $C = \{ g \in \mathfrak{A}^* : g \geq 0 \}$ is a weak*-closed cone. The pure states lie on extreme rays. If $1 \in \mathfrak{A}$ then
$$S(\mathfrak{A}) = C \cap \{ g \in \mathfrak{A}^* : g(1) = 1 \} = C \cap \{ g \in \mathfrak{A}^* : \|g\| = 1 \}$$

Proof of Proposition 8.20.

($\implies$) Suppose f is not extreme; say $f = \frac{1}{2}(g + h)$ for $g, h \in S(\mathfrak{A})$ and $g, h \neq f$. Then $0 \leq \frac{1}{2}g \leq f$ but $g \neq tf$ for $t \in [0, 1]$; so f is not pure.

($\Longleftarrow$) Suppose f is not pure; then there is g with $0 \leq g \leq f$ with $g \notin \mathbb{R}_+ f$. Let $h = f - g \geq 0$. Then

$$f = \|g\|(\|g\|^{-1}g) + \|h\|(\|h\|^{-1}h)$$

with $\|g\|^{-1}g, \|h\|^{-1}h \in S(\mathfrak{A})$; furthermore if $(e_\lambda)_\lambda$ is an approximate identity then

$$\|g\| + \|h\| = \lim_\lambda g(e_\lambda) + h(e_\lambda) = \lim_\lambda f(e_\lambda) = 1$$

So f is not extreme. ☐ Proposition 8.20

Lemma 8.22. $\operatorname{ext}(Q(\mathfrak{A})) = \{ 0 \} \cup \operatorname{ext}(S(\mathfrak{A}))$. *So* $\overline{\operatorname{Conv}(\operatorname{ext}(S(\mathfrak{A})))}^{w^*} \supseteq S(\mathfrak{A})$.

Proof. If $f \in Q(\mathfrak{A})$ with $0 < \|f\| < 1$ then it is clear that f is not an extreme point. Clearly $0 \in \operatorname{ext}(Q(\mathfrak{A}))$, and by a triangle inequality argument we get that $\operatorname{ext}(S(\mathfrak{A})) \subseteq \operatorname{ext}(Q(A))$. So $\operatorname{ext}(Q(\mathfrak{A})) = \{ 0 \} \cup \operatorname{ext}(S(\mathfrak{A}))$.

By Krein-Milman we have $\overline{\operatorname{Conv}(\{ 0 \} \cup \operatorname{ext}(S(\mathfrak{A})))}^{w^*} = Q(\mathfrak{A}) \supseteq S(\mathfrak{A})$. So if $f \in S(\mathfrak{A})$ there is $(f_\lambda)_\lambda$ in $\operatorname{Conv}(\{ 0 \} \cup \operatorname{ext}(S(\mathfrak{A})))$ such that $f_\lambda \xrightarrow{w^*} f$. Write $f_\lambda = (1 - t_\lambda) \cdot 0 + t_\lambda g_\lambda$ with $g \in \operatorname{Conv}(\operatorname{ext}(S(\mathfrak{A}))) \subseteq S(\mathfrak{A})$ and $0 \leq t_\lambda \leq 1$; then $\|f_\lambda\| = t_\lambda$. But $\{ f \in Q(\mathfrak{A}) : \|f\| \leq r \}$ is weak*-compact; so $\lim_\lambda t_\lambda = 1$, and $g_\lambda \xrightarrow{w^*} f$. So $f \in \overline{\operatorname{Conv}(\operatorname{ext}(S(\mathfrak{A})))}$. ☐ Lemma 8.22

Lemma 8.23. *If f is a state with GNS representation $\langle \pi_f, \xi_f, \mathcal{H}_f \rangle$ then $\{ g : 0 \leq g \leq f \} \leftrightarrow \{ H \in \pi_f(\mathfrak{A})' : 0 \leq H \leq I \}$ with $g(a) = \langle \pi_f(a)\xi_f | H\xi_f \rangle \leftarrow\!\shortmid H$*

Proof. If $H \in \pi_f(\mathfrak{A})'$ with $0 \leq H \leq I$ then

$$g(a) = \langle \pi_f(a)\xi_f | H\xi_f \rangle = \langle H^{\frac{1}{2}}\pi_f(a)\xi_f | H^{\frac{1}{2}}\xi_f \rangle = \langle \pi_f(a)H^{\frac{1}{2}}\xi_f | H^{\frac{1}{2}}\xi_f \rangle$$

so $g \geq 0$. Then

$$(f - g)(a) = \langle \pi_f(a)\xi_f | (I - H)\xi_f \rangle = \cdots = \langle \pi_f(a)(I - H)^{\frac{1}{2}}\xi_f | (I - H)^{\frac{1}{2}}\xi_f \rangle \geq 0$$

for $a \geq 0$. So $0 \leq f \leq g$.

Conversely if $0 \leq g \leq f$ we define a sesquilinear form on $\mathfrak{A}/N$ (where $N = \{ a \in \mathfrak{A} : f(a^*a) = 0 \}$) by $[\dot{a}|\dot{b}]_g = g(b^*a)$. This is positive as $g \geq 0$, and well defined as if $a \in N$ then $0 \leq g(a^*a) \leq f(a^*a) = 0$ and the same proof from before applies. Also $[\dot{a}|\dot{a}]_g = g(a^*a) \leq f(a^*a) = \|\dot{a}\|^2_{\mathcal{H}_f}$, so our form is of norm ≤ 1. Thus there is $H \in \mathcal{B}(\mathcal{H})$ such that $[\dot{a}|\dot{b}] = \langle H\dot{a} \rangle \dot{b}_{\mathcal{H}_f}$; since our form is positive and norm ≤ 1, we get $H \geq 0$ and $\|H\| \leq 1$. So $0 \leq H \leq I$. Now for $a \in \mathfrak{A}$ we have

$$\langle (H\pi(a) - \pi(a)H)\dot{c} | \dot{b} \rangle = \langle H\pi(a)\dot{c} | \dot{b} \rangle - \langle H\dot{c} | \pi(a^*)\dot{b} \rangle = g(b^*\pi(a)c) - g(b^*\pi(a)c) = 0$$

So $H \in \pi(\mathfrak{A})'$. ☐ Lemma 8.23

Theorem 8.24. *If $f \in S(\mathfrak{A})$ then π_f is irreducible if and only if f is pure.*

Proof. Note that

$$\begin{aligned} \pi_f \text{ irreducible} &\iff \pi_f(\mathfrak{A})' = \mathbb{C}I \\ &\iff \{ g : 0 \leq g \leq f \} = \{ tf : 0 \leq t \leq 1 \} \\ &\iff f \text{ is pure} \end{aligned}$$

as desired. ☐ Theorem 8.24

Lemma 8.25. *If $a = a^* \in \mathfrak{A}$ then there is a pure state f such that $|f(a)| = \|a\|$.*

Proof. Since $a = a^*$ we get $C_0^*(a) \cong \mathcal{C}_0(\sigma(a) \setminus \{ 0 \})$.

TODO 50. *$C^*(a)$?*

The "evaluation at $\lambda = \|a\|$ or $\lambda = -\|a\|$" functional is a state on $C_0^*(a)$ that norms a; i.e. $f_0 \in S(C_0^*(a))$ and $f_0(a) = \pm\|a\|$. By Hahn-Banahch this extends to $f \in \mathfrak{A}^*$ of norm 1. If $(e_\lambda)_\lambda$ is an approximate identity for $C_0^*(a)k$ then $f_0(e_\lambda) \to 1$; so $f(e_\lambda) \to 1$, and f is a state. If $(d_\mu)_\mu$ is an approximate identity for $\mathfrak{A}$ then for all $r < 1$ there is λ such that $f(e_\lambda) > r$; so there is $d_\mu > e_\lambda$ such that $f(d_\mu) > r$.

Let $\mathcal{F} = \{ f \in S(\mathfrak{A}) : f(a) = \|a\| \}$ or $\mathcal{F} = \{ f \in S(\mathfrak{A}) : f(a) = -\|a\| \}$. Then $\mathcal{F}$ is non-empty, weak*-closed, and convex.

Claim 8.26. *$\mathcal{F}$ is a face of $Q(A)$.*

Proof. Suppose $f \in \mathcal{F}$ with $f = \frac{1}{2}(g+h)$ for $g, h \in Q(\mathfrak{A})$. Then

$$\pm a = f(a) = \frac{g(a) + h(a)}{2} \leq \frac{\|a\| + \|a\|}{2}$$

TODO 51. *Last inequality may need slight modification*

So $g(a) = h(a) = \pm\|a\|$; thus $g, h \in \mathcal{F}$. ☐ Claim 8.26

By Krein-Milman we get that $\mathcal{F}$ has an extreme point f_0. But a face of a face is a face; so $f_0 \in \operatorname{ext}(Q(\mathfrak{A}))$ and $f_0 \neq 0$. So $f_0 \in \operatorname{ext}(S(\mathfrak{A}))$. ☐ Lemma 8.25

Theorem 8.27 (GNS)**.** *If $\mathfrak{A}$ is a C^*-algebra then*

$$\pi = \bigoplus_{f \text{ pure}} \pi_f$$

*is a faithful *-representation. If $\mathfrak{A}$ is separable then a countable collection of pure states is sufficient.*

Proof. By lemma if $a = a^*$ there is a pure state f with $|f(a)| = \|a\|$. ($f(a) = \langle \pi_f(a)\xi_f | \xi_f \rangle$.) So $\|\pi_f(a)\| = \|a\|$, and $\|\pi(a)\| = \|a\|$.

For a arbitrary we have

$$\|\pi(a)\|^2 = \|\pi(a^*a)\| = \|a^*a\| = \|a\|^2$$

so π is isometric.

If $\mathfrak{A}$ is separable choose $\{ a_n : n \in \mathbb{N} \}$ dense in $b_1(\mathfrak{A}_{\text{sa}})$. For each a_n choose f_n pure such that $\|\pi_f(a_n)\| = \|a_n\|$. Let

$$\sigma = \bigoplus_n \pi_{f_n}$$

Then $\|\sigma(a_n)\| = \|a_n\|$ for all n, so $\|\sigma(a)\| = \|a\|$ for all $a = a^*$ with $\|a\| \leq 1$. So σ is isometric. ☐ Theorem 8.27

Corollary 8.28. *C^*-algebras are semisimple.*

Proof. We have

$$\operatorname{rad}(\mathfrak{A}) = \bigcap_{\pi \text{ irreducible}} \ker(\pi) \subseteq \bigcap_{f \text{ pure}} \ker(\pi_f) = \{ 0 \}$$

as desired. ☐ Corollary 8.28

8.2 Representations and ideals

Proposition 8.29. *Suppose $\mathfrak{A}$ is a C^*-algebra and $J \lhd$ is an ideal. If π is a non-degenerate *-representation of J on $\mathcal{H}$ then there is a unique $\widetilde{\pi} = \operatorname{ind}(\pi) \colon \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ such that $\widetilde{\pi} \restriction = \pi$. Moreover if π is irreducible then so is $\widetilde{\pi}$.*

Proof. We have $\mathcal{H} = \overline{\pi(J)\mathcal{H}}$. Define $\widetilde{\pi}(a)\pi(j)x = \pi(aj)x$. (This is forced, and thus unique.) Is this well-defined? Suppose $\pi(j_1)x_1 = \pi(j_2)x_2$. Let $(e_\lambda)_\lambda$ be an approximate identity for J; we need to show that $\pi(aj_1)x_1 = \pi(aj_2)x_2$ for all $a \in \mathfrak{A}$. But

$$\pi(aj_1)x_1 = \lim_\lambda \pi(ae_\lambda j_1)x_1 = \lim_\lambda \pi(ae_\lambda)\pi(j_1)x_1 = \lim_\lambda \pi(ae_\lambda)\pi(j_2)x_2 = \lim_\lambda \pi(ae_\lambda j_2)x_2 = \pi(aj_2)x_2$$

Also $\widetilde{\pi}$ is linear and multiplicative. Also

$$\begin{aligned}\langle \widetilde{\pi}(a^*)\pi(j_1x_1)|\pi(j_2x_2)\rangle &= \langle \pi(j_2^*)\pi(a^*j_1)x_1|x_2\rangle \\ &= \langle \pi(j_2^*a^*j_1)x_1|x_2\rangle \\ &= \langle \pi(j_2^*a^*)\pi(j_1)x_1|x_2\rangle \\ &= \langle \pi(j_1)x_1|\pi(aj_2)x_2\rangle \\ &= \langle \pi(j_1)x_1|\widetilde{\pi}(a)\pi(j_2)x_2\rangle \\ &= \langle \widetilde{\pi}(a)^*\pi(j_1)x_1|\pi(j_2)x_2\rangle\end{aligned}$$

(on a dense subset at least). Finally, we have

$$\begin{aligned}\|\widetilde{\pi}(a)\| &= \sup_{\|\pi(j)x\|\le 1} \|\pi(aj)x\| \\ &= \sup\sup_\lambda \|\pi(ae_\lambda j)x\| \\ &= \sup\sup_\lambda \|\pi(ae_\lambda)\pi(j)x\| \\ &\le \sup\sup_\lambda \|ae_\lambda\| 1 \\ &\le \|a\|\end{aligned}$$

so $\widetilde{\pi}(a)$ is bounded. So $\widetilde{\pi}$ is bounded as well, and extends to a *-representation on all of $\mathcal{H}$. □ Proposition 8.29

Proposition 8.30. *Suppose $\pi\colon \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a *-representation and $J \triangleleft \mathfrak{A}$. Let $M = \overline{\pi(J)\mathcal{H}}$. Then M is a subrepresentation, so $\pi \cong \pi_1 \oplus \pi_2$ for $\pi_1\colon \mathfrak{A} \to \mathcal{B}(M)$ and $\pi_2\colon \mathfrak{A} \to \mathcal{B}(M^\perp)$. Then $\pi_1 = \operatorname{ind}(\pi \restriction J)$ and $\pi_2 \restriction J = 0$ (so π_2 factors through $\mathfrak{A}/J$).*

Proof. It is clear that M is invariant, hence reducing by taking adjoints we can write $\pi = \pi_1 \oplus \pi_2$. Then $\pi_1 \restriction J\colon J \to \mathcal{B}(M)$ is non-degenerate; so $\pi_1 = \operatorname{ind}(\pi_1 \restriction J)$ by lemma. Also $\pi(J) \restriction M^\perp = 0$, so $\ker(\pi_2) \supseteq J$; thus π_2 factors through $\mathfrak{A}/J$. □ Proposition 8.30

Example 8.31. Let $\mathfrak{A} = \mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ separable. Let $\mathcal{K} = \mathcal{K}(\mathcal{H})$; this is the only proper ideal of $\mathfrak{A}$.

Indeed, if $\mathcal{J} \triangleleft \mathcal{B}(\mathcal{H})$ with $0 \ne J \in \mathcal{J}$ then there is x, y such that $Jx = y \ne 0$; then given u, v there are rank one R, S such that $R(u) = x$ and $S(y) = v$. Then SJR is rank one and sends $u \mapsto v$, and SJR lies in $\mathcal{J}$; so $\mathcal{K} \subseteq \mathcal{J}$. If $J \in \mathcal{J} \setminus \mathcal{K}$ then $T = J^*J \in \mathcal{J}$ is not compact; without loss of generality assume $\sigma(T) \subseteq [0,1]$. If $K = K^* \ge 0$ compact, then $\sigma(K) = \{0, \lambda 1, \lambda_2, \dots\}$ with the $\lambda_n \to 0$; the eigenspaces $E_K(\lambda_n)$ are finite dimensional. Conversely if there is $\lambda \in \sigma(T)$ with $\dim(E - \{\lambda\}) = \infty$ ($= P$ an infinite rank projection, $C^*(T) \subseteq \mathcal{J}$) then there is an isometry S such that $S\mathcal{H} = P\mathcal{H}$, and $X \in \mathcal{B}(\mathcal{H})$ such that $SXS^* = P(SXS^*)P \in \mathcal{J}$. So $X = S^*(SXS^*)S \in \mathcal{J}$. Then $\sigma(T) \cap [r, 1]$ is uncountable. There is a projection

$$P = \sup_{0 \le f \le \chi_{[r,1]}} f(T) \in W^*(T)$$

with $PT \ge rP$ of infinite rank. So there is $Y \in \mathcal{B}(\mathcal{H})$ with $YT = P$, etc.

Assume $\pi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ and $\pi_1 = \operatorname{ind}(\pi \restriction K)$ and $\pi_2\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K} \to \mathcal{B}(\mathcal{K}_2)$. Then $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}_1)$, with $\mathcal{K}_1 = \overline{\pi(\mathcal{K})\mathcal{K}}$. $\mathcal{K}$ has only 1 irreducible represnetation up to unitary equivalence, namely id. Then

$$\pi = \operatorname{id}(\alpha) \oplus \pi_2$$

where the former is weak*-continuous and the latter is not.

9 Spectral theory for normal operators

Recall that if N is normal (i.e. $N^*N = NN^*$) then $C^*(N) \cong \mathcal{C}(\sigma(N))$ is abelian. More generally if the N_α are commuting normal operators, then by Fuglede's theorem $C^*(\{ N_\alpha \}_\alpha)$ is abelian, so is isomorphic to $\mathcal{C}(X)$ for some compact Hausdorff space X; if it is separable, then X is compact and metrizable.

Example 9.1. If μ is a Borel probability measure on X, there is a $*$-representation $\pi_\mu \colon \mathcal{C}(X) \to \mathcal{B}(L^2(\mu))$ given by $\pi_\mu(f)h = fh$. Then

$$\langle \pi_\mu(\overline{f})h, k\rangle = \langle \overline{f}h, k\rangle = \int (\overline{f}h)\overline{k}\mathrm{d}\mu = \int h(\overline{fk})\mathrm{d}\mu = \langle h, fk\rangle = \langle h, \pi_\mu(f)k\rangle = \langle \pi_\mu(f)^*h, k\rangle$$

So $\pi_\mu(\overline{f}) = \pi_\mu(f)^*$, and π_μ is a $*$-homomorphism. Also

$$\|\pi_\mu(f)\| = \underbrace{\operatorname{ess.sup}|f(x)|}_{\text{w.r.t. } \mu} \leq \|f\|_\infty$$

and 1 is a cyclic vector: $\overline{\pi_\mu(\mathcal{C}(X))1} = \overline{\mathcal{C}(X)}^{L^2(\mu)} = L^2(\mu)$.

Theorem 9.2. *Suppose $\pi \colon \mathcal{C}(X) \to \mathcal{B}(\mathcal{H})$ is a representation with cyclic vector x with $\|x\| = 1$. Then there is a regular Borel probability measure μ on X such that π is unitarily equivalent to π_μ (i.e. there is unitary $U \colon L^2(\mu) \to \mathcal{H}$ such that $\pi(f) = U\pi_\mu(f)U^*$).*

Proof. Define a state in $\mathcal{C}(X)$ by $\varphi(f) = \langle \pi(f)x, x\rangle$. (It is positive and linear, and $\|\varphi\| = \varphi(1) = \|x\|^2 = 1$.) By Riesz representation theorem there is a positive regular Borel measure μ on X such that $\varphi(f) = \int f\mathrm{d}\mu$. Then $\|\mu\| = \int 1\mathrm{d}\mu = \varphi(1) = 1$; so μ is a probability measure.

Define $U \colon \mathcal{C}(X) \to \mathcal{H}$ by $Uf = \pi(f)x$. Then

$$\|Uf\|^2 = \langle \pi(f)x, \pi(f)x\rangle = \langle \pi(|f|^2)x, x\rangle = \varphi(|f|^2) = \int |f|^2\mathrm{d}\mu = \|f\|^2_{L^2(\mu)}$$

Since $\mathcal{C}(X)$ is dense in $L^2(\mu)$ and U is isometric on $(\mathcal{C}(X), \|\cdot\|_{L^2(\mu)})$ we get that U extends by continuity to $U \colon L^2(\mu) \to \mathcal{H}$ which is isometric. But $\operatorname{Ran}(U)$ is closed, and thus contains $\overline{\pi(\mathcal{C}(X))x} = \mathcal{H}$; so U is unitary.

If $f, g \in \mathcal{C}(X)$ then

$$U\pi_\mu(f)g = Ufg = \rho(fg)x = \rho(f)\rho(g)x = \rho(f)Ug$$

TODO 52. *ρ? Mean π?*

This holds for $g \in \mathcal{C}(X)$, and $\mathcal{C}(X)$ is dense in $L^2(\mu)$; so by continuity we get $U\pi_\mu(f) = \rho(f)U$, and so $\rho(f) = U\pi_\mu(f)U^*$. □ Theorem 9.2

Lemma 9.3. *Suppose $\mathfrak{A}$ is a C^*-algebra and $\pi \colon \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ a non-degenerate $*$-representation. Then there is a decomposition $\mathcal{H} = \bigoplus_\alpha H_\alpha$ where each $\mathcal{H}_\alpha$ is a reducing subspace for $\pi(\mathfrak{A}$ and $\pi(\mathfrak{A}) \restriction \mathcal{H}_\alpha$ has a cyclic vector x_α.*

TODO 53. *Reducing subspace?*

Proof. If $0 \neq x$ then $\mathcal{H}_x = \overline{\pi(\mathfrak{A})x} = \overline{\pi(\mathfrak{A})''x}$

TODO 54. *Single $'$? Double?*

is a reducing subspace, and contains $Ix = x$. If $0 \neq y \perp \mathcal{H}_x$ then $\mathcal{H}_y \perp \mathcal{H}_x$: indeed, if $a, b \in \mathfrak{A}$ then $\langle \pi(a)y, \pi(b)x\rangle = \langle y, \pi(a^*b)x\rangle = 0$.

So by Zorn's lemma there is a maximal collection of vectors $\{ x_\alpha \}_\alpha$ in $\mathcal{H}$ such that $\mathcal{H}_{x_\alpha} \perp \mathcal{H}_{x_\beta}$ for all $\alpha \neq \beta$. Let $M = (\sum \mathcal{H}_\alpha)^\perp$. Suppose M were not $\{ 0 \}$; then there is $0 \neq y \in M$, so that $y \perp \mathcal{H}_{x_\alpha}$ for all α. So $\mathcal{H}_y \perp \mathcal{H}_{x_\alpha}$ for all α; so $\mathcal{H}_y \subseteq M$. So $\{ x_\alpha \}_\alpha \cup \{ y \}$ is a larger family, contradicting maximality. So $M = 0$, and $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_{x_\alpha}$. □ Lemma 9.3

Theorem 9.4 (Spectral theorem v1)**.** *If N is a normal operator on a separable Hilbert space then N is unitarily equivalent to a multiplication operator.*

Proof. $C^*(N) \cong \mathcal{C}(X)$ (in fact $X = \sigma(N)$) via $f \in \mathcal{C}(X) \mapsto f(N)$ by the continuous functional calculus; this is a *-representation. By lemma we get

$$\mathcal{H} = \bigoplus_{1 \leq i < \alpha} \mathcal{H}_i$$

(where $\alpha \in \mathbb{N} \cup \{ \omega \}$) such that $\pi_i(f) = f(N) \restriction \mathcal{H}_i$ is a cyclic representation. Then there are probability measures μ_i on $\sigma(N)$ such that $\pi_i(f) \cong M_f^{\mu_i}$ on $L^2(\mu_i)$. In particular $\pi(\mathrm{id}) = N$. So $N \cong \bigoplus_i \pi_i(\mathrm{id}) \cong \bigoplus M_z^{\mu_i}$ on $\bigoplus_i L^2(\mu_i)$.

Let $Y = \sigma(N) \times \mathbb{N}$. Suppose $\mu \in M(Y)$ with $\mu \restriction \sigma(N) \times \{ i \} = 2^{-i}\mu_i$; then μ is a probability measure. Then

$$L^2(\mu) = \bigoplus L^2(\sigma(N) \times \{ i \}, \mu) = \bigoplus L^2(2^{-i}\mu_i \cong \bigoplus L^2(\mu_i)$$

Let $h(x, i) = x$; then $M_h \cong \bigoplus M_{\mathrm{id}}^{2^{-i}\mu_i} \cong \bigoplus M_{\mathrm{id}}^{\mu_i} \cong N$. (If $U_i \colon L^2(\mu_i) \to L^2(2^{-i}\mu_i)$ is $U_i h = 2^{\frac{i}{2}} h$ then

$$\|U_i h\|_2^2 = \int 2^i |h|^2 \mathrm{d}(2^{-i}\mu_i) = \|h\|_{L^2(\mu)}$$

and $U_i M_f h = U_i f h = 2^{\frac{i}{2}} f h = M_f 2^{\frac{i}{2}} h = M_f U_i h$.) $\square$ Theorem 9.4

Example 9.5. Suppose N is normal and compact. Then $\sigma(N)$ is finite or an infinite sequence converging to $0 \in \sigma(N)$. If N is cyclic then $N \cong M_z$ on $L^2(\sigma(N)) = \ell^2(\sigma(N))$. If $\mu \in M(\sigma(N))$ then since $\sigma(N))$ is countable we can write

$$\mu = \sum \varepsilon_i \delta_{\lambda_i}$$

where λ_i range over $\sigma(N)$. So

$$N \cong \bigoplus \lambda_i$$

acting on $L^2(\mu) \cong \ell^2$. So N is diagonalizable. In general if N is a direct sum of diagonals it is diagonalizable. If $\sigma(N) = \{ \lambda_n : n \geq 1 \} \cup \{ 0 \}$ then there are $d_n = \dim(\ker(N - \lambda_n I)) < \infty$ so that

$$N \cong \mathrm{diag}(\underbrace{\lambda_1, \lambda_1, \ldots, \lambda_1}_{d_1}, \underbrace{\lambda_2, \lambda_2, \ldots, \lambda_2}_{d_2}, \ldots$$

Definition 9.6. If $\mathfrak{A}$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$, we say $x \in \mathcal{H}$ is a *separating vector* if whenever $A \in \mathfrak{A}$ has $Ax = 0$ then $A = 0$.

Remark 9.7. If x is a cyclic vector for $\mathfrak{A}$ then it is a separating vector for $\mathfrak{A}'$.

Proof. Suppose $B \in \mathfrak{A}'$ and $Bx = 0$. Then for all $A \in \mathfrak{A}$ we have $B(Ax) = A(Bx) = 0$. So $B \restriction \underbrace{\overline{\mathfrak{A}x}}_{=\mathcal{H}} = 0$, and thus $B = 0$. $\square$

Definition 9.8. We say $\pi \colon \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is *multiplicity-free* if $\pi(\mathfrak{A})'$ is abelian.

The idea is that if $\pi \cong \pi_0 \oplus \rho \oplus \rho$ then it has multiplicity; then the operators

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{11}I & a_{12}I \\ 0 & a_{21}I & a_{22}I \end{pmatrix}$$

lie in $\pi(\mathfrak{A})'$.

Definition 9.9. A *masa* (maximal abelian self-adjoint subalgebra) is an abelian C*-subalgebra of $\mathcal{B}(\mathcal{H})$ not contained in any larger abelian C*-algebra.

Remark 9.10. If $\mathfrak{A}$ is a masa then $\mathfrak{A} \subseteq \mathfrak{A}'$ and $\mathfrak{A} \subseteq \mathfrak{A}'' = \overline{\mathfrak{A}}^{\mathrm{WOT}}$, and $\overline{\mathfrak{A}}^{\mathrm{WOT}}$ is still abelian. So $\mathfrak{A}$ is WOT-closed (and thus a von Neumann algebra).

If $\mathfrak{A}'' \subsetneq \mathfrak{A}'$, pick $B \in \mathfrak{A}' \setminus \mathfrak{A}''$; then B commutes with $\mathfrak{A}''$, so $C^*(B, \mathfrak{A}'')$ is abelian. So $C*(B, \mathfrak{A}'')''$ is abelian, and contains $\mathfrak{A}''$, a contradiction. So $\mathfrak{A}' = \mathfrak{A}'' = \mathfrak{A}$.

Lemma 9.11. *Suppose $\mathfrak{A}$ is an abelian subalgebra of $\mathcal{B}(\mathcal{H})$ and $\mathcal{H}$ is separable. Then $\mathfrak{A}'$ has a cyclic vector, so $\mathfrak{A}''$ has a separating vector.*

Proof. Decompose $\mathcal{H} = \bigoplus \mathcal{H}_i$ where $\mathfrak{A}' \restriction \mathcal{H}_i$ has a cyclic vector x_i. Let $x = \sum_{i=1}^{\infty} 2^{-i} x_i$. Then $\mathcal{H}_i$ reduces $\mathfrak{A}'$, so $P_{\mathcal{H}_i} \in \mathfrak{A}''$. But $\mathfrak{A}'' = \overline{\mathfrak{A}}^{\mathrm{WOT}} \subseteq \mathfrak{A}'$ (where this last is because $\mathfrak{A}$ is abelian). So $P_{\mathcal{H}_i} \in \mathfrak{A}'$.

Then x is cyclic. Indeed, $x_i = 2^i P_{\mathcal{H}_i} x \in \mathfrak{A}'$, so $\mathcal{H}_i = \overline{\mathfrak{A}' x_i} \subseteq \overline{\mathfrak{A}' x}$; so $\mathcal{H} = \overline{\mathfrak{A}' x}$. Thus x is separating for $\mathfrak{A}''$. ☐ Lemma 9.11

Theorem 9.12. *Suppose $\rho \colon \mathcal{C}(X) \to \mathcal{B}(\mathcal{H})$ is a *-representation where $\mathcal{H}$ is separable. Then the following are equivalent:*

1. *$\rho(\mathcal{C}(X))$ has a cyclic vector.*
2. *ρ is multiplicity free.*
3. *$\rho(\mathcal{C}(X))''$ is a masa.*
4. *$\rho(\mathcal{C}(X))''$ is unitarily equivalent to $L^\infty(\mu)$ acting on $L^2(\mu)$ by multiplication for some probability measure μ on X.*

Proof.

(1) $\Longrightarrow$ (2 and 4) Suppose $\rho(\mathcal{C}(X))$ has a cyclic vector; then there is a regular Borel probability measure μ on X such that $\rho \cong \pi_\mu$. Suppose $T \in \pi_\mu(\mathcal{C}(X))'$; let $h = T1 \in L^2(\mu)$. For $g \in \mathcal{C}(X)$ we have $Tg = TM_g 1 = M_g T1 = gh$. So $\|gh\|_2 = \|Tg\|_2 \leq \|T\| \|g\|_2$. Then $\|h\|_{L^2(\mu)} \leq \|T\|$; indeed, otherwise there is $r \geq \|T\|$ such that $A = \{ x : |h(x)| \geq r \}$ has $\mu(A) > 0$. But $\mathcal{C}(X)$ is dense in $L^2(\mu)$; so there is $g_n \in \mathcal{C}(X)$ such that $\|g_n\|_2 \leq \sqrt{\mu(A)}$ and $g_n \to \chi_A$ in $L^2(\mu)$. So $\|T\| \|\chi_A\| < r \|\chi_A\| < \|\chi_A h\| = \lim \|g_n h\| \leq \|T\| \sup \|g_n\|_2 = \|T\| \|\chi_A\|$, a contradiction. So by continuity $T = M_h$ and $h \in L^\infty(\mu)$. So $\mathfrak{A}' = \{ M_h : h \in L^\infty(\mu) \} \supseteq \mathfrak{A}$, and we have shown (2).

But also $\mathfrak{A}'' = \overline{\mathfrak{A}}^{\mathrm{WOT}} \subseteq \mathfrak{A}'$, and $\mathfrak{A}'$ is abelian, so $\mathfrak{A}' \subseteq \mathfrak{A}''$. Thus $\mathfrak{A}'' = \mathfrak{A}' = \{ M_h : h \in L^\infty(\mu) \}$, and we have shown (4).

(2) $\Longrightarrow$ (3) $\mathfrak{A}'$ abelian, so the same argument shows that $\mathfrak{A}' = \mathfrak{A}''$; so $\mathfrak{A}''$ is a masa.

(4) $\Longrightarrow$ (1) 1 is a cyclic vector for $\pi_\mu(\mathcal{C}(X))$.

(3) $\Longrightarrow$ (1) $\rho(\mathcal{C}(X))$ is abelian, so lemma says $\rho(\mathcal{C}(X))' = \mathfrak{A}'$ has a cyclic vector. But $\mathfrak{A}'' = \mathfrak{A}'$; so $\mathfrak{A}''$ has a cyclic vector x. Then

$$\overline{\rho(\mathcal{C}(X))x} = \overline{\overline{\rho(\mathcal{C}(X))}^{\mathrm{WOT}} x} = \overline{\mathfrak{A}'' x} = \mathcal{H}$$

so x is a cyclic vector for $\rho(\mathcal{C}(X))$. ☐ Theorem 9.12

Lemma 9.13. *We have $L^\infty(\mu)$ acting on $L^2(\mu)$ by multiplication (where μ is a regular Borel probability measure). The weak* topology on $L^\infty(\mu) = L^1(\mu)^*$ coincides with the WOT and the ultraweak topology on $\mathcal{M}(L^\infty(\mu)) = \{ M_h : h \in L^\infty(\mu) \}$.*

Proof. All these topologies are the weakest topologies making certain linear functionals continuous. The weak* topology on $L^\infty(\mu)$ corresponds to the maps

$$h \mapsto \int h f \,\mathrm{d}\mu$$

for $f \in L^1(\mu)$; the WOT on $\mathcal{M}(L^\infty(\mu))$ corresponds to the maps

$$h \mapsto \langle M_h x, y \rangle$$

for $x, y \in \mathcal{H}$;

TODO 55. *$L^2(\mu)$?*

the ultraweak topology on $\mathcal{M}(L^\infty(\mu))$ corresponds to the maps

$$h \mapsto \sum_i \langle M_h x_i, y_i \rangle$$

where

$$\sum_i \|x_i\| \|y_i\| < \infty$$

If $f \in L^1(\mu)$ and $x = |f|^{\frac{1}{2}} \operatorname{sgn}(f), y = |f|^{\frac{1}{2}} \in L^2(\mu)$ then

$$\langle M_h x, y \rangle = \int h x \overline{y} \mathrm{d}\mu = \int h f \mathrm{d}\mu$$

So WOT-continuous implies ultraweak continuous.

Consider

$$h \mapsto \langle M_h x, y \rangle = \int h x \overline{y} \mathrm{d}\mu$$

Consider $f = x\overline{y} \in L^1(\mu)$. Then $\|x\overline{y}\|_1 \leq \|x\|_2 \|y\|_2$. Consider the ultraweak continuous functional

$$h \mapsto \sum_{i=1}^{\infty} \langle M_h x_i, y_i \rangle = \sum_{i=1}^{\infty} \int h f_i \mathrm{d}\mu = \int h \sum_i f_i \mathrm{d}\mu$$

where $f_i = x_i \overline{y_i}$, so $\|f_i\|_1 \leq \|x_i\|_2 \|y_i\|_2$ and $\sum_i f_i \in L^1$.

TODO 56. *some words*

$\square$ Lemma 9.13

Lemma 9.14. *Suppose μ, ν are regular Borel probability measures on X a compact metric space. Then there is a *-isomorphism $\sigma \colon L^\infty(\mu) \to L^\infty(\nu)$ such that $\sigma \restriction \mathcal{C}(X)$ is the "identity" if and only if μ and ν are mutually absolutely continuous. Moreover the *-isomorphism is weak*-continuous.*

Proof.

($\Longleftarrow$) By the Radon-Nikodym theorem $\nu = k\mu$ for some $k \in L^1($ with $k > 0$ almost everywhere. So define $U \colon L^2(\mu) \to L^2(\nu)$ by $Uf = k^{-\frac{1}{2}} f$. Then

$$\|Uf\|_{L^2(\nu)}^2 = \int |k^{-\frac{1}{2}} f|^2 \mathrm{d}\nu = \int k^{-1} |f|^2 k \mathrm{d}\mu = \|f\|_{L^2(\mu)}^2$$

So U is isometric and surjective. If $h, f \in L^\infty(\mu) = L^\infty(\nu)$ then

$$U M_h^\mu f = U h f = k^{-\frac{1}{2}} h f = M_h^\nu (k^{-\frac{1}{2}} f) = M_h^\nu U f$$

So $M_h^\nu = U M_h^\mu U^*$. This is a *-isomorphism between $L^\infty(\mu)$ and $L^\infty(\nu)$ which is WOT-continuous, and thus weak*-continuous.

($\Longrightarrow$) Suppose $\sigma \colon L^\infty(\mu) \to L^\infty(\nu)$ is a *-isomorphism such that if $f \in \mathcal{C}(X)$ then $\sigma(f) = f$. We view σ as a map $\mathcal{M}(L^\infty(\mu)) \to \mathcal{M}(L^\infty(\nu))$.

Claim 9.15. *σ is* normal*: if $(f_\alpha)_\alpha$ is a bounded increasing net in $L^\infty(\mu)$ with $\sup_\alpha f_\alpha = f \in L^\infty$ then $\sigma(f) = \sup_\alpha \sigma(f_\alpha)$.*

Proof. Note that $f \geq 0$ implies $\sigma(f) \geq 0$ because it is a *-homomorphism. So $(\sigma(f_\alpha))_\alpha$ is an increasing net, and is bounded. Let $g = \sup_\alpha \sigma(f_\alpha)$; let $h = \sigma^{-1}(g)$. We know that

$$\sigma(f_\alpha) \leq g = \sup_\alpha \sigma(f_\alpha) \leq \sigma(f)$$

where the last inequality is because $f \geq f_\alpha$ implies $\sigma(f) \geq \sigma(f_\alpha)$. So $f_\alpha \leq h \leq f$. So $f = \sup_\alpha f_\alpha \leq h \leq f$, and $h = f$. So $g = \sigma(f) = \sup_\alpha \sigma(f_\alpha)$. $\square$ Claim 9.15

Suppose $\mathcal{O} \subseteq X$ is open. For $n \geq 1$ let

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \mathcal{O} \text{ and } \operatorname{dist}(x\mathcal{O}^c) \geq \frac{1}{n} \\ n \operatorname{dist}(x, \mathcal{O}^c) \text{if } \operatorname{dist}(x, \mathcal{O}^c) \leq \frac{1}{n} \end{cases}$$

Then $f_n \leq f_{n+1}$ with $\sup_n f_n = \chi_{\mathcal{O}}$. So

$$\sigma(\chi_{\mathcal{O}}) = \sup \sigma(f_n) = \sup f_n = \chi_{\mathcal{O}}$$

Now let $\Sigma = \{ E \subseteq X : E \text{ measurable}, \sigma(\chi_E) = \chi_E \}$.

Claim 9.16. *Σ is a σ-algebra.*

Proof. For closure under complements, we have

$$\sigma(\chi_{E^c}) = \sigma(1 - \chi_E) = 1 - \chi_E = \chi_{E^c}$$

for $E, F \in \Sigma$. For closure under intersection, we have

$$\sigma(\chi_{E\cap F}) = \sigma(\chi_E \chi_F) = \sigma(\chi_E)\sigma(\chi_F) = \chi_E \chi_F = \chi_{E\cap F}$$

for $E, F \in\in \Sigma$. If $(E_i : i \geq 1)$ are pairwise disjoint and

$$E = \bigcup_{i\in\mathbb{N}} E_i$$

then

$$\sigma(\chi_E) = \sigma(\sup_{n\geq 1} \chi_{E_1} \cup \cdots \cup \chi_{E_n}) = \sup_{n\geq 1} \sigma(\chi_{E_1\cup\cdots\cup E_n}) = \sup_{n\geq 1} \sigma(\chi_{E_1} + \cdots + \chi_{E_n}) = \sup_{n\geq 1} \chi_{E_1\cup\cdots\cup E_n} = \chi_E$$

So Σ is a σ-algebra. ☐ Claim 9.16

But Σ contains all open sets and all sets of measure 0 (since $\sigma(0) = 0$). So Σ is all measurable sets. So σ is the *identity* on all simple functions, which are norm-dense. So σ is the "identity". ☐ Lemma 9.14

Theorem 9.17. *Suppose $\sigma\colon \mathcal{C}(X) \to \mathcal{B}(\mathcal{H})$ is a non-degenerate representation with $\mathcal{H}$ separable. Let $\mathcal{M} = \sigma(\mathcal{C}(X))''$. Then there is a regular Borel probability measure μ on X such that $L^\infty(\mu) \cong \mathcal{M}$ via a *-isomorphism $\widetilde{\sigma}$ which extends σ and is a weak*-WOT homeomorphism.*

Proof. $\mathcal{M}$ is an abelian von Neumann algebra; so since $\mathcal{M}'$ has a cyclic vector we get that $\mathcal{M}$ has a separating vector x. Let $\mathcal{K} = \overline{\mathcal{M}x}$. The restriction map $\rho\colon \mathcal{M} \to \mathcal{B}(\mathcal{K})$ (with $\rho(T) = T \restriction \mathcal{K}$) is a WOT-continuous *-isomorphism. Since x is a separating vector we get that ρ is injective, and thus isometric.

Claim 9.18. *$\rho(\mathcal{M})$ is WOT-closed.*

Proof. Suppose $A \in b_1(\overline{\rho(\mathcal{M})}^{\text{WOT}})$. Then by Kaplansky's density theorem there are $A_\alpha = \rho(T_\alpha)$ such that $T_\alpha \in \mathcal{M}$ with $\|T_\alpha\| \leq 1$ and $\rho(T_\alpha) \xrightarrow{\text{WOT}} A$. Drop to a subnet so $T_{\alpha_\beta} \xrightarrow{\text{WOT}} T$ (possibly since $(b_1(\mathcal{B}(\mathcal{H}), \text{WOT})$ is compact by Banach-Alaoglu). Then ρ is WOT-WOT-continuous; so $\rho(T) = A \in \rho(\mathcal{M})$. ☐ Claim 9.18

$\rho(\mathcal{M})$ has x as a cyclic vector; so there is μ_1 a regular probability measure such that $\rho(\mathcal{M}) \cong L^\infty(\mu_1)$ acting on $L^2(\mu_1)$. Then $\sigma'\colon \mathcal{M} \to \mathcal{M} \restriction \mathcal{K}^\perp$ can be written as a direct sum of cyclic representations. So

$$\mathcal{H} \cong \mathcal{K} \oplus \bigoplus_{n\geq 2} \mathcal{K}_n$$

such that each $\mathcal{M} \restriction \mathcal{K}_n$ is cyclic. So there are probability measures μ_n such that $\mathcal{M} \restriction \mathcal{K} \cong L^\infty(\mu_n)$ on $L^2(\mu_n)$. Let

$$\mu = \sum_{n\geq 1} 2^{-n} \mu_n$$

Then $\sigma\colon \mathcal{C}(X) \to \mathcal{B}(\bigoplus_n \mathcal{K}_n)$ by $\sigma(f) = \bigoplus_n \sigma_n \mapsto \bigoplus M_f^{\mu_n}$. So $\sigma_n(f) = M_f^{\mu_n}$ for all $f \in L^\infty(\mu_n)$. We get a map $\widetilde{\sigma}\colon L^\infty(\mu) \xrightarrow{*\text{-isomorphism}} \mathcal{B}(\mathcal{H})$ given by

$$\widetilde{\sigma}(f) = \bigoplus_{n\geq 1} M_f^{\mu_n}$$

Thus $\mu_n << \mu_1$. So $\mu \cong \mu_1$.

TODO 57. *Following claim somewhere above?*

Claim 9.19. *$\mu_n << \mu_1$.*

Proof. Otherwise there is E measurable such that $\mu_n(E) > 0$ but $\mu(E) = 0$. Then $\chi_E \neq 0$ in $\mathcal{M} \restriction \mathcal{K}_n$; so $\chi_E \in L^\infty(\mu)$ with $\widetilde{\sigma}(\chi_E) \neq 0$ since $\sigma_n(\chi_E) \neq 0$ But σ_1 is injective, so $\sigma_1(\chi_E) \neq 0$, a contradiction. ☐ Claim 9.19

So $\mathcal{M} = \widetilde{\sigma}(L^\infty(\mu)) \cong L^\infty(\mu)$ (which is also isomorphic to $L^\infty(\mu_1)$). ☐ Theorem 9.17

Theorem 9.20 (L^∞ functional calculus)**.** *Suppose N is a normal operator on a separable Hilbert space. Then there is a Borel probability measure μ on $\sigma(N)$ such that the continuous functional calculus $\sigma\colon \mathcal{C}(\sigma(N)) \to \mathcal{B}(\mathcal{H})$ extends to a weak*-WOT continuous *-homomorphism $\widetilde{\sigma}\colon L^\infty(\mu) \to \mathcal{B}(\mathcal{H})$. (One thinks of this as mapping $f \mapsto M_h$.)*

Proof. $\sigma(\mathcal{C}(\sigma(N)))'' = \mathcal{M} \cong L^\infty(\mu)$ for some probability measure μ on $\sigma(N)$, and the map $\widetilde{\sigma}\colon L^\infty(\mu) \to \mathcal{M}$ extends σ and is weak*-WOT-continuous. ☐ Theorem 9.20

9.1 Spectral measures

Suppose N is normal on a separable Hilbert space $\mathcal{H}$. Then $\widetilde{\sigma}\colon L^\infty(\mu) \xrightarrow{*\text{-isomorphism}} \{ N \}''$. Let Σ be the set of measurable subsets of $\sigma(N)$ (or $\mathbb{C}$); let $E_N\colon \Sigma \to \mathcal{B}(\mathcal{H})$ be $E_N(A) = \chi_A(N) = \widetilde{\sigma}(\chi_A)$. This is a *projection valued measure.*

(Countable additivity) Suppose the A_i are pairwise disjoint and measurable. Then

$$\widetilde{\sigma}(\chi_{\bigcup A_i}) = \widetilde{\sigma}(\sup \chi_{A_1\cup\cdots\cup A_n}) = \sup \widetilde{\sigma}(\chi_{A_1\cup\cdots\cup A_n}) = \sup \sum \widetilde{\sigma}(\chi_{A_i}) = \sum \widetilde{\sigma}(\chi_{A_i})$$

So

$$E_N\left(\bigsqcup_i A_i\right) = \mathrm{SOT}\sum_{i=1}^\infty E_N(A_i)$$

If $f = \sum a_i \chi_{E_i}$ with the E_i pairwise disjoint then

$$\int f \mathrm{d}E_N := \sum a_i E_N(E_i) = \widetilde{\sigma}(f)$$

extend to $f \in L^\infty$ by

$$\int f \mathrm{d}E_N := \widetilde{\sigma}(f)$$

Lemma 9.21. *If $\mathcal{M}$ is an abelian von Neumann algebra on a separable $\mathcal{H}$ then there is $A = A^* \in \mathcal{M}$ such that $\mathcal{M} = C^*(A)''$.*

Proof. $\mathcal{M} \cong L^\infty(\mu)$. Find a collection $\{ E_n \}_{n\geq 1}$ of orthogonal projections in $\mathcal{M}$ such that $\mathcal{M} = \overline{\mathrm{span}\{ E_n \}}^{\mathrm{WOT}}$. Pull out (countably many) atoms. Technical part: take $\{ \mathcal{O}_n \}$ open that determine the topology of X, and make sure that we can approxiate $\chi_{\mathcal{O}_n}$.

Let

$$A = \sum_{n=1}^\infty 3^{-n} \chi_{E_n}$$

Then

$$\frac{1}{3}\chi_{E_1} \leq A \leq \frac{1}{2}$$

Then

$$A\chi_{E_1}^c = \sum_{n=2}^{\infty} 3^{-n}\chi_{E_n \cap E_1^c} \leq \frac{1}{6}\chi_{E_1^c}$$

So

$$A = \underbrace{A\chi_{E_1}}_{\geq \frac{1}{3}\chi_{E_1}} + \underbrace{A\chi_{E_1^c}}_{\leq \frac{1}{6}\chi_{E_1^c}}$$

But

$$\sigma(A\chi_{E_1} \upharpoonright E_1\mathcal{H}) \subseteq \left[\frac{1}{3}, \frac{1}{2}\right]$$
$$\sigma(A\chi_{E_1^c} \upharpoonright E_1^c\mathcal{H}) \subseteq \left[0, \frac{1}{6}\right]$$

So if we let

$$f(x) = \begin{cases} 1 & \text{if } x \in \left[\frac{1}{3}, \frac{1}{2}\right] \\ 0 & \text{if } x \in \left[0, \frac{1}{6}\right] \end{cases}$$

then $f \in \mathcal{C}(\sigma(A))$, and

$$f(A) = f(A\chi_{E_1} \oplus A\chi_{E_1^c}) = \widetilde{\sigma}(\chi_{E_1}) \oplus 0 = E_1$$

Thus $E_1 \in \sigma^*(A)$. So

$$3\left(A - \frac{1}{3}E_1\right) = \sum_{n=2}^{\infty} 3^{1-n}\chi_{E_n}$$

etc. $E_n \in C^*(A)$ for $n \geq 1$. So

$$C^*(A) = C^*(E_n)$$

and

$$C^*(A)'' = C^*(E_n)'' = \mathcal{M}$$

□ Lemma 9.21

Corollary 9.22. *$\mathcal{M}$ an abelian von Neumann algebra in a separable hilbert space $\mathcal{H}$ there is probability measure μ on $[0,1]$ such that $\mathcal{M} \cong L^\infty(\mu)$.*

Proof. $\mathcal{M} = C^*(A)'' \cong L^\infty(\mu)$ with μ a probability measure on $\sigma(A) \subseteq \mathbb{R}$. □ Corollary 9.22

9.2 Multiplicity

For us $\mathcal{H}$ is separable.

Definition 9.23. We say a representation π has *multiplicity* n for $1 \leq n \leq \aleph_0$ if $\pi \cong \underbrace{\sigma \oplus \cdots \oplus \sigma}_{n} = \sigma^{(n)}$ where σ is multiplicity-free (i.e. $\sigma(\mathfrak{A})'$ is abelian).

Recall that if $\mathfrak{A} = \mathcal{C}(X)$ then σ is multiplicity free if and only if $\sigma \cong \sigma_\mu$ on $L^2(\mu)$ by multiplication; so $\sigma(\mathcal{C}(X))' = \sigma(\mathcal{C}(X))'' \cong L^\infty(\mu)$ acting on $L^2(\mu)$.

Theorem 9.24. *If $\sigma \cong \sigma_\mu$ is a multiplicity-free rerpesentation of $\mathcal{C}(X)$ and $\pi = \sigma^{(n)}$, then $\pi(\mathcal{C}(X))' \cong M_n(L^\infty(\mu))$. Hence the multiplicity of π is well-defined.*

Proof. We have $\pi \cong \sigma_\mu^{(n)}$ acting on $L^2(\mu)^{(n)} = \underbrace{L^2(\mu) \oplus \cdots \oplus L^2(\mu)}_{n}$ via $\pi(h) = \operatorname{diag}(M_h, M_h, \ldots, M_h)$. If $A \in \pi(\mathcal{C}(X))'$, we write A as an $n \times n$ matrix $A = [A_{ij}]_{ij}$ with respect to this decomposition. Then

$$0 = \pi(h)A - A\pi(h) = [M_h A_{ij} - A_{ij} M_h]_{ij}$$

if and only if each $A_{ij} \in \sigma_\mu(\mathcal{C}(X))' = L^\infty(\mu)$. So $\pi(\mathcal{C}(X))' = M_n(L^\infty(\mu))$.

What if $n = \aleph_0$? Then A has a matrix $[A_{ij}]_{i,j\geq 1}$. Then the same argument shows $A_{ij} \in L^\infty(\mu)$ and

$$\pi(\mathcal{C}(X))' = \{\, B = [M_{h_{ij}}]_{ij} : h_{ij} \in L^\infty(\mu), \|A\| < \infty \,\} = \mathcal{B}(\mathcal{H}) \overline{\otimes} L^\infty(\mu)$$

where we take the WOT-closure of the tensor product.

Suppose π also has multiplicity $m < n$; so $\pi \cong \sigma_\nu^{(m)}$. Then

$$M_n(L^\infty(\mu)) \cong \pi(\mathcal{C}(X))' \cong M_m(L^\infty(\nu))$$

Suppose φ is a multiplicative linear functional on $L^\infty(\nu)$; it induces a map $\varphi^{(m)} \colon M_m(L^\infty(\nu)) \to M_m$ given by $\varphi^{(m)}([M_{h_{ij}}]_{ij}) = [\varphi(h_{ij})]_{ij}$. Then $\varphi^{(m)}$ is a homomorphism: it is linear and multiplicative. Indeed, we have

$$[M_{h_{ij}}][M_{g_{ij}}] = [M_{\sum_{k=1}^m h_{ij} g_{kj}}]$$

and

$$\varphi^{(m)}([M_{h_{ij}}])\varphi^{(m)}([M_{g_{ij}}]) = [\varphi(h_{ij})][\varphi(g_{ij})] = \Big[\sum \varphi(h_{ik})\varphi(g_{kj})\Big] = \Big[\varphi\Big(\sum h_{ij} g_{kj}\Big)\Big] = \varphi^{(m)}([M_{h_{ij}}][M_{g_{ij}}])$$

So we get unital *-homomorphisms

$$M_n(\mathbb{C}1 \hookrightarrow M_n(L^\infty(\mu)) \cong M_m(L^\infty(\nu)) \underbrace{\xrightarrow{\varphi^{(m)}}}_{\text{surjective}} M_m(\mathbb{C})$$

So we get a unital *-homomorphism $M_n \to M_m$ with $m < n$.

If $n < \infty$ then M_n is simple; so $n^2 = \dim(M_n) \leq \dim(M_m) = m^2$, a contradiction. If $n = \aleph_0$ then $M_{\aleph_0} = \mathcal{B}(\mathcal{H})$ only has one proper ideal: the compact operators $\mathcal{K}$. Also $\dim(\mathcal{B}(\mathcal{H})) = \dim(\mathcal{B}(\mathcal{H})/\mathcal{K}) = 2^{\aleph_0}$. So there are no finite dimensional quotients, a contradiction. So multiplicity is well-defined. ☐ Theorem 9.24

Definition 9.25. Suppose $\pi \colon \mathcal{C}(X) \to \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is separable. A projection $P \in \pi(\mathcal{C}(X))''$ has *multiplicity* n if $\pi(\mathcal{C}(X)) \restriction PH$ has multiplicity n.

Proposition 9.26. *There is a largest projection P_n of multiplicity n.*

Proof. Let $(P_\alpha)_\alpha$ be the collection of all multiplicity n projections in $\pi(\mathcal{C}(X))'' \cong L^\infty(\mu)$. So there are measurable sets A_α such that $P_\alpha \cong M_{\chi_{A_\alpha}}$. Let t be the supremum over all finite subsets of $\mu(A_{\alpha_1} \cup \cdots \cup A_{\alpha_m})$. Choose $F_i = A_{\alpha_{i,1}} \cup \cdots \cup A_{\alpha_{i,m}}$ such that $\mu(F_i) \to t$. Let $F = \bigcup_{i=1}^\infty F_i$. Then

$$t = \sup \mu(F_i) \leq \mu(F) = \lim_{m\to\infty} \mu(F_1 \cup \cdots \cup F_m) \leq t$$

So $\mu(F) = t$.

Claim 9.27. $\mu(A_\alpha \setminus F) = 0$ *for all* α.

Proof. Say $\mu(A_\alpha \setminus F) = \delta > 0$. Pick i_0 such that $\mu(F_{i_0}) > t - \frac{\delta}{2}$. Then

$$t \geq \mu(F_{i_0} \cup A_\alpha) \geq \mu(F_{i_0}) + \mu(A_\alpha \setminus F) > t - \frac{\delta}{2} + \delta > t$$

a contradiction. ☐ Claim 9.27

So there is a countable set $(P_i)_i$ of multiplicity n such that

$$\bigvee_{i\geq 1} P_i = M_{\chi_F} = \bigvee P_\alpha$$

Let $Q_1 = P_1$ and

$$Q_{n+1} = P_{n+1}\left(\bigvee_{i=1}^{n} P_i\right)^{\perp}$$

Then $Q_iQ_j = 0$ if $i \neq j$ and

$$\sum_{i=1}^{n} Q_i = \bigvee_{i=1}^{n} P_i$$

So

$$Q = \sum_{i=1}^{\infty} Q_i = \bigvee_{i\geq 1} P_i = \bigvee P_\alpha$$

and $Q_i \cong M_{\chi_{B_i}}$ with the B_i pairwise disjoint, measurable. Each P_i has multiplicity n and $Q_i \leq P_i$, so each Q_i has multiplicity n. Then

$$P_i\mathcal{C}(X)' \cong M_n(L^\infty(A_i))$$
$$Q_i\mathcal{C}(X)' \cong M_n(L^\infty(B_i))$$

with each $B_i \subseteq A_i$. Then since the Q_i are pairwise orthogonal we get

$$Q\pi(\mathcal{C}(X)') = \sum Q_i\pi(\mathcal{C}(X))' \cong \sum M_n(L^\infty(B_i)) \cong M_n\Big(L^\infty\Big(\bigcup B_i\Big)\Big) = M_n(L^\infty(F))$$

So Q has multiplicity n and is the biggest. □ Proposition 9.26

Lemma 9.28. *If $\pi\colon \mathcal{C}(X) \to \mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ separable then there is $0 \neq P$ a projection in $\pi(\mathcal{C}(X))''$ of uniform multiplicity. (i.e. P has a multiplicity.)*

Proof. $\pi(\mathcal{C}(X))''$ is an abelian von Neumann algebra; so there is a separating vector x_1; let $M_1 = \overline{\pi(\mathcal{C}(X))x_1}$. Then M_1 is reducing so $\pi(\mathcal{C}(X))'' \restriction M_1$ is maximal abelian, isomorphic to $L^\infty(\mu_1)$; call $\mu_1 = \mu$. Then $\pi(\mathcal{C}(X))'' \restriction M_1^\perp$ is an abelian von Neumann algebra, so there is a separating vector x_2; let $M_2 = \overline{\pi(\mathcal{C}(X))x_2}$. Then M_2 is reducing so $\pi(\mathcal{C}(X))'' \restriction M_2$ is maximal abelian, isomorphic to $L^\infty(\mu_2)$ with $\mu_2 \ll \mu_1 = \mu$. Recursively find separating x_{n+1} of $\pi(\mathcal{C}(X))'' \restriction (M_1 + \cdots + M_n)^\perp$ and let $M_{n+1} = \overline{\pi(\mathcal{C}(X))}$; then $\pi(\mathcal{C}(X))'' \restriction M_{n+1} \cong L^\infty(\mu_{n+1})$ with $\mu_{n+1} \ll \mu_n$.

There is A_n measurable such that $\mu_n \approx \mu \restriction A_n$; then $X = A_1 \supseteq A_2 \supseteq \cdots$. Suppose there is a smallest $n+1$ such that $\mu_{n+1} \not\approx \mu_{n+1}$. Then $\mu \approx \mu_1 \approx \cdots \approx \mu_n \not\cong \mu_{n+1}$. Then $A_1, \ldots, A_n$ have full measure but $\mu(A_{n+1}) < 1$. Let $P \in \pi(\mathcal{C}(X))'' \cong L^\infty(\mu)$ correspond to $\chi_{A_{n+1}^c} \in L^\infty(\mu)$. Let $B = A_{n+1}^c$. Then $P \restriction M_i \cong M_{\chi_B}$ for $1 \leq i \leq n$ and $P \restriction M_i = 0$ if $i \geq n+1$. So

$$P \cong M_{\chi_B}^{(n)} \oplus 0$$

Also $P((\sum M_i)^\perp) = 0$. Then

$$P\pi(\mathcal{C}(X))'' \cong P\left(\left(\bigoplus_{i\geq 1} \pi(\mathcal{C}(X))'' \restriction M_i\right) \oplus \left(\pi(\mathcal{C}(X))'' \restriction (\sum M_i)^\perp\right)\right)$$

Then

$$P \restriction \left(\sum_{i=1}^{n} M_i\right)^\perp = 0$$

since x_{n+1} is separating; but $M_{\chi_B} \restriction M_{n+1} = 0$, so $M_{\chi_B} \restriction (\sum_{i=1}^{n} M_i)^\perp = 0$.

$$P_{\pi(\mathcal{C}(X))''} = \underbrace{\bigoplus_{i=1}^{n} L^\infty(B) \restriction PM_i}_{\text{multiplicity } n} \oplus 0$$

So P has multiplicity n. This is fine if $n < \infty$; suppose then that $n = \aleph_0$. Then $\mu_n \approx \mu$ for all $n \geq 1$ and $\{ x_n : n \geq 1 \}$ are separating vectors for $\pi(\mathcal{C}(X))'' \cong L^\infty(\mu)$. By Zorn's lemma we can extend this to a maximal family of separating vectors $\{ y_j \}$ such that $N_j = \overline{\pi(\mathcal{C}(X))y_j}$ are pairwise orthogonal. Then $\pi(\mathcal{C}(X)) \restriction N_j \cong L^\infty(\mu)$ a masa on J_j. Let $R = (\sum N_J)^\perp$; we know $\pi(\mathcal{C}(X))'' \restriction R$ does not have a separating vector for $L^\infty(\mu)$. So $\pi(\mathcal{C}(X))'' \restriction R \cong L^\infty(\nu)$ with $\nu \ll \mu$ but $\nu \not\approx \mu$. So $\nu \approx \chi_D\mu$ with $\mu(D) < 1$. Let $P \in \pi(\mathcal{C}(X))''$ correspond to $\chi_{D^c} \in L^\infty(\mu)$; so $P \restriction R = 0$. Thus $P\mathcal{H} = \bigoplus PN_j$ and $\pi(\mathcal{C}(X))'' \restriction PN_j \cong L^\infty(D^c)$; so P has multiplicity $\aleph_0$. □ Lemma 9.28

Theorem 9.29. *Suppose $\pi\colon \mathcal{C}(X) \to \mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ separable. Then there are pairwise orthogonal projections P_n with $1 \leq n \leq \aleph_0$ the maximal projections of multiplicity n. The SOT sum*

$$\sum_{n=1}^{\infty} P_n + P_{\aleph_0} = I$$

So $\pi \cong \bigoplus_{n=1}^\infty \sigma_{\mu_n}^{(n)} \oplus \sigma_{\mu_{\aleph_0}}^{(\aleph_0)}$ with $\mu_n \perp \mu_m$ if $n \neq m$ and

$$\mu = \sum \mu_n + \mu_{\aleph_0}$$

Proof. By lemma there is a largest projection P_n of multiplicity n. Then $\pi(\mathcal{C}(X)) \restriction P_n\mathcal{H} \cong \sigma_{\mu_n}^{(n)}(\mathcal{C}(X))$. Then $P_nP_m = 0$ if $n \neq m$ because on the intersection we have two multiplicities, a contradiction. If the SOT sum

$$\sum_{n=1}^{\infty} P_n + P_{\aleph_0} = Q < I$$

then look at $\pi(\mathcal{C}(X)) \restriction Q^\perp\mathcal{H}$. By last lemma we get $Q^\perp \geq P$ and P has multiplicity n; but this contradicts maximality of P_n. So $Q = I$. □ Theorem 9.29

Theorem 9.30 (Weyl-von Neumann-Berg)**.** *Suppose N is a normal operator on separable $\mathcal{H}$ and $\varepsilon > 0$. Then there is an orthonormal basis $\{ e_n \}$ and a diagonal operator $D = \operatorname{diag}(d_1, d_2, \ldots)$ with respect to $\{ e_n \}$ such that $K = N - D$ is compact and $\|N - D\| < \varepsilon$; so $N = D + K$ is the sum of a diagonal and a small compact.*

Suppose A and B are approximately unitarily equivalent (a.u.e.). If there is a sequence of unitary U_n such that $B = \lim_{n\to\infty} U_n^*AU_n$ in norm then $A \sim_{a.u.e.} B$ if and only if $\overline{\mathcal{U}(A)} = \overline{\mathcal{U}(B)}$ (where $\mathcal{U}(A) = \{ U^*AU : U \text{ unitary} \}$). In this case for all $\varepsilon > 0$ there is U such that $B - U^*AU$ is compact and has norm $< \varepsilon$.

Done in Ken's book, same chapter as normal operators. See also Voiculescu's theorem for a non-commutative version.