Course notes for PMATH 810

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Contents

1 Banach algebras

Definition 1. A *Banach algebra* is an associative algebra $\mathfrak{A}$ over $\mathbb{C}$ (or $\mathbb{R}$, but not for us) which has a norm that makes $(\mathfrak{A}, \|\cdot\|)$ a Banach space and satisfies

$$\|xy\| \leq \|x\|\|y\|$$

and if $\mathfrak{A}$ has a unit (which we will denote e or 1) then $\|e\| = 1$.

Remark 2. The above implies that multiplication is jointly continuous. Indeed, we have

$$x_1y_1 - x_2y_2 = x_1y_1 - x_2y_1 + x_2y_1 - x_2y_2 = (x_1 - x_2)y_1 + x_2(y_1 - y_2)$$

so

$$\|x_1y_1 - x_2y_2\| \leq \|x_1 - x_2\|\|y_1\| + \|x_2\|\|y_1 - y_2\|$$

Hence if $x_n \to x$ and $y_n \to y$ then $x_ny_n \to x_1y_1$.

Example 3.

1. If $\mathfrak{X}$ is a Banach space then $\mathcal{B}(\mathfrak{X})$ is a Banach algebra (with $\|T\| = \sup\{\, \|Tx\| : \|x\| \leq 1 \,\}$).

2. If X is a compact Hausdorff space then $C(X)$ is a Banach space where $\|f\|_\infty = \sup\{\, |f(x)| : x \in X \,\}$. If X is locally compact and Hausdorff then we define $C_0(X)$ to consist of the continuous functions f on X such that for all $\varepsilon > 0$ the set $\{\, x \in X : |f(x)| \geq \varepsilon \,\}$ is compact; we define $C_b(X)$ to consist of the bounded continuous functions. For both $C_0(X)$ and $C_b(X)$ the norm $\|f\|_\infty = \sup\{\, |f(x)| : x \in X \,\}$ confers a Banach algebra structure.

3. Consider the set $C^{(n)}[a, b]$ of functions on $[a, b]$ with n continuous derivatives. Our product rule is

$$(fg)^{(k)} = \sum \binom{k}{j} f^{(j)} g^{(k-j)}$$

The norm

$$\|f\|_{C^n} = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_\infty}{k!}$$

makes $C^{(n)}[a, b]$ into a Banach algebra.

Exercise 4. Check that $\|fg\|_{C^n} \leq \|f\|_{C^n}\|g\|_{C^n}$.

4. Suppose G is a locally compact abelian grape (e.g. $\mathbb{R}^n, \mathbb{T}^k, \mathbb{T}^k \times \mathbb{R}^n, \ldots$). We get a Haar measure m on G: a regular Borel measure that is translation-invariant (i.e. $m(A+s) = m(A)$ for Borel $A \subseteq G$ and $s \in G$). We define $L^1(G)$ to be the set of measurable f on G such that
$$\|f\|_1 = \int |f| \mathrm{d}m < \infty$$
The product on $L^1(G)$ is given by convolution:
$$(f * g)(t) = \int_G f(s)g(t-s)\mathrm{d}m(s)$$
One can check that
   - $g * f = f * g$
   - $(f * g) * h = f * (g * h)$ (this follows form Fubini).

   For the norm bound, note that
$$\begin{aligned} \|f * g\|_1 &= \int_G |(f*g)(t)|\mathrm{d}m(t) \\ &= \int_G \left| \int_G f(s)g(t-s)\mathrm{d}m(s) \right| \mathrm{d}m(t) \\ &\le \int_G \int_G |f(s)||g(\underbrace{t-s}_{u})|\mathrm{d}m(s)\mathrm{d}m(t) \\ &= \int_G \int_G |f(s)||g(u)|\mathrm{d}m(s)\mathrm{d}m(u) \\ &= \|f\|_1 \|g\|_1 \end{aligned}$$
(since the Jacobian of $(s,t) \mapsto (s,u)$ is
$$\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$
).

5. Consider $A(\mathbb{D})$ the *disk algebra* consisting of $f(z)$ continuous on $\overline{\mathbb{D}}$ and analytic on $\mathbb{D} = \{ z \in \mathbb{Z} : |z| < 1 \}$. Together with the norm
$$\|f\| = \sup_{|z| \le 1} |f(z)| = \sup_{|z|=1} |f(z)|$$
(where the second equality is by the maximum modulus principle) forms a Banach algebra. Then $A(\mathbb{D}) \subseteq C(\mathbb{D})$; in fact $A(\mathbb{D}) \subseteq C(\mathbb{T})$ where $\mathbb{T} = \{ z : |z| = 1 \} = \partial\overline{\mathbb{D}}$. Indeed the map $f \mapsto f \restriction \mathbb{T}$ is isometric.

6. For $T \in \mathcal{B}(\mathfrak{X})$ where $\mathfrak{X}$ is a Banach space, we define $\mathcal{A}(T) = \overline{\{ p(T) : p \in \mathbb{C}[z] \}}^{\|\cdot\|} \subseteq \mathcal{B}(\mathfrak{X})$. If $T \in \mathcal{B}(\mathcal{H})$ for $\mathcal{H}$ a Hilbert space we define $C^*(T) = \overline{\mathrm{alg}\{ I, T, T^* \}}^{\|\cdot\|}$. (Here alg is "the algebra generated by".)

7. If (X, μ) is a measure space we define $L^\infty(\mu)$ to be the set of measurable f such that f is essentially bounded (i.e. there is t such that $\mu(\{ x : |f(x)| > t \}) = 0$) modulo $f \sim g$ if $f - g = 0$ almost everywhere. The norm is given by
$$\|f\|_\infty = \inf\{ t : \mu(\{ x : |f(x)| > t \}) = 0 \} = \operatorname{ess.sup}|f|$$
We have an embedding $L^\infty(\mu) \hookrightarrow \mathcal{B}(L^2(\mu))$ given by $f \mapsto M_f$ where $M_f(h) = fh$.

Remark 5. If $\mathfrak{A}$ is a Banach algebra without unit we define $\mathfrak{A}^+ = \{ (a, \lambda) : a \in \mathfrak{A}, \lambda \in \mathbb{C} \}$; we write $(a, \lambda) = a + \lambda e$. We define

$$(a + \lambda e)(b + \mu e) = (ab + \lambda b + \mu a) + \lambda\mu e$$
$$\|a + \lambda e\| = \|a\| + |\lambda|$$

so

$$\|(a + \lambda e)(b + \mu e)\| \leq \|a\|\|b\| + |\lambda|\|b\| + |\mu|\|a\| + |\lambda\mu| = (\|a\| + |\lambda|)(\|b\| + |\mu|)$$

In fact $\mathfrak{A}$ is a (closed) maximal ideal in $\mathfrak{A}^+$.

Proposition 6. *Every Banach algebra $\mathfrak{A}$ is isometrically isomorphic to a subalgebra of $\mathcal{B}(\mathfrak{X})$ for some Banach space $\mathfrak{X}$.*

Proof. We map $\mathfrak{A}$ into $\mathcal{B}(\mathfrak{A}^+)$ by $a \mapsto L_a$ where $L_a x = ax$. Then

$$\|a\| = \|ae\| \leq \|L_a\| = \sup\{ \|ax\| : x \in \mathfrak{A}^+, \|x\| \leq 1 \} \leq \sup\{ \|a\|\|x\| : x \in \mathfrak{A}^+, \|x\| \leq 1 \} = \|a\|$$

so this is indeed an isometry. □ Proposition 6

Definition 7. Suppose $\mathfrak{A}$ is a unital Banach algebra and $a \in \mathfrak{A}$.

- The *spectrum* of a is $\sigma_{\mathfrak{A}}(a) = \{ \lambda \in \mathbb{C} : \lambda 1 - a \text{ is not invertible} \}$. (If the $\mathfrak{A}$ is clear from context we will sometimes omit it and write $\sigma(a)$.)
- The *resolvent* of a is $\rho(a) = \mathbb{C} \setminus \sigma(a)$.
- The *resolvent function* $R(a, \lambda) = (\lambda - a)^{-1}$ is defined on $\rho(a)$.

Definition 8. Suppose $T \in \mathcal{B}(\mathfrak{X})$ for some Banach space $\mathfrak{X}$.

- We define the *point spectrum* $\sigma_p(T)$ to be the set of eigenvalues of T: those λ for which there is $x \neq 0$ such that $Tx = \lambda x$.
- We define the *approximate point spectrum* $\sigma_\pi(T)$ to be the set of $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not bounded below. (An operator T is *bounded below* if there is $\varepsilon > 0$ such that $\|Tx\| \geq \varepsilon\|x\|$ for all $x \in \mathfrak{X}$.)
- We define the *compression spectrum* $\gamma(T)$ to be $\{ \lambda : \overline{(\lambda I - T)\mathfrak{X}} \neq \mathfrak{X} \}$; i.e. the λ for which $\lambda I - T$ does not have dense range.

Theorem 9. *For $T \in \mathcal{B}(\mathfrak{X})$ with $\mathfrak{X}$ a Banach space, the following are equivalent:*

1. *T is invertible.*
2. *T maps $\mathfrak{X}$ bijectively to itself.*
3. *T is bounded below and has dense range.*
4. *T and T^* are bounded below ($T^* \in \mathcal{B}(\mathfrak{X}^*)$).*
5. *T^* is invertible in $\mathcal{B}(\mathfrak{X}^*)$.*

Proof.

(1) $\implies$ (2) Immediate.

(2) $\implies$ (1) Banach isomorphism theorem.

(1) $\implies$ (3) Note that $x = T^{-1}(Tx)$; so $\|x\| \leq \|T^{-1}\|\|Tx\|$, and $\|Tx\| \geq (\|T^{-1}\|)^{-1}\|x\|$, and T is bounded below. (Surjectivity implies dense range.)

(3) $\implies$ (2) If $x \neq 0$ then $\|Tx\| \geq \varepsilon\|x\| > 0$; hence $Tx \neq 0$, and T is injective. For surjectivity, suppose $y \in \mathfrak{X}$; then since T has dense range there are x_n such that $y_n = Tx_n \to y$. Then in particular the y_n are Cauchy; since

$$\|y_n - y_m\| = \|T(x_n - x_m)\| \geq \varepsilon\|x_n - x_m\|$$

we get that the x_n are also Cauchy, and thus have a limit $x \in \mathfrak{X}$. Then

$$Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n = y$$

and T is surjective.

(1) $\implies$ (5) By hypothesis we have $I_{\mathfrak{X}} = T^{-1}T = TT^{-1}$; so

$$I_{\mathfrak{X}^*} = I_{\mathfrak{X}}^* = T^*(T^{-1})^* = (T^{-1})^*T^*$$

so T^* is invertible in $\mathcal{B}(X^*)$.

(5) $\implies$ (4) If T^* is invertible then T^* is bounded below (by $1 \implies 3$); also ($1 \implies 5$) implies that T^{**} is invertible and thus bounded below. But $T = T^{**} \restriction \mathfrak{X}$; so T is bounded below.

(4) $\implies$ (3) T is bounded below by hypothesis. Note that

$$(\operatorname{Ran} T)^\perp = \{\, f \in \mathfrak{X}^* : \underbrace{f(Tx)}_{(T^*f)(x)} = 0 \text{ for all } x \in \mathfrak{X} \,\} = \{\, f : T^*f = 0 \,\} = \ker(T^*) = \{\, 0 \,\}$$

(since T^* is bounded below). By the Hahn-Banach theorem if $\overline{\operatorname{Ran} T}$ were a proper subspace then there would be $0 \neq f \in \mathfrak{X}^*$ such that $f \restriction \overline{\operatorname{Ran} T} = 0$, a contradiction. So $\overline{\operatorname{Ran} T} = \mathfrak{X}$, and T has dense range. ☐ Theorem 9

Corollary 10. *If $T \in \mathcal{B}(\mathfrak{X})$ then $\sigma(T) = \sigma_\pi(T) \cup \gamma(T)$.*

Proposition 11. *Suppose $\mathfrak{A}$ is a unital Banach algebra. If $\|a\| < 1$ then $1 - a$ is invertible.*

Proof. If $x \in \mathbb{C}$ and $|x| < 1$ then

$$\frac{1}{1-x} = \sum_{n=}^{\infty} x^n$$

If $\|a\| < 1$, define

$$b = \sum_{n=0}^{\infty} a^n$$

(where $a^0 = 1$). To see that this is well-defined, note that

$$\sum_{n=0}^{\infty} \|a^n\| \leq \sum_{n=0}^{\infty} \|a\|^n < \infty$$

So the sequence

$$b_k = \sum_{n=0}^{k} a^n$$

is a convergent sequence, and b is well-defined in $\mathfrak{A}$ as the limit of the b_k. Since multiplication is continuous we get that

$$(1-a)b = \lim_{k\to\infty} (1-a)b_k = \lim_{k\to\infty} (1-a)\sum_{n=0}^{k} a^n = \lim_{k\to\infty} (1 - a^{k+1}) = 1$$

(since $\|a^{k+1}\| \leq \|a\|^{k+1} \to 0$). Also $(1-a)b_k = b_k(1-a)$, so $b(1-a) = (1-a)b = 1$, as desired. ☐ Proposition 11

Corollary 12. *$\mathfrak{A}^{-1}$ is open and $a \mapsto a^{-1}$ is a continuous antihomomorphism $\mathfrak{A}^{-1} \to \mathfrak{A}^{-1}$. (Note that $\mathfrak{A}^{-1}$ is a grape under multiplication and $(ab)^{-1} = b^{-1}a^{-1}$.)*

Proof. The previous proposition says that $b_1(1) = \{ a : \|1-a\| < 1 \} \subseteq \mathfrak{A}^{-1}$. Suppose $a \in \mathfrak{A}^{-1}$ and $b \in \mathfrak{A}$ with $\|b\| < \frac{1}{\|a^{-1}\|}$. Then $a - b = a(1 - a^{-1}b)$ and $\|a^{-1}b\| \leq \|a^{-1}\|\|b\| < 1$. So $1 - a^{-1}b$ is invertible (in fact the inverse is

$$\sum_{n=0}^{\infty} (a^{-1}b)^n$$

). So $a - b$ is invertible with

$$(a-b)^{-1} = (1 - a^{-1}b)^{-1}a^{-1} = \sum_{n=0}^{\infty} (a^{-1}b)^n a^{-1}$$

So $b_{\|a^{-1}\|^{-1}}(a) \subseteq \mathfrak{A}^{-1}$, and $\mathfrak{A}^{-1}$ is open.

$(ab)^{-1} = b^{-1}a^{-1}$ shows that $a \mapsto a^{-1}$ is an antihomomorphism; bijectivity follows from $a = (a^{-1})^{-1}$. It remains to check continuity. If $\|a\| < 1$ then

$$\|(1-a)^{-1} - 1\| = \left\| \sum_{n=0}^{\infty} a^n - 1 \right\| = \left\| \sum_{n=1}^{\infty} a^n \right\| \leq \sum_{n=1}^{\infty} \|a\|^n = \frac{\|a\|}{1 - \|a\|}$$

As $a \to 0$ we have

$$\frac{\|a\|}{1 - \|a\|} \to 0$$

(uniform estimate). Thus if $b_n \to 1$ then $a_n = 1 - b_n \to 0$, and $b_n^{-1} = (1 - a_n)^{-1} \to 1$. So inversion is continuous at 1. So if $a \in \mathfrak{A}^{-1}$ and $a_n \in \mathfrak{A}^{-1}$ converge to a, eventually $\|a - a_n\| < \frac{1}{\|a^{-1}\|}$. Then write $a_n = a - b_n = a(1 - a^{-1}b_n)$ so $a^{-1}b_n \to 0$. Then $a_n^{-1} = (1 - a^{-1}b_n)^{-1}a^{-1} \to a^{-1}$, and inversion is indeed continuous. □ Corollary 12

Proposition 13. *Suppose $\mathfrak{A}$ is a unital Banach algebra and $a \in \mathfrak{A}$. Then $\rho(a)$ is open and $\sigma(a)$ is a compact subset of $\{ \lambda \in \mathbb{C} : |\lambda| \leq \|a\| \}$.*

Proof. Note that

$$\rho(a) = \{ \lambda : \lambda 1 - a \text{ is invertible} \} = \varphi^{-1}(\underbrace{\mathfrak{A}^{-1}}_{\text{open}})$$

where $\varphi \colon \lambda \mapsto \lambda 1 - a$. Alternatively, if $\lambda_0 - a$ is invertible then

$$b_{\|(\lambda_0 - a)^{-1}\|^{-1}}(\lambda_0 - a)$$

is contained in $\mathfrak{A}^{-1}$ and $\{ \lambda : |\lambda - \lambda_0| < \|(\lambda_0 - a)^{-1}\|^{-1} \}$. So $\sigma(a) = \mathbb{C} \setminus \rho(a)$ is closed.

If $|\lambda| > \|a\|$ then

$$\lambda - a = \lambda(1 - \frac{a}{\lambda})$$

But $\|\frac{a}{\lambda}\| = \frac{\|a\|}{|\lambda|} < 1$, so $1 - \frac{a}{\lambda}$ is invertible. So $\lambda - a$ is invertible; so $\sigma(a) \subseteq \{ \lambda : |\lambda| \leq \|a\| \}$; so it is closed and bounded, and thus compact.

TODO 1. *Connectives?*

□ Proposition 13

Example 14.

1. Let $\mathcal{H} = L^2(0,1)$, $f \in \mathcal{H}$, and $M_f h = fh$ for $h \in L^2(0,1)$.

   Claim 15. $\|M_f\| = \|f\|_\infty = \operatorname{ess.sup}|f|$.

Proof. Note that

$$\begin{aligned}\|M_f\|^2 &= \sup\{\, \|fh\|_2^2 : \|h\|_2 \le 1\,\} \\ &= \sup\left\{ \int |fh|^2 : \|h\|_2 \le 1 \right\} \\ &\le \sup\left\{ \int \|f\|_\infty^2 |h|^2 : \|h\|_2 \le 1 \right\} \\ &= \|f\|_\infty^2 \sup\{\, \|h\|_2^2 : \|h\|_2 \le 1\,\} \\ &= \|f\|_\infty^2\end{aligned}$$

So $\|M_f\| \le \|f\|_\infty$.
For $\varepsilon > 0$, let $A_\varepsilon = \{\, x : |f(x)| > \|f\|_\infty - \varepsilon \,\}$; then $m(A_\varepsilon) > 0$. Let $h_\varepsilon = \frac{\chi_{A_\varepsilon}}{m(A_\varepsilon)^{\frac{1}{2}}}$. Then

$$\|h_\varepsilon\|_2^2 = \int \frac{\chi_{A_\varepsilon}}{m(A_\varepsilon)} = 1$$

$$|fh_\varepsilon| \ge (\|f\|_\infty - \varepsilon)\frac{\chi_\varepsilon}{m(A_\varepsilon)^{\frac{1}{2}}}$$

So

$$\|fh_\varepsilon\| \ge (\|f\|_\infty - \varepsilon)\|h_\varepsilon\| = \|f\|_\infty - \varepsilon$$

and

$$\|Mf\| \ge \sup_{\varepsilon > 0} \|f\|_\infty - \varepsilon = \|f\|_\infty$$

□ Claim 15

Note that $f \mapsto M_f$ is an algebra homomorphism of $L^\infty(0,1)$ into $B(L^2(0,1))$ which is isometric. What is M_f^*? Well, for $h, k \in L^2(0,1)$ we have

$$\begin{aligned}\langle M_f^* h, k\rangle &= \langle h, M_f k\rangle \\ &= \langle h, fk\rangle \\ &= \int h\overline{fk} \\ &= \int (\overline{f}h)\overline{k} \\ &= \langle \overline{f}h, k\rangle \\ &= \langle M_{\overline{f}} h, k\rangle\end{aligned}$$

So $M_f^* = M_{\overline{f}}$.

Claim 16. $\sigma(M_f) = \sigma_{L^\infty}(f) = \operatorname{ess.ran}(f) = \{\, \lambda : m(f^{-1}(b_\varepsilon(\lambda))) > 0$ *for all* $\varepsilon > 0 \,\}$.

Proof. Note that

$$\mathbb{C} \setminus \operatorname{ess.ran}(f) = \{\, \lambda : \exists \varepsilon > 0 \text{ such that } m(f^{-1}(b_\varepsilon(\lambda))) = 0 \,\}$$

If $\lambda \notin \operatorname{ess.ran}(f)$ then there is ε such that $|f(x) - \lambda| > \varepsilon$ almost everywhere; so $\frac{1}{f-\lambda} \in L^\infty$ (since $\left|\frac{1}{f-\lambda}\right| \le \frac{1}{\varepsilon}$ almost everywhere). So $f - \lambda$ is invertible in L^∞.
Note that $I = M_1$ and $\lambda I - M_f = M_{\lambda - f}$. So

$$M_{\lambda-f} M_{\frac{1}{\lambda-f}} = M_{\frac{1}{\lambda-f}} M_{\lambda-f} = M_1 = I$$

So if $\lambda \notin \operatorname{ess.ran}(f)$ then $\lambda - f$ is invertible in L^∞ and $M_{\lambda-f}$ is invertible in $\mathcal{B}(L^2(0,1))$.

If $\lambda \in \operatorname{ess.ran}(f)$ then $\frac{1}{\lambda - f}$ is not essentially bounded and may take value $+\infty$ somewhere; so $\lambda - f$ is not invertible in L^∞.

For $\varepsilon > 0$ let $A_\varepsilon = \{ x : |f(x) - \lambda| < \varepsilon \}$; then $m(A_\varepsilon) > 0$. Let $h_\varepsilon = \frac{\chi_{A_\varepsilon}}{m(A_\varepsilon)^{\frac{1}{2}}}$. Then $|M_{\lambda - f} h_\varepsilon| = |(\lambda - f)h_\varepsilon| < \varepsilon |h_\varepsilon|$; so $\|M_{\lambda - f} h_\varepsilon\| < \varepsilon$. So $M_{\lambda - f}$ is not bounded below, and $M_{\lambda - f}$ is not invertible. □ Claim 16

Example 17. Consider M_x. We have $\overline{\operatorname{Ran}(x)} = \operatorname{ess.ran}(x) = [0,1]$ and $\sigma_p(M_x) = \emptyset$. If $M_x h = xh = \lambda h$ then $(x - \lambda)h = 0$ almost everywhere; since $x - \lambda \neq 0$ almost everywhere, we get that $h = 0$ almost everywhere.

If $\lambda \in [0,1]$, then $M_{\lambda - x}$ is not bounded below.

We have

$$\overline{\operatorname{Ran} M_{\lambda - x}} \supseteq \bigcup M_{\lambda - x} L^2([0, \lambda - \varepsilon] \cup [\lambda + \varepsilon, 1])$$

Since $|\lambda - x| \geq \varepsilon$ on $B_\varepsilon = [0, \lambda - \varepsilon] \cup [\lambda + \varepsilon, 1]$ and $M_{\lambda - f} \colon L^2(B_\varepsilon) \to L^2(B_\varepsilon)$, we get that $M_{\lambda - x}$ is invertible on $L^2(B_\varepsilon)$ and $M_{\lambda - x} L^2(B_\varepsilon) = L^2(B_\varepsilon)$. So

$$\overline{\operatorname{Ran} M_{\lambda - x}} \supseteq \overline{\bigcup_{\varepsilon > 0} L^2(B_\varepsilon)} = L^2(0,1)$$

2. Let $\mathcal{H} = \ell_2$ with orthonormal basis $\{ e_n : n \geq 0 \}$. If $(d_n : n \in \mathbb{N})$ is bounded we let $D = \operatorname{diag}((d_n : n \in \mathbb{N}))$ so
$$D\Big(\sum a_n e_n\Big) = \sum d_n a_n e_n$$
So $\|D\| = \sup |d_n|$, and $\sigma(D) = \overline{\{ d_n \}}$.

3. Let S be the unilateral shift on ℓ_2 so
$$S \sum_{n \geq 0} a_n e_n = \sum_{n \geq 0} a_n e_{n+1}$$
The adjoint has
$$\begin{aligned} \Big\langle S^* \sum a_n e_n, \sum b_n e_n \Big\rangle &= \Big\langle \sum a_n e_n, S \sum b_n e_n \Big\rangle \\ &= \Big\langle \sum a_n e_n, \sum b_n e_{n+1} \Big\rangle \\ &= \sum_{n=0}^\infty a_{n+1} \overline{b_n} \\ &= \Big\langle \sum_{n=0}^\infty a_{n+1} e_n, \sum b_n e_n \Big\rangle \end{aligned}$$
So
$$S^* e_n = \begin{cases} e_{n-1} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$$
is the backwards shift.

Proposition 18. *If $\mathcal{H}$ is a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ then $\sigma(T^*) = \sigma(T)^*$ (where the latter is pointwise complex conjugation).*

Proof. If $\lambda \notin \sigma(T)$ then $(\lambda I - T)(\lambda I - T)^{-1} = I = (\lambda I - T)^{-1}(\lambda I - T)$. Taking adjoints we find that

$$((\lambda I - T)^{-1})^* (\overline{\lambda} I - T^*) = I^* = I = (\overline{\lambda} I - T^*)((\lambda I - T)^{-1})^*$$

so $(\overline{\lambda} I - T^*)^{-1} = ((\lambda I - T)^{-1})^*$. Since $T = T^{**}$ this is reversible. So $\rho(T^*) = \rho(T)^*$. □ Proposition 18

Note that $S^*S = I$ but $SS^* = I - P_{\mathbb{C}e_0}$ where $P_{\mathbb{C}e_0} = e_0 e_0^*$.

Notation 19. If $x, y \in \mathcal{H}$ then $xy^* \in \mathcal{B}(\mathcal{H})$ of rank 1 is given by $(xy^*)(z) = x(y^*z) = \langle z, y\rangle x$.

So S, S^* are not invertible. We have that S is injective but not surjective, with $\mathrm{Ran}(S) = (\mathbb{C}e_0)^\perp$; also S^* is surjective but not injective with $S^*e_0 = 0$, and $\ker(S^*) = \mathbb{C}e_0$. So $0 \in \sigma(S)$.

We have $\|S\| = \|S^*\| = 1$ and S is an isometry ($\|Sx\| = \|x\|$ for all x). So $\sigma(S) \subseteq \overline{\mathbb{D}} = \{\,\lambda : |\lambda| \le 1\,\}$.

If $S^*x = \lambda x$ where $x = (x_0, x_1, \ldots)$ then $x_{n+1} = \lambda x_n$ for all n; so $x = x_0(1, \lambda, \lambda^2, \ldots)$. Then

$$\|x\|_2^2 = |x_0|^2 \sum_{n=0}^{\infty} |\lambda|^{2n} = \begin{cases} \frac{|x_0|^2}{1-|\lambda|^2} < \infty & \text{if } |\lambda| < 1 \\ 0 & \text{if } x_0 = 0 \\ \infty & \text{else} \end{cases}$$

So if $x_\lambda = (1, \lambda, \lambda^2, \ldots)$ for $|\lambda| < 1$ then $S^*x_\lambda = \lambda x_\lambda$. So $\sigma_p(S^*) = \mathbb{D}$. So $\sigma(S^*) = \overline{\mathbb{D}}$ and $\sigma(S) = \overline{\mathbb{D}}$.

If $Sx = \lambda x$ for $\lambda \neq 0$ then $x_0 = 0 = x_1 = x_2 = \cdots$; so $\lambda \notin \sigma_p(S)$. Also $0 \notin \sigma_p(S)$ because S is isometric. So $\sigma_p(S) = \emptyset$.

Suppose $|\lambda| = 1$; let $x_n = \frac{1}{\sqrt{n}}(1, \lambda, \lambda^2, \ldots, \lambda^{n-1}, 0, 0, 0 \ldots)$. Then

$$S^*x_n = \frac{1}{\sqrt{n}}(\lambda, \lambda^2, \ldots, \lambda^{n-1}, 0, 0, \ldots)$$

so

$$S^*x_n - \lambda x_n = \frac{1}{\sqrt{n}}(0, \ldots, 0, -\lambda^n, 0, 0, \ldots)$$

and $\|(S^* - \lambda)x_n\| = \frac{1}{\sqrt{n}} \to 0$, so $S^* - \lambda$ isn't bounded below. Also

$$Sx_n = \frac{1}{\sqrt{n}}(0, 1, \lambda, \lambda^2, \ldots, \lambda^{n-2}, \lambda^{n-1}, 0, \ldots)$$

and

$$\overline{\lambda} x_n \frac{1}{\sqrt{n}}(\overline{\lambda}, 1, \lambda, \ldots, \lambda^{n-2}, 0, 0, \ldots)$$

so

$$\|(S - \overline{\lambda} I)x_n\| = \left\| \frac{1}{\sqrt{n}}(-\overline{\lambda}, 0, \ldots, 0, \lambda^{n-1}, 0, \ldots) \right\| = \sqrt{\frac{2}{n}} \to 0$$

and $S - \overline{\lambda}$ is not bounded below.

Definition 20. Suppose $\Omega \subseteq \mathbb{C}$ is open and $\mathfrak{X}$ is a Banach space. We say $f \colon \Omega \to \mathfrak{X}$ is *strongly analytic* on Ω if for all $z_0 \in \Omega$ there is $r > 0$ and $(x_n : n \ge 0)$ in $\mathfrak{X}$ such that

$$f(z) = \sum_{n=0}^{\infty} x_n (z - z_0)^n$$

converges absolutely and uniformly on $\{\, z : |z - z_0| \le r \,\}$. We say f is *weakly analytic* if for all $\varphi \in \mathfrak{X}'$ we have that $\varphi \circ f \colon \Omega \to \mathbb{C}$ is analytic.

Exercise 21 (Homework). Weakly analytic implies strongly analytic. (I think he said something about Banach-Steinhaus?)

Theorem 22. *Suppose $\mathfrak{A}$ is a unital Banach algebra and $a \in \mathfrak{A}$.*

1. *For $\lambda, \mu \in \rho(a)$ we have*
$$\frac{R(a, \lambda) - R(a, \mu)}{\lambda - \mu} = -R(a, \lambda)R(a, \mu)$$

2. *$R(a,\lambda)$ is a strongly analytic function on $\rho(a)$.*

3. *$R'(a,\lambda) = -R(a,\lambda)^2$.*

4. *$\|R(a,\lambda)\| \to 0$ as $|\lambda| \to \infty$.*

Proof.

1. We have$(R(a,\lambda) - R(a,\mu)(\lambda - a)(\mu - a) = (\mu - a) - (\lambda - a) = \mu - \lambda$; multiply by $\frac{R(a,\lambda)-R(a,\mu)}{\lambda-\mu}$ to get the desired result.

2. If $\lambda_0 \in \rho(a)$ and $|\lambda - \lambda_0| < \frac{1}{\|(\lambda_0-a)^{-1}\|}$.
$$\begin{aligned}\lambda - a &= (\lambda_0 - a) - (\lambda_0 - \lambda)\\ &= (\lambda_0 - a)(1 - (\lambda_0 - \lambda)(\lambda_0 - a)^{-1})\end{aligned}$$
$$\|(\lambda_0 - \lambda)(\lambda_0 - a)^{-1}\| = |\lambda_0 - \lambda|\|(\lambda_0 - a)^{-1}\| < 1$$
So
$$(\lambda - a)^{-1} = \sum_{n=0}^{\infty}((\lambda_0 - \lambda)(\lambda_0 - a)^{-1})^n(\lambda_0 - a)^{-1} = \sum_{n=0}^{\infty}(-1)^n(\lambda_0 - a)^{-n-1}(\lambda - \lambda_0)^n$$
If $0 < R < \frac{1}{\|(\lambda_0-a)^{-1}\|}$ then if $|\lambda - \lambda_0| \le R$ then $\|(\lambda - \lambda_0)(\lambda_0 - a)^{-1}\| \le \frac{R}{\|(\lambda_0-a)^{-1}\|} = r < 1$. So
$$\sum \|((\lambda_) - \lambda)(\lambda_0 - a)^{-1}\|\|(\lambda_0 - a)^{-1}\| \le \sum r^n \|(\lambda_0 - a)^{-1}\| = \frac{\|(\lambda_0 - a)^{-1}\|}{1 - r} < \infty$$
So convergence is absolute and uniform (by M-test) on $\{\, \lambda : |\lambda - \lambda_0| \le R \,\}$. So $R(a,\lambda)$ is strongly analytic.

3. We note that
$$R'(a,\mu) = \lim_{\lambda\to\mu} \frac{R(a,\lambda) - R(a,\mu)}{\lambda - \mu} = -R(a,\mu)^2$$

4. If $|\lambda| = 2\|a\|$ then $(\lambda - a)^{-1} = \lambda^{-1}(1 - \lambda^{-1}a)^{-1} = \lambda^{-1}\sum(\lambda^{-1}a)^n$. So
$$\|(\lambda - a)^{-1}\| \le \frac{1}{|\lambda|}\sum_{n=0}^{\infty}\|(\lambda^{-1}a)^n\| \le \frac{1}{|\lambda|}\sum \frac{1}{2^n} = \frac{2}{|\lambda|}$$
So $\|R(a,\lambda)\| \le \frac{2}{|\lambda|} \to 0$ as $|\lambda| \to \infty$. □ Theorem 22

Theorem 23 (Liouville)**.** *If $f\colon \mathbb{C} \to \mathfrak{X}$ is a weakly analytic entire function which is bounded then it is constant.*

Proof. For all $\varphi \in \mathfrak{X}'$ we have $\varphi \circ f\colon \mathbb{C} \to \mathbb{C}$ is entire and bounded. So $\varphi \circ f$ is constant by Liouville's theorem. By Hahn-Banach we have that f is constant: if $f(z_1) \ne f(z_2)$ then there would be φ such that $\varphi(f(z_1) - f(z_2)) \ne 0$. □ Theorem 23

Theorem 24. *Suppose $\mathfrak{A}$ is a unital Banach algebra. Then $\sigma(a)$ is not empty.*

Proof. If $\sigma(a) = \emptyset$ then $R(a,\lambda)$ is entire, strongly analytic, and has $\|R(a,\lambda)\| \to 0$ as $|\lambda| \to \infty$, and is thus bounded. So by Liouville's theorem it is constant, a contradiction since $R(a,0) = -a^{-1} \ne (1-a)^{-1} = R(a,1)$. □ Theorem 24

If $K \subseteq \mathbb{C}$ is compact we let $\mathrm{Rat}(K)$ consist of rational functions $\frac{p(x)}{q(x)}$ with $p, q \in \mathbb{C}[x]$ such that the poles (zeroes of q) lie in $\mathbb{C} \setminus K$. If $\sigma(a) = K$ and $\frac{p}{q} \in \mathrm{Rat}(K)$ then we may write $q(x) = (x - \alpha_1)\cdots(x - \alpha_m)$ with each $\alpha_i \notin K$; then $q(a) = (a - \alpha_1 1)\cdots(a - \alpha_m 1)$, and $q(a)^{-1} = (a - \alpha_1 1)^{-1}\cdots(a - \alpha_m 1)^{-1}$ is well-defined because $\alpha_i \notin K = \sigma(a)$. We can then define $\frac{p}{q}(a) = p(a)q(a^{-1})$. This is a well-defined algebra homomorphism of $\mathrm{Rat}(\sigma(a))$ into $\mathfrak{A}$.

Theorem 25 (Spectral mapping theorem for rational functions)**.** *If $a \in \mathfrak{A}$ and $f = \frac{p}{q} \in \mathrm{Rat}(\sigma(a))$ then $\sigma(f(a)) = f(\sigma(a))$.*

Proof. Write $f = \frac{p}{q}$ with

$$q(x) = \prod_{i=1}^{m}(x - \alpha_i)$$

If $\lambda \in \mathbb{C}$ then we may write $f(x) - \lambda 1 = \frac{p_1(x)}{q(x)}$ with

$$p_1(x) = \prod_{j=1}^{n}(x - \beta_j)$$

Then

$$f(a) - \lambda 1 = p_1(a)q(a)^{-1} = \prod_{j=1}^{n}(a - \beta_j 1)q(a)^{-1}$$

So

$$\begin{aligned} \lambda \in \sigma(f(a)) &\iff f(a) - \lambda 1 \text{ is not invertible} \\ &\iff \exists j \text{ such that } a - \beta_j 1 \text{ is not invertible} \\ &\iff \exists j \text{ such that } \beta_j \in \sigma(a) \end{aligned}$$

and

$$\begin{aligned} \lambda \in f(\sigma(a)) &\iff \exists \beta \in \sigma(a) \text{ such that } f(\beta) - \lambda = 0 \\ &\iff \exists x \in \sigma(a) \text{ such that } \prod_{j=1}^{n}(x - \beta_j)q(x) = 0 \\ &\iff \exists j \text{ such that } x = \beta_j \end{aligned}$$

TODO 2. *Typo here?*

But the last equivalences are the same. □ Theorem 25

Definition 26. The *spectral radius* of a is $\mathrm{spr}(a) = \sup\{\, |\lambda| : \lambda \in \sigma(a) \,\}$.

Theorem 27. *Suppose $\mathfrak{A}$ is a unital Banach algebra and $a \in \mathfrak{A}$. Then*

$$\mathrm{spr}(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$$

Proof. By the spectral mapping theorem we have $\sigma(a^n) = \sigma(a)^n$. Since $\mathrm{spr}(a) \le \|a\|$ we have

$$\mathrm{spr}(a) = \mathrm{spr}(a^n)^{\frac{1}{n}} \le \|a^n\|^{\frac{1}{n}}$$

thus

$$\mathrm{spr}(a) \le \inf_{n\ge 1} \|a^n\|^{\frac{1}{n}}$$

Recall that $R(a, \lambda) = (\lambda - a)^{-1}$ is analytic on $\mathbb{C} \setminus \sigma(a)$. Hence for $|\lambda| > \|a\|$ we have

$$R(a, \lambda) = \sum_{n=0}^{\infty} a^n \lambda^{-n-1}$$

TODO 3. *why? Something about a power series around ∞?*

If $\varphi \in \mathfrak{A}'$ then

$$\varphi(R(a,\lambda)) = \sum_{n=0}^{\infty} \varphi(a^n)\lambda^{-n-1}$$

is scalar-valued and analytic on $\rho(a) \supseteq \mathbb{C} \setminus \{\, \lambda : |\lambda| \le \operatorname{spr}(a) \,\}$; note that this last set is the biggest disk around $\mathbb{C}$ on which R is defined. In particular, convergence is absolute and uniform over $|\lambda| \ge r + \varepsilon$ (with $r = \operatorname{spr}(a)$). So

$$\sup_{n\ge 0} |\varphi(a^n)|(r+\varepsilon)^{-n-1} < \infty$$

(as the terms in the series approach 0). So

$$\sup_{n\ge 0} \left| \varphi\left(\left(\frac{a}{r+\varepsilon} \right)^n \right) \right| \le \frac{C(\varphi)}{r+\varepsilon}$$

for some constant $C(\varphi)$ (depending on φ). Hence by the uniform boundedness principle we have

$$\sup_{n\ge 0} \left\| \left(\frac{a}{r+\varepsilon} \right)^n \right\| = C' < \infty$$

Thus $\|a^n\| \le C'(r+\varepsilon)^n$, and hence $\|a^n\|^{\frac{1}{n}} \le (C')^{\frac{1}{n}}(r+\varepsilon) \to r+\varepsilon$. So

$$\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \le r \le \inf \|a^n\|^{\frac{1}{n}}$$

TODO 4. *port limsup typesetting to essential range?*

So $r = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \inf \|a^n\|^{\frac{1}{n}}$. ☐ Theorem 27

Remark 28. $R(a,\lambda) = \sum_{n=0}^{\infty} a^n \lambda^{-n-1}$ converges absolutely and uniformly on $\{\, \lambda : |\lambda| \ge r+\varepsilon \,\}$.

Exercise 29. Check the details of this.

Proposition 30 (Mazur)**.** *If $\mathfrak{A}$ is a Banach field then $\mathfrak{A} = \mathbb{C}1$.*

Proof. If $a \in \mathfrak{A}$ then $\sigma(a) \ne 0$. Pick $\lambda \in \sigma(a)$; then $a - \lambda 1$ is not invertible, so since $\mathfrak{A}$ is a field we get that $a - \lambda 1 = 0$ and $a = \lambda 1$. ☐ Proposition 30

1.1 Riesz functional calculus

Suppose U is open and contains $\sigma(a)$. Suppose f is a holomorphic function on U and $\lambda \in \sigma(a)$. Cauchy's theorem tells us that to evaluate $f(\lambda)$ we can draw a rectifiable curve

TODO 5. *rectifiable?*

$\mathcal{C}$ such that $\mathcal{C} \subseteq U \setminus \sigma(a)$ and the winding number

$$\perp_{\mathcal{C}}(z) = \begin{cases} 0 & \text{if } z \in \mathbb{C} \setminus U \\ 1 & \text{if } z \in K \end{cases}$$

TODO 6. *ind?*

Then by Cauchy's theorem we have

$$f(\lambda) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)}{z-\lambda} \mathrm{d}z$$

for $z \in \sigma(a)$.

We can try to define $f(a)$ by

$$f(a) = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(z1 - a)^{-1} \mathrm{d}z$$

Note that $(z1-a)^{-1}$ is defined on $\mathbb{C} \setminus \sigma(a)$, and thus on $\mathcal{C}$; also $f(z)$ is defined and analytic on $U \supseteq \mathcal{C}$. So $f(z)(z-a)^{-1}$ is defined on $U \setminus \sigma(a)$; it is analytic, and thus continuous.

Theorem 31. *Suppose $\mathfrak{X}$ is a Banach space; suppose $\mathcal{C}$ is a rectifiable curve in $\mathbb{C}$ and $f\colon \mathcal{C} \to \mathfrak{X}$ is continuous. Then*

$$\int_{\mathcal{C}} f(z)\mathrm{d}z$$

makes sense as a Riemann integral.

Proof. Paramterize $\mathcal{C}$ by arc length s for $0 \le s \le L$. Take partitions Δ consisting of $0 = s_0 < s_1 < \cdots < s_n = L$ and Ξ consisting of $\xi_i \in \varphi([s_{i-1}, \ldots, s_i])$ for $1 \le i \le n$. If $\varphi\colon [0, L] \to \mathcal{C}$ is our parametrization then our Riemann sum is

$$J(\Delta, \Xi) = \sum_{i=1}^{n} f(\xi_i)(\varphi(s_i) - \varphi(s_{i-1}))$$

We define

$$\operatorname{mesh}(\Delta) = \max_{1 \le i \le n} (s_i - s_{i-1})$$

Claim 32. $\lim\limits_{\operatorname{mesh}(\Delta) \to 0} J(\Delta, \Xi)$ *converges; we call this limit* $\displaystyle\int_{\mathcal{C}} f(z)\mathrm{d}z$.

TODO 7. *I believe this is a limit of nets?*

Proof. Suppose $\varepsilon > 0$. By continuity of f there is $\delta > 0$ such that $|s - t| < \delta$ implies $\|f(\varphi(s)) - f(\varphi(t))\| < \varepsilon$. Suppose (Δ_1, Ξ_1) and (Δ_2, Ξ_2) both have mesh $< \delta$. Let $\Delta = \Delta_1 \cup \Delta_2 = \{ 0 = s_0 < s_1 \cdots < s_n = L \}$, and for $p \in \{ 1, 2 \}$ write $\Delta_p = \{ s_i : i \in J_p \}$ with $\{ 0, n \} \subseteq J_p \subseteq \{ 0, \ldots, n \}$. Let $\Xi = \{ \varphi(s_i) : 1 \le i \le n \}$. We compare $J(\Delta_p, \Xi_p$ to $J(\Delta, \Xi)$.

$$J(\Delta, \Xi) - J(\Delta_p, \Xi_p) = \sum_{i=1}^{n} f(\varphi(s_i))(\varphi(s_i) - \varphi(s_{i-1})) - \sum f(\xi_j)(\varphi(s_i) - \varphi(s_{i-1}))$$

where $j \in J_p$ satisfies $[s_{i-1}, s_i] \subseteq [s_j, s_{j'}]$ with $[s_j, s_{j'}]$ an interval in Δ_p. Hence

$$\begin{aligned}
\|J(\Delta, \Xi) - J(\Delta_p, \Xi_p)\| &= \left\| \sum_{i=1}^{n} f(\varphi(s_i))(\varphi(s_i) - \varphi(s_{i-1})) - \sum f(\xi_j)(\varphi(s_i) - \varphi(s_{i-1})) \right\| \\
&\le \sum_{i=1}^{n} \|f(\varphi(s_i)) - f(\xi_j)\| |\varphi(s_i) - \varphi(s_{i-1})| \text{ (note } \varphi(s_i) \text{ and } \xi_j \text{ are within } \delta \text{ of each other)} \\
&< \sum_{i=1}^{n} \varepsilon(s_i - s_{i-1}) \\
&= \varepsilon L
\end{aligned}$$

So $\|J(\Delta_1, \Xi_1) - J(\Delta_2, \Xi_2)\| < (2L)\varepsilon$. So the Riemann sums are Cauchy, and thus converge. ☐ TODO 7

☐ Theorem 31

Theorem 33 (Riesz functional calculus)**.** *If $f \in \operatorname{Hol}(U)$ with U an open set containing $\sigma(a)$, we define*

$$f(a) = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(z - a)^{-1}\mathrm{d}z$$

Then

1. *This definition is independent of the choice of $\mathcal{C}$; hence $f(a)$ is well-defined.*
2. *$(f + g)(a) = f(a) + g(a)$ and $(\lambda f)(a) = \lambda \cdot f(a)$.*
3. *$(fg)(a) = f(a)g(a)$. (Hence, combining all the above, we get that $f \mapsto f(a)$ is a homomorphism.)*

4. If

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is analytic in a disk $D_R(z_0) \supseteq \sigma(a)$ *then*

$$f(a) = \sum_{n=0}^{\infty} a_n (a - z_0 1)^n$$

Proof.

1. Suppose $\mathcal{C}_1, \mathcal{C}_2$ are permissable curves. Then $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$ (i.e. union of $\mathcal{C}_1$ and $\mathcal{C}_2$ with the orientation of $\mathcal{C}_2$ reversed) is a curve such that

$$\perp_{\mathcal{C}}(z) = \begin{cases} 0 & \text{if } z \in \mathbb{C} \setminus U \\ 0 & \text{if } z \in \sigma(a) \end{cases}$$

So $f(z)(z-a)^{-1}$ is analytic on $U \setminus \sigma(a)$, and $\mathcal{C} \subseteq U \setminus \sigma(a)$; so $\mathcal{C}$ is homologous to zero in $U \setminus \sigma(a)$. Taking $\varphi \in \mathfrak{A}'$ we have

$$\begin{aligned} \varphi\left(\frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(z-a)^{-1} \mathrm{d}x\right) &= \frac{1}{2\pi i} \int_{\mathcal{C}} \underbrace{f(z)\varphi((z-a)^{-1})}_{\text{scalar-valued and analytic in } U \setminus \sigma(a)} \mathrm{d}z \\ &= 0 \end{aligned}$$

by Cauchy's theorem. But this holds for all $\varphi \in \mathfrak{A}'$. So by the Hahn-Banach theorem we get

$$0 = \frac{1}{2\pi i} \int_{\mathcal{C}} f(z)(z-a)^{-1} \mathrm{d}z = \frac{1}{2\pi i} \int_{\mathcal{C}_1} f(z)(z-a)^{-1}) \mathrm{d}z - \frac{1}{2\pi i} \int_{\mathcal{C}_2} f(z)(z-a)^{-1} \mathrm{d}z$$

□ Theorem 33