Course notes for PMATH 833

Christopher Hawthorne

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1 Introduction

Rough outline:

- Locally compact grapes, Haar measures
- Abelian grapes, Pontryagin duality
- Compact grapes, Peter-Weyl, aspects of duality
- Amenable grapes, Hulanicki's theorem

2 Locally compact grapes

Recall:

Definition 2.1. Suppose $X \neq \varnothing$ is a set. A *topology* on X is a family $\tau \subseteq \mathcal{P}(X)$ satisfying the following:

- $\varnothing, X \in \tau$
- If $U, V \in \tau$ then $U \cap V \in \tau$ (and hence closed under finite intersections)
- If $\{ U_i \}_{i \in I} \subseteq \tau$ then
$$\bigcup_{i \in I} U_i \in \tau$$

We call the pair (X, τ) a *topological space*.

Example 2.2 (Initial topologies). Suppose $X \neq \varnothing$; suppose we have topological spaces $\{ (Y_i, \tau_i) \}_{i \in I}$ and maps $f_i \colon X \to Y_i$ for each i. We define

$$\sigma(X, \{ f_i \}_{i \in I}) = \left\{ U \in \mathcal{P}(X) : \substack{\text{for each } x \in U \text{ there are } i_1, \ldots, i_n \in I \text{ and} \\ V_{i_1} \in \tau_{i_1}, \ldots, V_{i_n} \in \tau_{i_n} \text{ such that } x \in \bigcap_{k=1}^n f_{i_k}^{-1}(V_{i_k}) \subseteq U} \right\}$$

Sets of the form
$$\bigcap_{k=1}^{n} f_{i_k}^{-1}(V_{i_k})$$
as above form a base for $\sigma(X, \{ f_i \}_{i \in I})$; sets of the form $f_i^{-1}(V_i)$ form a sub-base.

Example 2.3.

Product topology Suppose
$$X = \prod_{i \in I} Y_i$$

with projections $\pi_i \colon X \to Y_i$. We let
$$\bigtimes_{i \in I} \tau_i = \sigma(X, \{ \pi_i \}_{i \in I})$$

The basic open sets are of the form
$$\prod_{i \in I} V_i$$

where each $V_i \in \tau_i$ and all for all but finitely many i we have $V_i = Y_i$.

Metric topology If $\rho \colon X \times X \to [0, \infty)$ is a metric, then the metric topology is given by $\tau_\rho = \sigma(X, \{ \rho(x, \cdot) \}_{x \in X})$.

Recall:

Definition 2.4. If $(X, \sigma), (Y, \tau)$ are topological spaces and $f \colon X \to Y$, then we say f is *continuous* if $f^{-1}(V) \in \sigma$ for each $V \in \tau$. A subset $K \subseteq X$ is *compact* (with respect to σ) if whenever

$$K \subseteq \bigcup_{i \in I} U_i$$

for $U_i \in \sigma$, there are $i_1, \ldots, i_n \in I$ such that

$$K \subseteq \bigcup_{k=1}^{n} U_{i_k}$$

Definition 2.5. A topological space (X, τ) is *locally compact* if for any $x \in X$ there is $U \in \tau$ with $x \in U$ such that $\overline{U}$ is compact. (Recall
$$\overline{U} = \bigcap \{ X \backslash V : V \in \tau, V \cap U = \varnothing \}$$
is the *closure* of U.)

Example 2.6.

1. $(\mathbb{R}, \tau_{|\cdot|})$ is locally compact.
2. Suppose $X \neq \varnothing$; consider the *discrete topology* $(X, \mathcal{P}(X))$. This is locally compact.
3. Suppose $\{ (X_i, \tau_i) \}_{i \in I}$ is a family of locally compact spaces. Then
$$\left(\prod_{i \in I} X_i, \bigtimes_{i \in I} \tau_i \right)$$
is locally compact if and only if all but finitely many (X_i, τ_i) are compact.

   Rough.

   $(\Longleftarrow)$ Use Tychonoff's theorem.

   $(\Longrightarrow)$ Each basic open set is of the form
$$U = V_{i_1} \times \cdots V_{i_n} \times \prod_{i \in I \setminus \{ i_1, \ldots, i_n \}} X_i$$
   If (X_{i_0}, τ_{i_0}) is not compact for some $i_0 \in I \setminus \{ i_1, \ldots, i_n \}$ then $\pi_{i_0}(\overline{U}) = X_{i_0}$ is not compact, so $\overline{U_{i_0}}$ is not compact. □

4. Suppose $\mathcal{X}$ be an infinite dimensional vector space over $\mathbb{R}$. Suppose $\|\cdot\|$ is a norm on $\mathcal{X}$. A lemma of Riesz tells us that if $\mathcal{Y} \subseteq \mathcal{X}$ is a closed subspace, then there is $x \in b_1(\mathcal{X})$ (the unit ball) such that $\operatorname{dist}(x, \mathcal{Y}) > \frac{1}{2}$. (This is a good exercise; use the Hahn-Banach theorem.) Inductively, we can find a sequence $(x_n)_{n=1}^{\infty} \subseteq b_1(\mathcal{X})$ such that $\|x_n - x_m\| > \frac{1}{2}$ for $n \neq m$. Hence no ball $x + b_r(\mathcal{X}) = B(x, r)$ (where $r > 0$) is *pre-compact*; i.e. has compact closure.
5. Suppose $\mathcal{F} \subseteq \mathcal{X}'$ (the algebraic dual) be a subspace which separates points; i.e.
$$\bigcap_{f \in \mathcal{F}} \ker(f) = \{ 0 \}$$
Then $(\mathcal{X}, \sigma(\mathcal{X}, \mathcal{F}))$ is not locally compact. For example, if $V_1, \ldots, V_n$ are neighbourhoods of 0 in $\mathbb{R}$, then
$$U = \bigcap_{k=1}^{n} f_k^{-1}(V_k)$$
contains a subspace $\mathcal{Y}$ of $\mathcal{X}$. Using the Hahn-Banach theorem, we can find $f \in \mathcal{F}$ such that $f(\mathcal{Y}) = \mathbb{R}$; so $f(U)$ is not compact, so $\overline{U}$ is not compact.

Definition 2.7. Suppose G is a grape. A topology $\tau \subseteq \mathcal{P}(G)$ is called a *grape topology* if the following maps are continuous:

- $\cdot\colon (G \times G, \tau \times \tau) \to (G, \tau)$
- $(\cdot)^{-1}\colon (G, \tau) \to (G, \tau)$

Remark 2.8. In fact, this is equivalent to requiring that the map $G \times G \to G$ given by $(x, y) \mapsto xy^{-1}$ be continuous. Indeed, if this holds, then $y \mapsto (e, y) \mapsto ey^{-1} = y^{-1}$ is continuous, so $(x, y) \mapsto (x, y^{-1}) \mapsto x(y^{-1})^{-1} = xy$ is as well.

Proposition 2.9. *Suppose (G, τ) is a topological grape.*

1. *If $U \in \tau$ and $x \in G$ then*
$$xU = \{ xy : y \in U \}, Ux = \{ yx : y \in U \} \in \tau$$
and if $\varnothing \neq A \subseteq G$ then
$$AU = \{ ay : a \in A, y \in U \}, UA = \{ ya : y \in U, a \in A \} \in \tau$$

2. *If $U \in \tau$ with $e \in U$ then there is $V \in \tau$ with $e \in V$ such that $V^2 = VV \subseteq U$. Furthermore, we can arrange that V be* symmetric*: i.e. that $V^{-1} = \{ y^{-1} : y \in V \} = V$.*

3. *If H is a subgrape of G, then so too is $\overline{H}$.*

4. *If H is an open subgrape of G, then H is closed.*

5. *If K, L are compact subsets of G, then so too is KL.*

6. *If K is compact in G and C is closed, then KC is closed.*

Proof.

1. If $x \in G$, let $L_x : G \to G$ be $y \mapsto xy$; then L_x is continuous as the composition of $y \mapsto (x, y) \mapsto xy$. But $L_x^{-1} = L_{x^{-1}}$ is also continuous; so L_x is a homeomorphism. Hence $xU = L_x(U) \in \tau$. Furthermore

$$AU = \bigcup_{a \in A} aU \in \tau$$

   Right multiplication is similar.

2. Let $\mu : G \times G \to G$ be $(x, y) \mapsto xy$. Then $\mu^{-1}(U)$ is an open neighbourhood of (e, e), and hence contains a basic open set $V_1 \times V_2$ with $e \in V_1$ and $e \in V_2$. Let $V = V_1 \cap V_2$. We can replace V with $V^{-1} \cap V$ to get symmetry; V^{-1} is open, being the image of an open set by the homeomorphism $x \mapsto x^{-1}$.

3. If $x, y \in \overline{H}$, write

$$x = \lim_\alpha x_\alpha$$
$$y = \lim_\beta y_\beta$$

   where $(x_\alpha), (y_\beta)$ are nets in H. Then

$$xy = \lim_\beta x y_\beta = \lim_\beta \lim_\alpha \underbrace{x_\alpha y_\beta}_{\in H} \in \overline{H}$$

   By continuity of $x \mapsto x^{-1}$, we see that for $x \in \overline{H}$ we have $x^{-1} \in \overline{H}$ as well.

4. Note that

$$H = G \backslash \underbrace{\bigcup_{x \in G \backslash H} \underbrace{xH}_{\text{open}}}_{\text{open}}$$

   So H is closed.

5. Tychonoff's theorem tells us that $K \times L \subseteq G \times G$ is compact; hence $KL = \mu(K \times L)$ is compact.

6. Suppose $xk \in \overline{KC}$. Then $x = \lim_\alpha k_\alpha y_\alpha$ with $k_\alpha \in K$ and $y_\alpha \in C$. By dropping to subnet, we may assume that $k = \lim_a k_a \in K$. Then

$$k^{-1}x = \lim_\alpha k_\alpha^{-1} k_\alpha y_\alpha = \lim_\alpha y_\alpha$$

   So $\lim_\alpha y_\alpha = k^{-1}x \in C$. □ Proposition 2.9

Let (G, τ) be a topological grape and H a subgrape of G. The collection of left cosets G/H comes equipped with a *quotient topology* $\tau_{G/H} = \{ W \subseteq \mathcal{P}(G/H) : q^{-1}(W) \in \tau \}$, where $q : G \to G/H$ is $x \mapsto xH$. (This is the *final topology* determined by q.)

Notice that if $U \in \tau$ then $q^{-1}(q(U)) = UH \in \tau$. Hence $\{ q(U) : U \in \tau \} \subseteq \tau_{G/H}$; i.e. the map q is open.

Definition 2.10. The space $(G/H, \tau_{G/H})$ is called a *homogeneous space*.

Proposition 2.11. *Suppose (G,τ) is a topological grape, H a subgrape of G. Then*

1. *If H is closed in G then $(G/H, \tau_{G/H})$ is Hausdorff.*
2. *If H is normal in G then $(G/H, \tau_{G/H})$ is a topological grape.*
3. *If there is $x \in G$ such that $\{x\}$ is closed then (G,τ) is Hausdorff.*

Proof.

1. If $x, y \in G$ have $q(x) \neq q(y)$ then $e \notin xHy^{-1}$ (indeed if we had $e = xhy^{-1}$ then $y = xh$). Since H is assumed to be closed we have xHy^{-1} is closed. So by Proposition 2.9 there is some $V = V^{-1} \in \tau$ with $e \in V$ such that $V^2 \subseteq G \backslash (xHy^{-1})$. But then $e \notin VxHy^{-}V = (VxH)(VyH)^{-1}$; indeed, if we had $e = vxhy^{-1}v'$ for $h \in H$ and $v, v' \in V$, then $v^{-1}(v')^{-1} = xh^{1-}y \in V^2 \cap xHy^{-1}$, contradicting our choice of V. Hence $VxH \cap VyH = \varnothing$, so $q(Vx) \cap q(Vy) = \varnothing$ in G/H.
2. If H is normal, then q is a homomorphism:
$$q(x)q(y) = xHyH = xyHy^{-1}yH = xyH = q(xy)$$
If $x, y \in G$ and $W \in \tau_{G/H}$ with $q(x)q(y) \in W$ then $xy \in q^{-1}(W) \in \tau$; so, by continuity of multiplication in G, there are $U, V \in \tau$ such that $x \in U$, $y \in V$, and $UV \subseteq q^{-1}(W)$. So $q(U)q(V) = q(UV) \subseteq W$; this shows continuity of $(xH, yH) \mapsto xyH$ as a map $(G/H) \times (G/H) \to G/H$. Continuity of $xH \mapsto x^{-1}H$ is similar.
3. We have $\{e\} = L_{x^{-1}}(\{x\})$ is a closed subgrape, as the image of a closed set under a homeomorphism. So $G \cong G/\{e\}$ is Hausdorff by (1). □ Proposition 2.11

Remark 2.12. If $\{e\}$ is not closed then $\overline{\{e\}}$ is the smallest closed subgrape containing e. (This follows from Proposition 2.9.) Hence
$$\overline{\{e\}} = \bigcap_{x \in G} x\overline{\{e\}}x^{-1}$$
since the $x\overline{\{e\}}x^{-1}$ are closed subgrapes containing e; this is then normal. So $G/\overline{\{e\}}$ is a Hausdorff topological grape.

Our convention will then be to replace any topological grape (G,τ) with $(G/\overline{\{e\}}, \tau_{G/\overline{\{e\}}})$ and thus assume (G,τ) is Hausdorff.

Definition 2.13. A *locally compact* (Hausdorff) grape (abbreviated l.c.g.) is a topological grape (G,τ) which is also a locally compact (Hausdorff) space.

Remark 2.14.

1. If $x \in G$ and $U \in \tau$ has $x \in U$ and $\overline{U}$ is compact (in which case we say U is *relatively compact*), then for any $y \in G$ we have $yx^{-1}U = \overline{L_{yx^{-1}}(U)} \subseteq L_{yx^{-1}}(\overline{U})$. Hence to check local compactness of a topological grape, it suffices, to exhibit a compact neighbourhood of one point (usually e).
2. If G is a l.c.g. and H is a normal subgrape, then G/N is locally compact. Indeed, if $e \in U \in \tau$ with $\overline{U}$ compact, then $\overline{q(U)} \subseteq q(\overline{U})$ is compact in G/N.
3. If (X,τ) is a locally compact (Hausdorff) space, then any open subset $U \subseteq X$ and any closed subset $C \subseteq X$, each with the relativized topology, is itself locally compact.

Example 2.15.

1. Let G be any grape with $\tau_d = \mathcal{P}(G)$ the discrete topology. Then (G, τ_d) is a l.c.g.
2. Consider $((\mathbb{R}, +), \tau_{|\cdot|})$ is a l.c.g.

3. If $\{ (G_i, \tau_i) \}_{i\in I}$ are l.c.g.'s, then

$$\left(\prod_{i\in I} G_i, \bigtimes_{i\in I} \tau_i \right)$$

is a l.c.g. if and only if all but finitely many of the (G_i, τ_i) are compact.

In particular, $(\mathbb{R}^n, +)$ with the product topology (equivalently, any norm topology) is a locally compact grape. Also, if $\{ F_i \}_{i\in I}$ is a family of finite grapes, then

$$\prod_{i\in I} F_i$$

(where the F_i is endowed with the discrete topology) is a compact grape and hence a l.c.g.

If $F \subseteq I$ is finite then

$$G_F = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} F_i : x_i = e \text{ for all } i \in F \right\}$$

is an open normal subgrape.

4. We give a construction of the p-adic numbers.

Set construction Fix a prime number p. Let

$$R_p = \prod_{n=0}^{\infty} \mathbb{Z}/p^{n+1}\mathbb{Z}$$

which is a compact ring; i.e. $(x, y) \mapsto x + y$ and $(x, y) \mapsto xy$ are continuous. As a notational convention, we identify $\mathbb{Z}/p^n\mathbb{Z}$ with $\{ 0, 1, \ldots, p^n - 1 \}$. The quotient map $[\cdot]_{p^n} : \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ is a ring homomorphism which factors through $\mathbb{Z}/p^m\mathbb{Z}$ for $m \in \{ 0, \ldots, n \}$. We let

$$\mathbb{O}_p = \{ (x_n)_{n=0}^{\infty} \in R_p : [x_n]_{p^n} = x_{n-1} \text{ for all } n \in \mathbb{N} \}$$

This is clearly a subring of R_p. If $(x^\alpha)_{\alpha\in A} \subseteq \mathbb{O}_p$ is a net converging to $x \in R_p$, then for each $n \in \mathbb{N}$ there is $\alpha_n \in A$ such that for $k \in \{ 0, \ldots, n \}$ we have $x^\alpha_k = x_k$. Thus for $k \in \{ 1, \ldots, n \}$ we have $x_{k-1} = x^\alpha_{k-1} = [x^\alpha_k]_{p^k} = [x_k]_{p^k}$. Hence $x \in \mathbb{O}_p$, so $\mathbb{O}_p$ is closed, and is thus a compact subring of R_p.

Let $\mathbb{1} = (1, 1, \ldots)$, which is the identity in R_p and $\mathbb{O}_p$.

Density of $\mathbb{Z}\mathbb{1}$ (and $\mathbb{N}_0\mathbb{1}$) in $\mathbb{O}_p$ and p-series representations The map $\mathbb{Z} \to \mathbb{O}_p$ given by $m \mapsto m\mathbb{1} = ([m]_p, [m]_{p^2}, \ldots)$ is a ring homomorphism. If $x = (x_n)_{n=0}^{\infty} \in \mathcal{O}_p$ (where $x_n \in \mathbb{Z}/p^{n+1}\mathbb{Z} = \{ 0, \ldots, p^{n+1} - 1 \}$) then

$$x_k\mathbb{1} = ([x_k]_p, \ldots, [x_k]_{p^k}, x_k, x_k, \ldots) \xrightarrow{k\to\infty} x$$

and hence $\overline{\mathbb{N}_0\mathbb{1}} = \mathbb{O}_p$ (where $\mathbb{N}_0 = \{ 0 \} \cup \mathbb{N}$); hence $\overline{\mathbb{Z}\mathbb{1}} = \mathbb{O}_p$. We call $\mathbb{O}_p$ the *ring of p-adic integers*. Notice that if $x = (x_n)_{n=0}^{\infty} \in \mathbb{O}_p$ then each

$$x_n = x_0 + \sum_{k=1}^{n} \frac{x_k - [x_k]_{p^k}}{p^k} p^k = \sum_{k=0}^{n} a_k p^k$$

where each $a_k \in \{ 0, \ldots, p - 1 \}$ is uniquely determined. Hence we may think of

$$x \sim \sum_{k=0}^{\infty} a_k p^k$$

One can check that the map $\mathbb{O}_p \to (\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}_0}$ given by $x \mapsto (a_k)_{k=0}^{\infty}$ is a homeomorphism, though not a homomorphism. (Here the latter is endowed with the product topology.)

Valuation and norm Given $x \in \mathbb{O}_p$, we let

$$v_p(x) = \inf\{\, n \in \mathbb{N}_0 : x_n \neq 0 \,\} = \sup\{\, k \in \mathbb{N}_0 : p^k \mid x_n \text{ for all } n \in \mathbb{N}_0 \,\}$$

We have $v_p(0) = \inf \varnothing = \sup \mathbb{N}_0 = \infty$. We let $|x|_p = p^{-v_p(x)}$ (where $|0|_p = p^{-\infty} = 0$).

Proposition 2.16. *For $x, y \in \mathbb{O}_p$ we have*

(a) $v_p(x) = \infty$ *if and only if* $x = 0$*; i.e.* $|x|_p = 0$ *if and only if* $x = 0$.
(b) $v_p(xy) = v_p(x) + v_p(y)$*; i.e.* $|xy|_p = |x|_p |y|_p$.
(c) $v_p(x+y) \geqslant \min\{\, v_p(x), v_p(y) \,\}$*; i.e.* $|x+y|_p \leqslant \max\{\, |x|_p, |y|_p \,\}$.
(d) $\mathbb{O}_p^\times = \{\, u \in \mathbb{O}_p : u^{-1}$ *exists* $\} \subseteq \{\, u \in \mathbb{O}_p : |u|_p = 1 \,\}$.

Proof.

(a) Obvious.
(b) Notice that by the series representation we have

$$x_n = \begin{cases} 0 & \text{if } n < v_p(x) \\ \sum_{k=v_p(x)}^{n} a_k p^k & \text{if } n \geqslant v_p(x) \end{cases}$$

The result then follows.
(c) Also follows from the series representation.
(d) Notice that if $u \in \mathbb{O}_p^\times$ then

$$0 = v_p(\mathbb{1}) = v_p(uu^{-1}) = v_p(u) + v_p(u^{-1})$$

where $v_p(u), v_p(u^{-1}) \geqslant 0$. Hence $v_p(u) = 0$. ☐ Proposition 2.16

Corollary 2.17. *The map $\rho\colon \mathbb{O}_p \times \mathbb{O}_p \to [0,\infty)$ given by $(x,y) \mapsto |x-y|_p$ is a metric on $\mathbb{O}_p$ with*

$$\tau_\rho = \left(\bigtimes_{n \in \mathbb{N}_0} \tau_d \right) \restriction \mathbb{O}_p$$

(the restriction of the product topology).

Proof. $-\mathbb{1} \in \mathbb{O}_p^\times$, so if $x, y, z \in \mathbb{O}_p$, then

$$\rho(x,z) = |x-z|_p = |x-y+y-z|_p \leqslant \max\{\, |x-y|_p, |y-z|_p \,\} \leqslant \rho(x,y) + \rho(y,z)$$

and $\rho(x,y) = |x-y|_p = |(-\mathbb{1})(y-x)|_p = \rho(y,x)$. Also $\rho(x,y) = 0$ if and only if $x = y$. Finally, note

$$V_\rho(x, p^{-n}) = \{\, x_0 \,\} \times \cdots \times \{\, x_{n-1} \,\} \times \left(\prod_{k=n}^{\infty} \mathbb{Z}/p^{k+1}\mathbb{Z} \cap \mathbb{O}_p \right)$$

with the former a base for τ_ρ at x and the latter a base for the product topology at x. ☐ Corollary 2.17

Proposition 2.18.

(a) $\mathbb{O}_p^\times = \{\, u \in \mathbb{O}_p : |u|_p = \,\}$*; note the latter set is* $\{\, u \in \mathbb{O}_p : u_0 \neq 0 \,\} = \mathbb{O}_p \backslash p\mathbb{O}_p$.
(b) If $x \in \mathbb{O}_p \backslash \{\, 0 \,\}$ *then* $x = p^{v_p(x)} u$ *for some* $u \in \mathbb{O}_p^\times$.

Proof.

(a) The containment $\subseteq$ is given above. For the reverse containment, suppose $u \in \mathbb{O}_p$ with $u_0 \neq 0$. There is a unique $v_0 \in \mathbb{Z}/p\mathbb{Z} \backslash \{\, 0 \,\}$ such that $u_0 v_0 = 1$. Then, since $[u_1]_p = u_0$ we have $\gcd(u_1, p) = 1$; so u_1 is a unit in $\mathbb{Z}/p^2\mathbb{Z}$. Hence there is v_1 in $\mathbb{Z}/p^2\mathbb{Z}$ such that $v_1 u_1 = 1$, and we necessarily have that $[v_1]_p = v_0$ since $[v_1 u_1]_p = 1 = v_0 u_0$; we proceed inductively. We find for each $n \in \mathbb{N}$ a $v_n \in \mathbb{Z}/p^{n+1}\mathbb{Z}$ such that $\gcd(v_n, p) = 1$ and $v_n u_n = 1$; so $[v_n]_{p^n} = v_{n-1}$. Thus $v = (v_n)_{n=0}^{\infty} = u^{-1}$.

(b) This follows from the first part and our series representation of x_n. ☐ Proposition 2.18

Remark 2.19. If $m \in \mathbb{Z}$ with $\gcd(m, p) = 1$, then $m\mathbb{1} \in \mathbb{O}_p^\times$. Hence $\{ \frac{n}{m} : n \in \mathbb{Z}, m \in \mathbb{N}, \gcd(m,p) = 1 \} \subseteq \mathbb{Q}$ is in fact isomorphic to a dense subring of $\mathbb{O}_p$.

Corollary 2.20.

(a) $\mathbb{O}_p^\times$ *is open and closed in* $\mathbb{O}_p$*, and is a topological grape.*

(b) The family of non-trivial ideals, and hence of closed subgrapes of $\mathbb{O}_p$*, is* $p\mathbb{O}_p \supsetneqq p^2 \supsetneqq \mathbb{O}_p \supsetneqq \cdots$.

Proof.

(a) $p\mathbb{O}_p$ is the ρ-open ball around 0 of radius p^{-1}, and is a subgrape. Then $\mathbb{O}_p^\times = \mathbb{O}_p \backslash p\mathbb{O}_p$. It remains to check that $u \mapsto u^{-1}$ is continuous on $\mathbb{O}_p^\times$. If $u, u' \in \mathbb{O}_p$ with $|u - u'|_p = p^{-n}$, then $u_k = u'_k$ for $k \in \{ 0, \dots, n-1 \}$. Thus $|u^{-1} - (u')^{-1}| = p^{-n} = |u - u'|_p$.

(b) $p\mathbb{O}_p = \mathbb{O}_p \backslash \mathbb{O}_p^\times$ is clearly the unique maximal ideal. Using Proposition 2.18, we see that $p^{n+1}\mathbb{O}_p$ is the unique maximal ideal of $p^n\mathbb{O}_p$. Since $\overline{\mathbb{Z}\mathbb{1}} = \mathbb{O}_p$, we see that any closed subgrape is a (closed) ideal. ☐ Corollary 2.20

Remark 2.21. Note that $\mathbb{1} + p^n\mathbb{O}_p$ is an open subgrape of $\mathbb{O}_p^\times$ for $n \in \mathbb{N}$.

p-adic numbers Since $|\cdot|_p$ is multiplicative on $\mathbb{O}_p$ and $|x|_p = 0$ if and only if $x = 0$, we see that $\mathbb{O}_p$ is an integral domain. Hence we may consider the field of quotients

$$\mathbb{Q}_p = \left\{ \frac{x}{y} : x, y \in \mathbb{O}_p, y \neq 0 \right\}$$

with $\frac{x}{y} = \frac{u}{w}$ if and only if $xw = uy$. We have that any $y \in \mathbb{O}_p \backslash \{ 0 \}$ admits form $p^{v_p(y)}u$ for $u \in \mathbb{O}_p^\times$; hence

$$\frac{x}{y} = \frac{xu^{-1}}{p^{v_p(y)}}$$

Thus

$$\mathbb{Q}_p = \left\{ \frac{x}{p^k\mathbb{1}} : x \in \mathbb{O}_p k, k \in \mathbb{N}_0 \right\}$$

Recall that

$$x_n = x_0 + \sum_{k=1}^{n} \frac{x_k - [x_k]_{p^k}}{p^k} p^k$$

so

$$\frac{x_n}{p^m} = \frac{x_0}{p^m} + \sum_{k=1}^{n} \frac{x_k - [x_k]_{p^k}}{p^k} p^{k-m}$$

As before, we may thus write $r \in \mathbb{Q}_p$ as

$$r = \sum_{k=m}^{\infty} a_k p^k$$

for some $m \in \mathbb{Z}$ with each $a_k \in \{ 0, \dots, p-1 \}$. Consider the map

$$\begin{aligned} \mathbb{Q}_p &\to (\mathbb{Z}/p\mathbb{Z})^{\oplus(-\mathbb{N})} \times (\mathbb{Z}/p\mathbb{Z}^{\mathbb{N}_0} \\ r &\mapsto (\dots, 0, 0, a_m, a_{m+1}, \dots) \end{aligned}$$

where

$$(\mathbb{Z}/p\mathbb{Z})^{\oplus(-\mathbb{N})} = \bigoplus_{i \in -\mathbb{N}} \mathbb{Z}/p\mathbb{Z} = \{ (\dots, a_m, a_{m+1}, \dots, a_{-1} : m \in -\mathbb{N}, a_k = 0 \text{ for all but finitely many } k \}$$

is endowed with the discrete topology.

TODO 1. *Something about this being isomorphic to a dense subring?*

Hence $\mathbb{O}_p \subseteq \mathbb{Q}_p$ is an open subgrape, and determines the topology. We have that $\mathbb{Q}_p$ is a *topological field*; i.e. all reasonable field operations are continuous.

5. Suppose $(\mathbb{K}, \tau)$ is a locally compact topological field.

   Aside 2.22. If $\mathbb{F}$ is a finite field, then $\mathbb{F}((X))$ (the ring of Laurent series over $\mathbb{F}$) is a topological field. (Regarded as a subspace of $\mathbb{F}^{\mathbb{Z}}$ with power series operations.)

   TODO 2. *Does this work?*

   Then $\mathrm{GL}_n(\mathbb{K}) = \{ a \in M_n(\mathbb{K}) : \det(a) \neq 0 \}$ is open in $M_n(\mathbb{K}) \cong \mathbb{K}^{n^2}$ and hence locally compact. Multiplication is given by polynomials, and hence is continuous, and inversion is given by Cramer's rule via rational functions, and is thus continuous. Thus $\mathrm{GL}_n(\mathbb{K})$ is a locally compact grape.

6. $\mathrm{SL}_n(\mathbb{K}) = \{ a \in M_n(\mathbb{K}) : \det(a) = 1 \}$ is closed in $M_n(\mathbb{K}) = \mathbb{K}^{n^2}$, and hence is locally compact; it is a locally compact grape. Also $O_n(\mathbb{K}) = \{ u \in M_n(\mathbb{K}) : uu^T = e \}$ is a closed subgrape. (Note that $uu^T = e$ is given by polynomial equations.)

7. $U(n) = \{ u \in M_n(\mathbb{C}) : uu^* = e \}$ is a closed subgrape of $\mathrm{GL}_n(\mathbb{C})$. It is bounded, hence compact (by Heine-Borel).

3 Haar integral and Haar measures

Let G be a locally compact grape. If $f \colon G \to \mathbb{C}$, let $f \cdot x, x \cdot f \colon G \to \mathbb{C}$ be $(f \cdot x)(y) = f(xy)$ and $(x \cdot f)(y) = f(yx)$. (We write $f \cdot x(y)$ to mean $(f \cdot x)(y)$.) Notice if $x, x' \in G$ and $y \in G$, then $(f \cdot (xx'))(y) = f(xx'y) = (f \cdot x)(x'y) = ((f \cdot x) \cdot x')(y)$; i.e. $f \cdot (xx') = (f \cdot x) \cdot x'$. Likewise we get $(xx') \cdot f = x \cdot (x' \cdot f)$.

Let $C_c(G) = \{ f \colon G \to \mathbb{C} \mid f \text{ continuous}, \operatorname{supp}(f) = \overline{\{ x \in G : f(x) \neq 0 \}} \text{ compact} \}$. We call this the linear space of *compactly supported functions* on G. Thanks to Urysohn's lemma, we get $C_c(G) \supsetneqq \{ 0 \}$. By Tietze's extension theorem, given $K, E \subseteq G$ with K compact, E closed, and $K \cap E = \varnothing$, we have that there is $f \in C^+(G) = \{ f \in C_c(G) \setminus \{ 0 \} : f(x) \geqslant 0 \text{ for all } x \in G \}$ such that $f{\restriction}K =$ and $f{\restriction}E = 0$. (This is a strong form of "regularity".)

Exercise 3.1. Prove this in a locally compact metric space.

Proposition 3.2. *If $f \in C_c(G)$ then*

$$\lim_{x \to e} \| f \cdot x - f \|_\infty = 0 = \lim_{x \to e} \| x \cdot f - f \|_\infty$$

In this case we say that f is (left and right) uniformly continuous*.*

Proof. Suppose $\varepsilon > 0$. Let $K = \overline{W} \operatorname{supp}(f)$ where $W = W^{-1}$ is a relatively compact neighbourhood of e. For each $y \in K$ we have $|y \cdot f - f(y)1| \colon G \to \mathbb{C}$ (where 1 is the constant function) is continuous with value 0 at e; hence there is a neighbourhood U_y of e such that

$$|f(xy) - f(y)| = |y \cdot f(x) - f(y)| < \varepsilon$$

for $x \in U_y$. Find a neighbourhood $V_y = V_y^{-1}$ of e such that $V_y^2 \subseteq U_y$. Then

$$K \subseteq \bigcup_{y \in K} V_y y$$

so

$$K \subseteq \bigcup_{j=1}^{n} V_{y_j} y_j$$

Let

$$V = W \cap \bigcap_{j=1}^{n} V_y$$

so $e \in V$ and $V^{-1} = V$. Suppose now $x \in V$. If $y \in K$ then $y \in V_{y_j} y_j \subseteq U_{y_j} y_j$ for some j; in particular, we have $yy_j^{-1} \in V_y$. Thus

$$xy = xyy_j^{-1} y_j \in VV_{y_j} y_j \subseteq V_{y_j}^2 y_j \subseteq U_{y_j} y_j$$

Hence by our choice of U_{y_j} we have

$$|f(xy) - f(y)| \leqslant |f(xy) - f(y_j)| + |f(y_j) - f(y)| < 2\varepsilon$$

If $y \notin K$, suppose we had $W_y \cap \operatorname{supp}(f) \neq \varnothing$. Then there would be $z \in W_y \cap \operatorname{supp}(f)$; so $z = wy$ for some $w \in W$, and hence $y = w^{-1} z \in W \operatorname{supp}(f) \subseteq K$, a contradiction. So $Wy \cap \operatorname{supp}(f) = \varnothing$. Hence if $x \in V \subseteq W$ we would have $f(xy) = 0 = f(y)$, so $|f(xy) - f(y)| < \varepsilon$. ☐ Proposition 3.2

Theorem 3.3 (Existence of the left Haar integral)**.** *There exists a (linear) functional $I \colon C_c(G) \to \mathbb{C}$ satisfying:*

1. *$I(f) > 0$ if $f \in C_c^+(G) = \{ g \in C_c(G) \backslash \{0\} : g(x) \geqslant 0$ for all $x \in G \}$.*
2. *$I(f \cdot x) = I(f)$ for all $f \in C_c(G)$ and $x \in G$.*

Proof. We give a construction in stages.

1. Fix φ in $C_c^+(G)$. Then for f in $C_c^+(G)$, we let

$$(f : \varphi) = \inf \left\{ \sum_{j=1}^{n} c_j : \text{ there exist } x_1, \ldots, x_n \in G, c_1, \ldots, c_n > 0, n \in \mathbb{N} \text{ such that } f \leqslant \sum_{j=1}^{n} \varphi \cdot x_j \right\}$$

Notice that if $U = \{ x \in G : \varphi(x) > \frac{1}{2} \|\varphi\|_\infty \}$, we see that $\operatorname{supp}(f)$ is covered by finitely many translates $x^{-1}U$; it follows that $(f : \varphi) < \infty$.

Claim 3.4. *For $f, g \in C_c^+(G)$ and $c > 0$ we have the following:*

(a) $(f \cdot x : \varphi) = (f : \varphi)$.
(b) $(f + g : \varphi) \leqslant (f : \varphi) + (g : \varphi)$.
(c) $(cf : \varphi) = c(f : \varphi)$.
(d) $f \leqslant g \implies (f : \varphi) \leqslant (g : \varphi)$.
(e) $(f : \varphi) \leqslant (f : g)(g : \varphi)$.

Proof. The first four are straightforward; we sketch the last. If

$$f \leqslant \sum_{j=1}^{n} c_j g \cdot x_j$$
$$g \leqslant \sum_{i=1}^{m} b_i \varphi \cdot y_i$$

for $c_j, b_i > 0$ and $x_j, y_i \in G$, then

$$f \leqslant \sum_{j=1}^{n} \sum_{i=1}^{m} c_j b_i \varphi \cdot (y_i x_j)$$

and hence

$$(f : \varphi) \leqslant \sum_{j=1}^{n} c_j \sum_{i=1}^{m} b_i$$

and the result follows. ☐ Claim 3.4

Now, fix another $\psi \in C_c^+(G)$, and for $f \in C_c^+(G)$ let

$$I_\varphi(f) = \frac{(f : \varphi)}{(\psi : \varphi)}$$

Then the first three properties tell us that $I_\varphi \colon C_c^+(G) \to [0, \infty)$ is left translation-invariant, subadditive, and $\mathbb{R}^{>0}$-homogeneous. Furthermore, the last property yields that

$$(\psi : \varphi) \leqslant (\psi : f)(f : \varphi)$$
$$(f : \varphi) \leqslant (f : \psi)(\psi : \varphi)$$

whence it follows that

$$0 < \frac{1}{(\psi : f)} \leqslant I_\varphi(f) \leqslant (f : \psi) \tag{1}$$

2. A somewhat technical claim:

Claim 3.5. *If $f, g \in C_c^+(G)$ and $\varepsilon > 0$ then there is a neighbourhood V of e such that $I_\varphi(f) + I_\varphi(g) \leqslant I_\varphi(f + g) + \varepsilon$ whenever* $\operatorname{supp}(f) \subseteq V$.

Proof. Let $k \in C_c^+(G)$ satisfy $k \restriction \operatorname{supp}(f+g) = 1$; let $\delta > 0$, and set $h = f + g + \delta k$. We then let

$$f' = \frac{f}{h}$$
$$g' = \frac{g}{h}$$

(with each of them 0 outside of the supports of f, g). Then by Proposition 3.2 applied to f', g' we get a neighbourhood V of e such that

$$|f'(x) - f'(y)| < \delta, |g'(x) - g'(y)| < \delta \tag{2}$$

whenever $y^{-1}x \in V$. Suppose $\varphi \in C_c^+(G)$ with $\operatorname{supp}(\varphi) \subseteq V$; suppose $x_1, \ldots, x_n \in G$ and $c_1, \ldots, c_n > 0$ satisfy

$$h \leqslant \sum_{j=1}^{n} c_j \varphi \cdot x_j^{-1}$$

Then for $x \in G$ we have

$$f(x) = f'(x)h(x) \leqslant \sum_{j=1}^{n} f'(x) c_j \varphi(x_j^{-1}x) \leqslant \sum_{j=1}^{n} (f'(x_j) + \delta) c_j \varphi_j(x_j^{-1}x)$$

where the last inequality follows from the choice of φ and (2). Likewise we see that

$$g \leqslant \sum_{j=1}^{n} (g'(x_j) + \delta) c_j \varphi \cdot x_j^{-1}$$

Now

$$f' + g' = \frac{f+g}{h} = \frac{f+g}{f+g+\delta k} \leqslant 1$$

So

$$\begin{aligned}(f \cdot \varphi) + (g : \varphi) &\leqslant \sum_{j=1}^{n} (f'(x_j) + \delta) c_j + \sum_{j=1}^{n} (g'(x_j) + \delta) c_j \\ &\leqslant \sum_{j=1}^{n} (1 + 2\delta) c_j\end{aligned}$$

Recall that our ψ is fixed. Now, dividing by $(\psi : \varphi)$ and taking infimum in the c_j relative to the definition of $(h : \varphi)$, and applying Claim 3.4, we see that

$$I_\varphi(f) + I_\varphi(g) \leqslant (1+2\delta)I_\varphi(h) \leqslant (1+2\delta)(I_\varphi(f+g) + \delta I_\varphi(k))$$

Now, choose $\delta > 0$ (and hence V) small enough so that

$$2\delta I_\varphi(f+g) + (1+2\delta)\delta I_\varphi(k) < \varepsilon$$

and the claim follows. □ Claim 3.5

3. We are now ready to draw our conclusion. Consider

$$X = \prod_{f \in C_c^+(G)} \left[\frac{1}{(\psi : f)}, (\varphi : f)\right]$$

which is compact by Tychonoff's theorem. By Equation (1) we get $(I_\varphi(f))_{f\in C_c^+(G)} \in X$ for any $\varphi \in C_c^+(G)$.

Given a neighbourhood V of e we let

$$K(V) = \overline{\left\{ (I_\varphi(f))_{f\in C_c^+(G)} : \operatorname{supp}(\varphi) \subseteq V \right\}} \subseteq X$$

Then K is a closed set of a compact space, and is thus compact. Then if $V_1, \dots, V_n$ are neighbourhoods of e, then

$$\bigcap_{j=1}^n K(V_j) \supseteq K\left(\bigcap_{j=1}^n V_j\right) \neq \varnothing$$

Thus $S = \bigcap\{ K(V) : V \text{ a neighbourhood of } e \} \neq \varnothing$ by finite intersection property; let $(I(f))_{f\in C_c^+(G)} \in S$. Given $f, g \in C_c^+(G)$ and $\varepsilon > 0$ there is a neighbourhood V of e and $\varphi \in C_c^+(G)$ with $\operatorname{supp}(\varphi) \subseteq V$ such that

$$|I(f) - I_\varphi(f)| < \varepsilon$$
$$|I(g) - I_\varphi(g)| < \varepsilon$$
$$|I(f+g) - I_\varphi(f+g)| < \varepsilon$$

and further by Claim 3.5 and Claim 3.4 we can arrange V such that

$$|I_\varphi(f) + I_\varphi(g) - I_\varphi(f+g)| < \varepsilon$$

We then find that

$$|I(f) + I(g) - I(f+g)| < 4\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we find that $I\colon C_c^+(G) \to (0, \infty)$ is an additive functional. By Claim 3.4, we get that I is $\mathbb{R}^{>0}$-homogeneous.

We now extend I to all of $C_c(G)$. We set $I(0) = 0$. Suppose $f \in C_c^{\mathbb{R}}(G)$ (i.e. it is real-valued) and we can write $f = f_1 - f_2 = g_1 - g_2$ for $f_1, f_2, g_1 g_2 \geqslant 0$. Then $f_1 + g_2 = g_1 + f_2$, so $I(f_1 + g_2) = I(g_+ f_2)$, and by additivity we get that $I(f) = I(f_1) - I(f_2) = I(g_1) - I(g_2)$ is well-defined. This clearly is $\mathbb{R}$-homogeneous. Now for arbitrary $f \in C_c(G)$, we let

$$I(f) = I(\operatorname{Re} f) + iI(\operatorname{Im} f)$$

It is straightforward to check that I is $\mathbb{C}$-homogeneous. It then follows from Claim 3.4 and the definition of S that $I(f \cdot x) = I(f)$ for $f \in C_c^+(G)$ and $x \in G$. Hence this left-invariance holds generally. Finally, for $f \in C_c^+(G)$, we have $I(f) > 0$ by definition of $S \subseteq X$. □ Theorem 3.3

Theorem 3.6 (Existence of left Haar measure)**.** *Let $\mathcal{B}(G) = \sigma\langle\tau\rangle$ (the σ-algera on G generated by open sets) be the* Borel σ-algebra. *Then there is a measure $m \colon \mathcal{B}(G) \to [0,\infty]$ satisfying the following:*

1. *m is a* Radon measure*: it is outer regular ($m(E)$ is the infimum of the measures of the open sets containing E), inner regular on open sets ($m(E)$ is the supremum of the measures of compact sets contained in E, if E is open), finite on compact sets.*
2. *m is left-invariant: if $E \in \mathcal{B}(G)$ and $x \in G$ then $m(xE) = m(E)$.*
3. *$m(U) > 0$ for any $U \in \tau\backslash\{\varnothing\}$.*

Sketch of proof. The Riesz representation theorem provides a Radon measure m for which

$$I(f) = \int_G f \mathrm{d}m$$

for all $f \in C_c(G)$. We have for $x \in G$ that

$$\int_G f(xy)\mathrm{d}m(y) = I(f \cdot x) = I(f) = \int_G f\mathrm{d}m$$

In particular, if U is open then for $f \in C_c(G)$ we have $\operatorname{supp}(f) \subseteq U$ if and only if $\operatorname{supp}(f \cdot x) \subseteq x^{-1}U$, so

$$\begin{aligned} m(U) &= \sup\{\, I(f) : f \in C_c^{[0,1]}(G), \operatorname{supp}(f) \subseteq U \,\} \\ &= \sup\{\, I(f\cdot x) : f \in C_c^{[0,1]}(G), \operatorname{supp}(f\cdot x) \subseteq x^{-1}U \,\} \\ &= m(x^{-1}U) \end{aligned}$$

So we see that $m(U) = m(xU)$ for $x \in G$. Then if $E \in \mathcal{B}(G)$ we have

$$m(E) = \inf\{\, m(U) : E \subseteq U \in \tau \,\}$$

and it follows that $m(xE) = m(E)$. That $m(U) > 0$ for $U \in \tau\backslash\{\varnothing\}$ follows from

$$m(U) = \sup\{\, I(f) : f \in C_c^{[0,1]}(G), \operatorname{supp}(f) \subseteq U \,\}$$

and that $I(f) > 0$ for $f \in C_c^+(G)$. ☐ Theorem 3.6

Theorem 3.7 (“Uniqueness” of left Haar measure)**.** *If $m' \colon \mathcal{B}(G) \to [0,\infty]$ is a left-invariant measure, then there is $c \geqslant 0$ such that $m' = cm$.*

Proof. It suffices to show that the map

$$f \mapsto \frac{\int_G f\mathrm{d}m'}{\int_G f\mathrm{d}m}$$

is constant for f in $C_c^+(G)$. This constant $c \geqslant 0$ hence satisfies that

$$\int_G f\mathrm{d}m' = c\int_G f\mathrm{d}m$$

and it will follow that $m' = cm$. To this end, fix $f, g \in C_c^+(G)$ and $\varepsilon > 0$. By uniform continuity of f and g there is a neighbourhood $V = V^{-1}$ of e such that

$$\begin{aligned} |f(xy) - f(yx)| &< \varepsilon \\ |g(xy) - g(yx)| &< \varepsilon \end{aligned}$$

for $x \in V, y \in G$. Fix $h \in C_c^+(G)$ satisfying $h(x^{-1}) = h(x)$ for $x \in G$ and $\operatorname{supp}(h) \subseteq V$. (One could for example pick $h' \in C_c^+(G)$ with $\operatorname{supp}(h') \subseteq V$ and let $h(x) = h'(x) + h'(x^{-1})$.) We use Tonelli's theorem:

$$\int h\mathrm{d}m \int f\mathrm{d}m' = \int\int h(x)f(y)\mathrm{d}m(x)\mathrm{d}m'(y) = \int\int h(x)f(xy)\mathrm{d}m(x)\mathrm{d}m'(y)$$

and

$$\begin{aligned}\int h\mathrm{d}m' \int f\mathrm{d}m &= \int\int h(y)f(x)\mathrm{d}m'(y)\mathrm{d}m(x)\\ &= \int\int h(x^{-1}y)f(x)\mathrm{d}m'(y)\mathrm{d}m(x)\\ &= \int\int h(x^{-1}y)f(x)\mathrm{d}m(x)\mathrm{d}m'(y)\\ &= \int\int h(x^{-1})f(yx)\mathrm{d}m(x)\mathrm{d}m'(y)\\ &= \int\int h(x)f(yx)\mathrm{d}m(x)\mathrm{d}m'(y)\end{aligned}$$

Thus

$$\begin{aligned}\left|\int h\mathrm{d}m \int f\mathrm{d}m' - \int h\mathrm{d}m' \int f\mathrm{d}m\right| &\leqslant \int\int h(x)\underbrace{|f(xy)-f(yx)|}_{<\varepsilon}\mathrm{d}m'(y)\mathrm{d}m(x)\\ &\leqslant \varepsilon m'(\underbrace{V\operatorname{supp}(f)\cup\operatorname{supp}(f)V}_{S_{f,V}})\int h\mathrm{d}m\end{aligned}$$

So

$$\left|\frac{\int f\mathrm{d}m'}{\int f\mathrm{d}m} - \frac{\int h\mathrm{d}m'}{\int h\mathrm{d}m}\right| \leqslant \varepsilon\frac{m'(S_{f,V})}{\int f\mathrm{d}m}$$

Likewise we get

$$\left|\frac{\int g\mathrm{d}m'}{\int g\mathrm{d}m} - \frac{\int h\mathrm{d}m'}{\int h\mathrm{d}m}\right| \leqslant \varepsilon\frac{m'(S_{g,V})}{\int g\mathrm{d}m}$$

so

$$\left|\frac{\int f\mathrm{d}m'}{\int f\mathrm{d}m} - \frac{\int g\mathrm{d}m'}{\int g\mathrm{d}m}\right| \leqslant \varepsilon\left(\frac{m'(S_{f,V})}{\int f\mathrm{d}m} + \frac{m'(S_{g,V})}{\int g\mathrm{d}m}\right)$$

Notice though that if $V' \subseteq V$ then $S_{f,V'} \subseteq S_{f,V}$; thus if we shrink $\varepsilon > 0$ we shrink V. ☐ Theorem 3.7

TODO 3. *Missing stuff*

Last time: introduced $L^1(G) = \overline{S^1(G)}^{\|\cdot\|_1} = \overline{C_c(G)}^{\|\cdot\|_1}$ the closure of the simple integrable functions. (The latter equality because m is regular on open sets.)

4 The modular function

Given $E \in \mathcal{B}(G)$ we have that $Ex \in \mathcal{B}(G)$ for $x \in G$. (Since $R_x\colon G \to G$ is a homeomorphism and $Ex = R_{x^{-1}}^{-1}(E)$.) Define $m_x\colon \mathcal{B}(G) \to [0,\infty]$ by $m_x(E) = m(Ex)$. One can check that m_x is left-invariant and positive on open sets. Hence by Theorem 3.7 we get $m_x = \Delta(x)m$ for some $\Delta(x) \in (0,\infty)$.

Notice that if $y \in G$ then for E with $0 < m(E) < \infty$ we get

$$\Delta(xy)m(E) = m(Exy) = \Delta(y)m(Ex) = \Delta(x)\Delta(y)m(E)$$

so $\Delta\colon G \to (0,\infty) \subseteq \mathbb{R}^\times$ is a homomorphism.

Definition 4.1. We call this the *modular function*. We say G is *unimodular* if $\Delta = 1$.

Proposition 4.2.

1. *For $f \in L^1(G)$ (or $f \in C_c(G)$) we have for $x \in G$ that*

$$\int_G f\mathrm{d}m = \Delta(x)\int_G x\cdot f\mathrm{d}m$$

2. $\Delta\colon G \to (0,\infty) \subseteq \mathbb{R}^\times$ *is continuous.*

Proof.

1. If $E \in \mathcal{B}(G)$ with $m(E) < \infty$ then

$$\Delta(x)\int_G 1_E \mathrm{d}m = \Delta(x)m(E) = m(Ex) = \int_G 1_{Ex}\mathrm{d}m = \int_G x^{-1}\cdot 1_E \mathrm{d}m$$

So, replacing x by x^{-1}, we see that

$$\Delta(x)\int_G x\cdot 1_E \mathrm{d}m = \int_G 1_E \mathrm{d}m$$

Then, if $\varphi \in S^1(G)$, then

$$\int_G \varphi \mathrm{d}m = \Delta(x)\int_G x\cdot\varphi \mathrm{d}m$$

Now, if $f \in L^1_+(G)$ (i.e. $f \geqslant 0$ m-almost-everywhere), then there is $(\varphi_n)_{n=1}^\infty \subseteq S^1_+(G)$ such that $\varphi_n \nearrow f$ (increasing pointwise converges) m-almost-everywhere. Then by monotone convergence theorem we get

$$\int_G x\cdot f\mathrm{d}m = \lim_{n\to\infty}\int_G x\cdot\varphi_n \mathrm{d}m = \lim_{n\to\infty}\frac{1}{\Delta(x)}\int_G \varphi_n\mathrm{d}m = \frac{1}{\Delta(x)}\int_G f\mathrm{d}m$$

We are now done, since $L^1(G) = \mathrm{span}(L^1_+(G))$.

2. Suppose $f \in C_c^+(G)$, $\varepsilon > 0$, and $V = V^{-1}$ is a relatively compact neighbourhood of e such that $\|x\cdot f - f\|_\infty < \varepsilon$ for $x \in V$. Then for $x \in V$ we have

$$|\Delta(x) - 1| = \frac{\left|\int_G x\cdot f\mathrm{d}m - \int_G f\mathrm{d}m\right|}{\int_G f\mathrm{d}m} \leqslant \frac{\int_G |x\cdot f - f|\mathrm{d}m}{\int_G f\mathrm{d}m} \leqslant \varepsilon\frac{m(\mathrm{supp}(f)V)}{\int_G f\mathrm{d}m}$$

Picking $\varepsilon' < \varepsilon$ necessitates taking $V' \subseteq V$, so we see that Δ is continuous at e. Now if $y \in G$ and $x \in V$ then

$$|\Delta(xy) - \Delta(y)| = |\Delta(x) - 1|\Delta(y) \leqslant \varepsilon\Delta(y)$$

so Δ is continuous at y. ☐ Proposition 4.2

Notation 4.3. For the left integral we write

$$\int_G f(x)\mathrm{d}x$$

or less commonly

$$\int_G f(x)\mathrm{d}m(x)$$

to mean

$$\int_G f\mathrm{d}m$$

Proposition 4.4.

1. *The integral on $C_c(G)$ given by*

$$f \mapsto \int_G f(x)\frac{1}{\Delta(x)}\mathrm{d}x$$

is right-invariant.

2. *For $f \in L^1(G)$ we have*

$$\int_G f(x^{-1})\frac{1}{\Delta(x)}\mathrm{d}x = \int_G f(x)\mathrm{d}x$$

Proof.

1. If $y \in G$ and $f \in C_c(G)$ we have

$$\int_G y \cdot f(x) \frac{1}{\Delta(x)} \mathrm{d}x = \int_G f(xy) \frac{1}{\Delta(xy)} \Delta(y) \mathrm{d}x = \frac{\Delta(y)}{\Delta(y)} \int_G f(x) \frac{1}{\Delta(x)} \mathrm{d}x = \int_G f(x) \frac{1}{\Delta(x)} \mathrm{d}x$$

2. We have for $f \in C_c^+(G)$ and $y \in G$ that

$$0 < \int_G f \cdot y(x^{-1}) \frac{1}{\Delta(x)} \mathrm{d}x = \int_G f(yx^{-1}) \frac{1}{\Delta(x)} \mathrm{d}x = \int_G f((xy^{-1})^{-1}) \frac{1}{\Delta(x)} \mathrm{d}x = \int_G f(x^{-1}) \frac{1}{\Delta(x)} \mathrm{d}x$$

   by the first part. (Notice that $\iota \colon G \to G$ given by $x \mapsto x^{-1}$ is a homeomorphism, and hence Borel measurable, so $f \circ \iota$ is Borel measurable if f is.) Hence there is $c > 0$ such that

$$\int_G f(x^{-1}) \frac{1}{\Delta(x)} \mathrm{d}x = c \int_G f(x) \mathrm{d}x$$

   for $f \in C_c(G)$ (and hence $f \in L^1(G)$).

   Now, if $c \neq 1$ then there is a relatively compact neighbourhood $U = U^{-1}$ of e such that

$$\left| \frac{1}{\Delta(x)} - 1 \right| < \frac{1}{2} |c - 1|$$

   for $x \in U$. Then

$$\begin{aligned}
0 &= \left| \int_G \underbrace{1_U(x)}_{=1_U(x^{-1})} \frac{1}{\Delta(x)} \mathrm{d}x - c \int_G 1_U(x) \mathrm{d}x \right| \\
&= \left| \int_U \left(\frac{1}{\Delta(x)} - c \right) \mathrm{d}x \right| \\
&= \left| \int_U \left(1 - c + \frac{1}{\Delta(x)} - 1 \right) \mathrm{d}x \right| \\
&= \left| (1-c) m(U) + \int_U \left(\frac{1}{\Delta(x)} - 1 \right) \mathrm{d}x \right| \\
&\geqslant (1-c) m(U) - \left| \int_U \left(\frac{1}{\Delta(x)} - 1 \right) \mathrm{d}x \right| \\
&> m(U) \left(|1 - c| - \frac{1}{2} |c - 1| \right) \\
&= \frac{1}{2} |c - 1| m(U) \\
&> 0
\end{aligned}$$

   a contradiction. So $c = 1$. ☐ Proposition 4.4

Notation 4.5. If $x \in G$ and $f \in L^1(G)$, we define $x * f, f * x, f^* \in L^1(G)$ by declaring for m-almost-every y that

$$\begin{aligned}
x * f(y) &= f(x^{-1} y) \\
f * x(y) &= f(yx^{-1}) \frac{1}{\Delta(x)} \\
f^*(y) &= \overline{f(y^{-1})} \frac{1}{\Delta(y)}
\end{aligned}$$

The last proposition then tells us that

$$\|f\|_1 = \int_G |f(x)|\mathrm{d}x = \|x * f\|_1 = \|f * x\|_1 = \|f^*\|_1$$

Notice that

$$\begin{aligned} x * (y * f) &= (xy) * f \\ (f * x) * y &= f * (xy) \\ (f * x)^* &= x^{-1} * f \\ (f^*)^* &= f \\ x * f &= f \cdot x^{-1} \end{aligned}$$

Proposition 4.6. *For $f \in L^1(G)$ we have*

$$\lim_{x \to e} \|x * f - f\|_1 = 0 = \lim_{x \to e} \|f * x - f\|_1$$

Proof. First, consider $g \in C_c(G)$. Suppose $\varepsilon > 0$; let $V = V^{-1}$ be a relatively compact neighbourhood of e such that

$$\begin{aligned} \|x \cdot g - g\|_\infty &< \varepsilon \\ \left| \frac{1}{\Delta(x)} - 1 \right| &< \varepsilon \end{aligned}$$

for all $x \in V$. Then

$$\begin{aligned} \|g * x - g\|_1 &\leqslant \|g * x - g\|_\infty m(\mathrm{supp}(g)V) \\ &\leqslant \left(\frac{1}{\Delta(x)} \|x^{-1} \cdot g - g\|_\infty + \left| \frac{1}{\Delta(x)} - 1 \right| \|g\|_\infty \right) m(\mathrm{supp}(g)V) \\ &\leqslant \big((1+\varepsilon)\varepsilon + \varepsilon \|g\|_\infty \big) m(\mathrm{supp}(g)V) \end{aligned}$$

So we're done. Now if$f \in L^1(G)$ and $\varepsilon > 0$, we can find $g \in C_c(G)$ such that $\|f - g\|_1 < \varepsilon$; it then follows by the usual estimates that

$$\limsup_{x \to e} \|f * x - f\|_1 < 3\varepsilon$$

and so, as $\varepsilon > 0$ is arbitrary, we get the limit, as desired. □ Proposition 4.6

Theorem 4.7 (Weil's integral relation)**.** *Let N be a closed normal subgrape of G.*

1. *If $f \in C_c(G)$ then the map $x \mapsto \int_N f(xn)\mathrm{d}n$ is constant of cosets of N, and hence defines a map $T_N f$ on G/N. Furthermore $T_N f \in C_c(G)$, and the operator $T_N \colon C_c(G) \to C_c(G/N)$ satisfies*
   (a) *$T_N(C_c^+(G)) \subseteq C_c^+(G/N)$*
   (b) *$T_N(f \cdot y) = (T_N f) \cdot (yN)$ for $y \in G$.*
2. *The functional*

$$f \mapsto \int_{G/N} T_N f(xN)\mathrm{d}xN$$

is a left Haar integral. Hence we may write

$$\int_{G/N} \int_N f(xn)\mathrm{d}n\mathrm{d}xN$$

(Notice that the constant on m_G is thus dictated by choices of m_N and $m_{G/N}$.)

Proof.

1. Notice that if $n' \in N$ then
$$\int_N f(xn'n)\mathrm{d}n = \int_N f(xn)\mathrm{d}n$$
Hence we get a function $T_N f\colon G/N \to \mathbb{C}$.

We check continuity on G/N. Suppose $\varepsilon > 0$, fix $V = V^{-1}$ a relatively compact neighbourhood of e; so $\|f \cdot y - f\|_\infty < \varepsilon$ for $y \in V$. Then fix $x \in G$ and $h \in C_c^{[0,1]}(G)$ with $h \upharpoonright V x^{-1} \operatorname{supp}(f) = 1$. Then for $y \in V$ (so $yN \in q_N(V)$ where $q_N\colon G \to G/N$ is the quotient map) we have
$$|T_N f(\underbrace{yxN}_{yNxN}) - T_N f(xN)| = \left|\int_N (f(yxn) - f(xn))\mathrm{d}n\right| \leqslant \int_N |f(yxn) - f(xn)| h(n)\mathrm{d}n \leqslant \varepsilon m_N(\operatorname{supp}(h) \cap N)$$
which shows continuity since if $\varepsilon' < \varepsilon$ we can build h with smaller support. So $T_N f$ is continuous. Also $\operatorname{supp}(T_N f) \subseteq q_N(\operatorname{supp}(f))$ is compact, so $T_N f \in C_c(G/N)$.

If $f \in C_c^+(G)$ has $f(x) > 0$ for some $x \in G$, we can find an open neighbourhood U of e such that $f(xy) > \frac{1}{2} f(x)$ for $y \in U$. Then
$$T_N f(xN) = \int_N f(xn)\mathrm{d}n \geqslant \int_{U \cap N} \frac{1}{2} f(x)\mathrm{d}n = \frac{1}{2} f(x) m_N(U \cap N) > 0$$
(Clearly $f(xN) \geqslant 0$ for general x.) Finally
$$T_N(f \cdot y)(xN) = \int_N f \cdot y(xn)\mathrm{d}n = \int_N f(yxn)\mathrm{d}n = T_N f(yxN) = (T_N f) \cdot (yN)(xN)$$

2. Follows from the first part immediately. ☐ Theorem 4.7

Corollary 4.8. *The modular functions on G and N satisfy $\Delta_N = \Delta_G \upharpoonright N$.*

Proof. If $n' \in N$ and $f \in C_c^+(G)$ then
$$\begin{aligned}
\int_G n' \cdot f(x)\mathrm{d}x &= \int_{G/N} \int_N n' \cdot f(xn)\mathrm{d}n\mathrm{d}xN \\
&= \int_{G/N} \int_N f(xnn')\mathrm{d}n\mathrm{d}xN \\
&= \int_{G/N} \frac{1}{\Delta_N(n')} \int_N f(xn)\mathrm{d}n\mathrm{d}xN \\
&= \frac{1}{\Delta_n(n')} \int_G f(x)\mathrm{d}x
\end{aligned}$$
so $\Delta_n(n') = \Delta_G(n')$. ☐ Corollary 4.8

Unimodularity makes computing integrals simpler. Indeed,
$$\int_G f(x)\mathrm{d}x = \int_G f(yx)\mathrm{d}x = \int_G f(xy)\mathrm{d}x = \int_G f(x^{-1})\mathrm{d}x$$

Proposition 4.9. *G is unimodular in the following cases:*

1. *G is abelian, compact, or discrete*
2. *G is* perfect*: i.e. $G = \overline{[G,G]}$ (the closure of the grape generated by the commutators $[x,y] = xyx^{-1}y^{-1}$).*
3. *$G/Z(G)$ is unimodular ($Z(G)$ is the centre.*
4. *G admits a unimodular closed normal subgrape N for which G/N is compact.*

Proof.

1. Trivial for G abelian; for G compact, the (left) Haar measure is the counting measure.

   Let us fully consider the compact case. Here $\Delta(G)$ is a compact subgrape of $(0, \infty) \subseteq \mathbb{R}^\times$. The map $\log\colon (0,\infty) \to \mathbb{R}$ is an isomorphism. If $\alpha \in \mathbb{R}\backslash\{0\}$ then $\mathbb{Z}\alpha$ is not compact. Hence $\{0\}$ is the only compact subgrape of $\mathbb{R}$, and hence $\{1\}$ is the only compact subgrape of $(0,\infty)$.

2. It is clear that $\Delta([x_1, y_1]\cdots[x_n, y_n]) = 1$; by continuity, we then get $\Delta(x) = 1$ for all $x \in G$.

3. We should note that $Z = Z(G)$ is closed and normal. If $y \in G$ and $f \in C_c(G)$ then

$$\begin{aligned}
\int_G y \cdot f(x)\mathrm{d}x &= \int_{G/Z}\int_Z y \cdot f(xz)\mathrm{d}z\mathrm{d}xZ \\
&= \int_{G/Z}\int_Z f(xzy)\mathrm{d}z\mathrm{d}xZ \\
&= \int_{G/Z}\int_Z f(xyz)\mathrm{d}z\mathrm{d}xZ \\
&= \int_{G/Z} T_Z f(xZyZ)\mathrm{d}xZ \\
&= \int_{G/Z} T_Z f(xZ)\mathrm{d}xZ \\
&= \int_G f(x)\mathrm{d}x
\end{aligned}$$

   Hence $\Delta(y) = 1$.

4. Since $\Delta_G{\restriction}N = \Delta_N = 1$, we get a homomorphism $\overline{\Delta}\colon G/N \to (0,\infty)$ (by 1st isomorphism theorem) with $\overline{\Delta} \circ q_N = \Delta_G$. If $W \subseteq (0,\infty)$ is open, then

$$\overline{\Delta}^{-1}(W) = \underbrace{q_N}_{\text{open map}}(\underbrace{\Delta^{-1}(W)}_{\text{open in } G})$$

   Thus $\overline{\Delta}$ is continuous. By (1), we get that $\overline{\Delta}(G/N) = \{1\}$. ☐ Proposition 4.9

Example 4.10.

1. Suppose $\mathbb{K}$ is a locally compact field. Let $|\mathbb{K}| > 3$. (Aside: we will use capital letters for singular matrices and lower-case for invertible matrices.) Let $\{E_{ij}\}_{i,j=1}^n$ be the matrix unit for $M_n(\mathbb{K})$: i.e. $E_{ij}E_{k\ell} = \delta_{jk}E_{i\ell}$. We will show that $\mathrm{SL}_n(\mathbb{K})$ is perfect, and hence unimodular.

   (a) If $\lambda \in \mathbb{K}$ and i, j, k are distinct (for $n \geqslant 3$) then

$$[e + \lambda E_{ik}, e + E_{kj}] = (e + \lambda E_{ik})(e + E_{kj})(e - \lambda E_{ik})(e - E_{kj}) = e + \lambda E_{ij}$$

   If $n = 2$ we have
$$\left[\begin{pmatrix}\alpha & 0 \\ 0 & \alpha^{-1}\end{pmatrix}\begin{pmatrix}1 & \beta \\ 0 & 1\end{pmatrix}\right] = \begin{pmatrix}1 & (1-\alpha^2)\beta \\ 0 & 1\end{pmatrix}$$

   and the equation $\lambda = (1-\alpha^2)\beta$ always admits solutions for $|\mathbb{K}| > 3$.

   (b) We claim $S = \langle e + \lambda E_{ij} : \lambda \in \mathbb{K}, i, j \in \{1, \ldots, n\}, i \neq j\rangle$ is all of $\mathrm{SL}_n(\mathbb{K})$. Indeed, using only elementary operations of adding one row to another, for any $a \in \mathrm{SL}_n(\mathbb{K})$ there is $s \in S$ for which sa is diagonal:

$$sa = \mathrm{diag}(\alpha_1, \ldots, \alpha_n) = \begin{pmatrix}\alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n\end{pmatrix}$$

Then see that

$$(e + E_{12}\operatorname{diag}(\alpha_1,\ldots,\alpha_n))\left(e + \frac{1-\alpha_1}{\alpha_2}E_{21}\right) = \begin{pmatrix} 1 & \alpha_2 & & & \\ 1-\alpha_1 & \alpha_2 & & & \\ & & \alpha_3 & & \\ & & & \ddots & \\ & & & & \alpha_n \end{pmatrix}$$

and

$$(e + (\alpha_1 - 1)E_{21})\begin{pmatrix} 1 & \alpha_2 & & & \\ 1-\alpha_1 & \alpha_2 & & & \\ & & \alpha_3 & & \\ & & & \ddots & \\ & & & & \alpha_n \end{pmatrix}(e - \alpha_2 E_{12}) = \begin{pmatrix} 1 & & & & \\ & \alpha_1\alpha_2 & & & \\ & & \alpha_3 & & \\ & & & \ddots & \\ & & & & \alpha_n \end{pmatrix}$$

An evident induction shows that $a \in S$.

(c) Combining the two statements, we get $\mathrm{SL}_n(\mathbb{K}) = S \subseteq [\mathrm{SL}_n(\mathbb{K}), \mathrm{SL}_n(\mathbb{K})] \subseteq \mathrm{SL}_n(\mathbb{K})$.

2. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $G = \mathrm{GL}_n(\mathbb{K})$. We observe that $Z = Z(\mathrm{GL}_n(\mathbb{K})) = \mathbb{K}^\times e$. Also from the first example we have that $\mathrm{SL}_n(\mathbb{K}) = [G, G]$. Let $H = Z \cdot \mathrm{SL}_n(\mathbb{K})$.

   If n is odd and $\mathbb{K} = \mathbb{R}$ or n is arbitrary and $\mathbb{K} = \mathbb{C}$ then $H = G$. Ifn is even and $\mathbb{K} = \mathbb{R}$, then $H = \mathrm{GL}_n(\mathbb{R})_0 = \det^{-1}((0,\infty))$ is open, and thus closed; furthermore, we get $\mathrm{GL}_n(\mathbb{R})_0 \sqcup a\,\mathrm{GL}_n(\mathbb{R})$ where $\det(a) = -1$.

   Either way, we get that H is open and normal in G with G/H finite, and hence compact. We have $H/Z \cong \mathrm{SL}_n(\mathbb{K})/Z_\cap \mathrm{SL}_n(\mathbb{K})$. But $\mathrm{SL}_n(\mathbb{K})$ is perfect, and hence the quotient is perfect; so H/Z is unimodular. Thus so is H and hence G.

3. (Euclidean motion.) We let $E(n) = \mathbb{R} \rtimes \mathrm{SO}(n)$. ($\mathrm{SO}(n)$ is the orthogonal real matrices of determinant 1.) Then $N = \mathbb{R} \rtimes \{e\}$ is normal and unimodular, with $E(n)/N \cong \mathrm{SO}(n)$ compact. Hence $E(n)$ is unimodular.

4. (Heisenberg.) Let

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \subseteq \mathrm{GL}_3(\mathbb{R})$$

   a closed subgrape. We have

$$Z(\mathbb{H}) = \left\{ \begin{pmatrix} 1 & 0 & z \\ & 1 & 0 \\ & & 1 \end{pmatrix} : z \in \mathbb{R} \right\}$$

   and $\mathbb{H}/Z(\mathbb{H}) \cong \mathbb{R}^2$. Thus $\mathbb{H}$ is unimodular.

5. (Conjugation automorphism.) For $x \in G$, let $\gamma(x) \in \mathrm{Aut}(G)$ be $\gamma(x)(y) = xyx^{-1}$. Notice $\gamma(xx') = \gamma(x)\gamma(x')$. Then

$$\delta(\gamma(x)) = \frac{1}{\Delta(x)}$$

   (where δ is as in assignment 1).

   Suppose $\alpha \in \mathrm{Aut}(G)$. If G is compact, then $\alpha(G) = G$ implies $\delta(\alpha) = 1$. If G is discrete, then $|\alpha(F)| = |F|$ for each finite $F \subseteq G$ implies $\delta(\alpha) = 1$.

   Suppose G, A are unimodular and A acts continuously on G by automorphisms. Consider $S = G \rtimes A$. Then by assignment 1 we get $\Delta(y, \beta) = \delta(\beta)$.

6. If H is open in G and G is unimodular, then H is unimodular.

   However, if H is closed and non-open in G, we may have that G is unimodular and H is not. Consider for example $G = \mathrm{SL}_2(\mathbb{R})$ and

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : \alpha \in (0, \infty), b \in \mathbb{R} \right\}$$

   Then $H = \mathbb{R} \rtimes (0, \infty)$ with $a(b) = ab$ (non-unimodular action) so H is not unimodular thanks to the first item.

7. It is possible that N is a unimodular open normal subgrape of G yet G is not unimodular. Indeed, consider $G = \mathbb{R} \rtimes \{ 2^n : n \in \mathbb{Z} \}$; this is an open subgrape of $\mathbb{R} \rtimes (0, \infty)$.

5 The convolution algebra of measures

Let

$$M(G) = \{ \mu \colon \mathcal{B}(G) \to \mathbb{C} \mid \mu \text{ a Radon measure} \}$$
$$M_+(G) = \{ \mu \colon \mathcal{B}(G) \to [0, \infty) \mid \mu \text{ a (finite) measure} \}$$

Definition 5.1. If $E \in \mathcal{B}(G)$, we define the *total variation* to be

$$|\mu|(E) = \sup\left\{ \sum_{j=1}^{\infty} |\mu(E_j)| : E = \bigsqcup_{j=1}^{\infty} E_j, \text{ each } E_j \in \mathcal{B}(G) \right\}$$

Fact 5.2. *If $\mu \in M(G)$ then $|\mu| \in M_+(G)$.*

Fact 5.3 (Hahn-Jordan decomposition)**.** *Each $\mu \in M(G)$ can be written $\mu = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4$ where $\mu_1, \ldots, \mu_4 \in M_+(G)$. Furthermore, we can arrange that $\mu_1 \perp \mu_2$ and $\mu_3 \perp \mu_4$ (i.e. $G = E_1 \sqcup E_2$ such that $\mu_2 \restriction E_1 = 0$ and $\mu_1 \restriction E_2 = 0$), and in this context the decomposition is unique.*

Generally we have

$$\mu_1, \ldots, \mu_4 \leqslant |\mu| \leqslant |\mu_1 - \mu_2| + |\mu_3 - \mu_4|$$

and $|\mu_1 - \mu_2| \leqslant \mu_1 + \mu_2$, etc. If $\mu_1 \perp \mu_2$ then $|\mu_1 - \mu_2| = \mu_1 + \mu_2$, etc.

Theorem 5.4 (Riesz representation theorem)**.** *Let $C_0(G) = \overline{C_c(G)}^{\|\cdot\|_\infty}$; this is a Banach space. Then $C_0(G)^* \cong M(G)$ via the pairing*

$$\langle f, \mu \rangle = \mu(f) = \int_G f \mathrm{d}\mu$$

Furthermore,

$$\sup\left\{ \left| \int_G f \mathrm{d}\mu \right| : f \in C_0(G), \|f\|_\infty \leqslant 1 \right\} = |\mu|(G)$$

which we define to be $\|\mu\|_1$.

Remark 5.5 (Approximation by "compactly supported" measures)**.** Given $\mu \in M(G)$ and $\varepsilon > 0$, the inner regularity of $|\mu|$ provides compact $K \subseteq G$ such that$|\mu|(G) < |\mu|(K) + \varepsilon$; thus $|\mu|(G \backslash K) < \varepsilon$. If we let $\mu_K \colon \mathcal{B}(G) \to \mathbb{C}$ be $\mu_K(E) = \mu(E \cap K)$, then

$$\|\mu \mu_K\|_1 = \|\mu_{G \backslash K}\|_1 = |\mu_{G \backslash K}|(G) = |\mu|(G \backslash K) < \varepsilon$$

Theorem 5.6. *Given $\mu, \nu \in M(G)$ there is a unique measure $\mu * \nu$ such that for $f \in C_0(G)$ (or $f \in C_c(G)$) we have*

$$\int_G f \mathrm{d}(\mu * \nu) = \int_G \int_G f(xy) \mathrm{d}\mu(x) \mathrm{d}\nu(y)$$

Then $(\mu, \nu) \mapsto \mu * \nu$ is bilinear and associative (i.e. $(\mu * \nu) * \rho = \mu * (\nu * \rho)$ where $\rho \in M(G)$) and satisfies $\|\mu * \nu\|_1 \leqslant \|\mu\|_1 \|\nu\|_1$. Hence $(M(G), *)$ is a Banach algebra.

This product is called the *convolution product*.

Before we begin, we give some facts about the Radon product measure.

Our setup: suppose X, Y are locally compact Hausdorff spaces. We define the product of the Borel σ-algebras by

$$\mathcal{B}(X) \otimes \mathcal{B}(Y) = \sigma\langle E \times F : E \in \mathcal{B}(X), F \in \mathcal{B}(Y)\rangle$$

Clearly $\mathcal{B}(X) \otimes \mathcal{B}(Y) \subseteq \mathcal{B}(X \times Y)$.

A problem: unless both X and Y are separable, we cannot guarantee equality.

Example 5.7. Let $X = Y = \{0,1\}^I$ where $|I| > \aleph_0$ or $X = Y = \mathbb{R}_d$. Nico suspects that $\subsetneq$ holds in both cases.

Theorem 5.8. *Given two Radon measures $\mu\colon \mathcal{B}(X) \to [0,\infty]$ and $\nu\colon \mathcal{B}(Y) \to [0,\infty]$, there is a unique measure $\mu \times \nu$ on $\mathcal{B}(X \times Y)$ such that*

$$\int_{X \times Y} f \mathrm{d}(\mu \times \nu) = \int_Y \int_X f(x,y) \mathrm{d}\mu(x) \mathrm{d}\nu(y) = \int_X \int_Y f(x,y) \mathrm{d}\nu(y) \mathrm{d}\mu(x)$$

for $f \in C_c(X \times Y)$. (We call this the restricted Fubini property (F_c)*.) This is the unique measure on $\mathcal{B}(X \times Y)$ such that $(\mu \times \nu)(E \times F) = \mu(E)\nu(F)$ for $E \in \mathcal{B}(X)$ and $F \in \mathcal{B}(Y)$. (We call this the* product property (P)*.)*

We call this the *Radon product measure*.

Corollary 5.9. *If $\mu \in M(X), \nu \in M(Y)$ are complex Radon measures, then there is $\mu \times \nu \in M(X \times Y)$ for which (F_c) and (P) hold.*

Fact 5.10 (Fubini for Radon products)**.** *For $\mu \in M(X)$, $\nu \in M(Y)$, and $f \in \mathcal{B}^\infty(X \times Y)$ (i.e. f is uniformly bounded and Borel measurable), we have that*

$$x \mapsto \int_Y f(x,y) \mathrm{d}\nu(y)$$
$$y \mapsto \int_X f(x,y) \mathrm{d}\mu(x)$$

are Borel measurable on X and Y, respectively, and

$$\int_{X \times Y} f \mathrm{d}(\mu \times \nu) = \int_Y \int_X f(x,y) \mathrm{d}\mu(x) \mathrm{d}\nu(y) = \int_X \int_Y f(x,y) \mathrm{d}\nu(y) \mathrm{d}\mu(x)$$

Proof of Theorem 5.6.

1. We define "actions" of $M(G)$ on $C_c(G)$. Given $f \in C_0(G)$ and $\mu \in M(G)$ we let $f \cdot \mu, \mu \cdot f \colon G \to \mathbb{C}$ be

$$\begin{aligned}(f \cdot \mu)(x) &= \mu(x \cdot f) \\ &= \int_G f(yx) \mathrm{d}\mu(y) \\ (\mu \cdot f)(x) &= \mu(f \cdot x) \\ &= \int_G f(xy) \mathrm{d}\mu(y)\end{aligned}$$

   Let us see that $\mu \cdot f \in C_0(G)$. Let V be a neighbourhood of e such that $|f(x) - f(x')| < \varepsilon$ if $x'x^{-1} \in V$. Then for such x, x' we have

$$\begin{aligned}|(\mu \cdot f)(x) - (\mu \cdot f)(x')| &= \left|\int_G (f(xy) - f(x'y)) \mathrm{d}\mu(x)\right| \\ &\leqslant \int_G \underbrace{|f(xy) - f(x'y)|}_{<\varepsilon} \mathrm{d}|\mu|(y) \\ &\leqslant \varepsilon |\mu|(G)\end{aligned}$$

(Note that complex measures are by definition finite.) So $\mu \cdot f$ is continuous. Furthermore, we have

$$|(\mu \cdot f)(x)| \leqslant \int_G \underbrace{|f(xy)|}_{\leqslant \|f\|_\infty} \mathrm{d}|\mu|(y) \leqslant \|f\|_\infty |\mu|(G) = \|f\|_\infty \|\mu\|_1$$

Again, for $\varepsilon > 0$, let $K \subseteq G$ be compact and $f' \in C_c(G)$ satisfy $\|\mu - \mu_K\|_1 < \varepsilon$ and $\|f - f'\|_\infty < \varepsilon$. Then

$$\begin{aligned}
\|\mu \cdot f - \mu_K \cdot f'\|_\infty &\leqslant \|\mu \cdot f - \mu_K \cdot f\|_\infty + \|\mu_K \cdot f - \mu_K \cdot f'\|_\infty \\
&\leqslant \|\mu - \mu_K\|_1 \|f\|_\infty + \underbrace{\|\mu_K\|_1}_{\leqslant \|\mu\|_1} \|f - f'\|_\infty \\
&< \varepsilon(\|f\|_\infty - \|\mu\|_1)
\end{aligned}$$

It is clear that $\operatorname{supp}(\mu_K \cdot f') \subseteq \operatorname{supp}(f)K^{-1}$; hence $\mu \cdot f \in C_0(G)$. The case $f \cdot \mu$ is similar.

2. We check an "associativity": that if $\mu, \nu \in M(G)$ and $f \in C_0(G)$, then $\mu \cdot (f \cdot \nu) = (\mu \cdot f) \cdot \nu$.

   For $x \in G$ we have

$$\begin{aligned}
(\mu \cdot (f \cdot \nu))(x) &= \int_G (f \cdot \nu)(xy) \mathrm{d}\mu(y) \\
&= \int_G \int_G f(zxy) \mathrm{d}\nu(z) \mathrm{d}\mu(y) \\
&= \int_G \int_G f(zxy) \mathrm{d}\mu(y) \mathrm{d}\nu(z) \text{ (by Fubini)} \\
&= ((\mu \cdot f) \cdot \nu)(x)
\end{aligned}$$

   as desired.

3. We now come to the finale. We define for $\mu, \nu \in M(G)$ and $f \in C_0$

$$\int_G f \mathrm{d}(\mu * \nu) = (\mu * \nu)(f) = \mu \cdot (\nu \cdot f)$$

   (By Riesz representation theorem this specifies $\mu * \nu$.) The map $(\mu, \nu) \mapsto \mu * \nu$ is bilinear and also

$$|(\mu * \nu)(f)| = |\mu \cdot (\nu \cdot f)| \leqslant \|\mu\|_1 \|\nu \cdot f\|_\infty \leqslant \|\mu\|_1 \|\nu\|_1 \|f\|_\infty$$

   so it follows that $\mu * \nu$ defines a bounded linear functional on $C_0(X)$, and hence an element of $M(G)$ with $\|\mu * \nu\|_1 \leqslant \|\mu\|_1 \|\nu\|_1$.

   It remains to check associativity. Let also $\rho \in M(G)$. We have for $f \in C_0(G)$ that

$$\begin{aligned}
(\mu * (\nu * \rho))(f) &= \int_G \int_G f(xy) \mathrm{d}\mu(x) \mathrm{d}(\nu * \rho)(y) \\
&= (\nu * \rho)(f \cdot \mu) \\
&= \nu \cdot (\rho \cdot (f \cdot \mu)) \\
&= \nu \cdot ((\rho \cdot f) \cdot \mu) \text{ (by associativity above)} \\
&= (\mu * \nu)(\pi \cdot f) \\
&= ((\mu * \nu) * \rho)(f)
\end{aligned}$$

   as desired. □ Theorem 5.6

Remark 5.11.

1. Fix $\nu \in M(G)$. Then both $\mu \mapsto \mu * \nu$ and $\mu \mapsto \nu * \mu$ are weak*-weak* continuous on $M(G) \cong C_0(G)^*$. Indeed, let $R_\nu \colon C_0(G) \to C_0G)$ be $R_\nu(f) = f \cdot \nu$. Then $\nu * \mu = R_\nu^*(\mu)$.

2. For $x \in G$ let $\delta_x : \mathcal{B}(G) \to \{ 0, 1 \} \subseteq \mathbb{C}$ be given by

$$\delta_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{else} \end{cases}$$

(We call this a *Dirac measure*.) If $f \in C_0(G)$ then $f = f(x)1_{\{ x \}}$ δ_x-almost-everywhere. So

$$\int_G f \mathrm{d}\delta_x = f(x)$$

Then if $x, y \in G$ and $f \in C_0(G)$, then

$$(\delta_x \delta_y)(f) = \int_G \int_G f(x'y') \mathrm{d}\delta_x(x') \mathrm{d}\delta_y(y') = f(xy) = \delta_{xy}(f)$$

i.e. $\delta_x * \delta_y = \delta_{xy}$. Also $\delta_x \cdot f = x \cdot f$ and $f \cdot \delta_x = f \cdot x$.

3. Let $B_1^+(M(G)) = \{ \mu \in M_+(G) : \mu(G) \leqslant 1 \}$.

Exercise 5.12. Thisis a convex set with $\mathrm{Ext}(B_1^+(M(G))) = \{ 0 \} \cup \{ \delta_x : x \in G \}$.

Then by Krein-Milman theorem, we have that convolution is the unique weak*-weak* continuous product on $M(G)$ satisfying Item 2.

6 Atomic/continuous and Lebesgue decompositions

Let $\mu \in M(G)$. Let

$$A(\mu) = \{ x \in G : |\mu|(\{ x \}) > 0 \} = \bigcup_{n=1}^{\infty} \left\{ x \in G : |\mu|(\{ x \}) > \frac{1}{n} \right\}$$

So $A(\mu)$ is countable, and hence Borel. Furthermore, we have

$$\infty > |\mu|(A(\mu)) = \sum_{x \in A(\mu)} |\mu|(\{ x \}) = \sum_{x \in A(\mu)} |\mu(\{ x \})|$$

It follows that

$$\mu_d = \sum_{x \in A(\mu)} \mu(\{ x \}) \delta_x$$

is a measure. We let $\mu_c = \mu - \mu_d$; so $\mu_c \perp \mu_d$ (with $G = A(\mu) \sqcup (G \backslash A(\mu))$). Hence $\mu = \mu_d + \mu_c$ and $|\mu| = |\mu_d| + |\mu_c|$; so

$$\|\mu\|_1 = |\mu|(G) = \|\mu_d\|_1 + \|\mu_c\|_1$$

Let

$$\begin{aligned} M_d(G) &= \overline{\mathrm{span}}\{ \delta_x : x \in G \} \\ &\cong \ell^1(G) \\ M_c(G) &= \{ \mu \in M(G) : \mu(\{ x \}) = 0 \text{ for any } x \in G \} \end{aligned}$$

Then $M_d(G)$ is a closed subspace and $M_c(G)$ is a subspace, which is closed since the defining formula of convolution yields that $\mu \mapsto \mu_c$ is a bounded idempotent map on $M(G)$ with range $M_c(G)$. We write $M(G) = M_d(G) \oplus_1 M_c(G)$ since all $\mu \in M(G)$ admit a decomposition $\mu = \mu_d + \mu_c$ with $\|\mu\|_1 = \|\mu_d\|_1 + \|\mu_c\|_1$.

Theorem 6.1 (Lebesgue decomposition). *Let $\mu \in M(G)$. We have $\mu = \mu_s + \mu_a$ where $\mu_s \perp m$, $\mu_a \ll m$ with $\frac{\mathrm{d}\mu}{\mathrm{d}m} = \frac{\mathrm{d}\mu_a}{\mathrm{d}m} \in L^1(G)$. i.e. for $f \in C_0(G)$ we have*

$$\int_G f \mathrm{d}\mu = \int_G f \mathrm{d}\mu_s + \int_G f \frac{\mathrm{d}\mu_a}{\mathrm{d}m} \mathrm{d}m$$

We have $\mu_s \perp \mu_a$ so $\|\mu\|_1 = \|\mu_s\|_1 + \|\mu_a\|_1$. Write

$$M(G) = \underbrace{M_s(G)}_{\text{space of singular}} \oplus_1 \underbrace{M_a(G)}_{\text{space of absolutely continuous}}$$

Suppose G is discrete; then

$$|\mu_c|(G) = \sup\{\,\underbrace{|\mu_c|(K)}_{=0} : K \subseteq G \text{ compact (hence finite)}\,\} = 0$$

So $\mu = \mu_d$, and $M(G) = M_d(G) = \ell^1(G)$. One can check that $\ell^1(G) = \overline{\operatorname{span}}\{\,\delta_x : x \in G\,\}$ is a Banach algebra.

Suppose G is not discrete. Then$m(\{\,x\,\}) = m(x\{\,x\,\}) = m(\{\,e\,\}) = 0$. ($\{\,e\,\}$ is a non-open closed set, and hence locally null.) Thus $M_a(G) \subseteq M_c(G)$. Thus if $\nu \in M_c(G)$ we get the Lebesgue decomposition $\nu = \nu_{cs} + \nu_a$ with $\nu_{cs} \perp m$ and $\nu_a \ll m$.

In summary, if $\mu \in M(G)$, we write

$$\mu = \mu_d + \mu_c = \mu_d + \mu_{cs} + \mu_d$$

all mutually singular. We then have

$$M(G) = M_d(G) \oplus_1 \underbrace{M_{cs}(G) \oplus_1 M_d(G)}_{M_c(G)} \cong \ell^1(G) \oplus_1 M_{cs}(G) \oplus_1 L^1(G)$$

Fact 6.2. *$M_d(G) = \ell^1(G)$ is a closed subalgebra.*

Question 6.3. What about $M_c(G)$, $M_a(G) \cong L^1(G)$, or $M_{cs}(G)$?

7 More convolutions

What does $\mu * \nu$ look like as a measure?

Theorem 7.1. *If $\mu, \nu \in M(G)$ and $E \in \mathcal{B}(G)$, then $(\mu * \nu)(E) = (\mu \times \nu)(\pi^{-1}(E))$, where $\pi\colon G \times G \to G$ is the product map.*

Remark 7.2.

1. π is continuous, and hence Borel measurable; so $\pi^{-1}(E) \in \mathcal{B}(G \times G)$ for $E \in \mathcal{B}(G)$.
2. Fubini's theorem yields that

$$\begin{aligned}(\mu \times \nu)(\pi^{-1}(E)) &= \int_{G \times G} 1_{\pi^{-1}(E)} \mathrm{d}(\mu \times \nu) \\ &= \int_{G \times G} 1_E \circ \pi \mathrm{d}(\mu \times \nu) \\ &= \int_{G \times G} 1_E(xy) \mathrm{d}(\mu \times \nu)(x, y) \\ &= \int_G \int_G 1_E(xy) \mathrm{d}\mu(x) \mathrm{d}\nu(y)\end{aligned}$$

Proof of Theorem 7.1. We have

$$\mu = (\mu_0 - \mu_2) + i(\mu_1 - \mu_3) = \sum_{k=0}^{3} i^k \mu_k$$

where $\mu_k \in M_+(G)$; likewise for ν. So

$$\mu * \nu = \sum_{k=0}^{3} \sum_{\ell=0}^{3} i^{k+\ell} \mu_k * \nu_\ell$$

We can thus assume that $\mu * \nu \in M_+(G)$.

1. Let us first consider compact $K \subseteq G$. Let $\varepsilon > 0$; let U be open with $U \supseteq K$ and $(\mu * \nu)(U \backslash K) < \varepsilon$. Let $f \in C_c^{[0,1]}(G)$ satisfy $f \restriction K = 1$ and $\operatorname{supp}(f) \subseteq U$ (by Urysohn's lemma). Then

$$\begin{aligned}
(\mu \times \nu)(\pi^{-1}(K)) &= \int_G \int_G 1_K(xy) \mathrm{d}\mu(x) \mathrm{d}\nu(y) \\
&\leqslant \int_G \int_G f(xy) \mathrm{d}\mu(x) \mathrm{d}\nu(y) \\
&= \int_G f \mathrm{d}(\mu * \nu) \\
&\leqslant \int_G 1_U \mathrm{d}(\mu * \nu) \\
&= (\mu * \nu)(U) \\
&< (\mu * \nu)(K) + \varepsilon
\end{aligned}$$

   Since ε was arbitrary, we get that

$$(\mu \times \nu)(\pi^{-1}(K)) \leqslant (\mu * \nu)(K)$$

2. Now consider a $(\mu * \nu)$-null set $N \in \mathcal{B}(G)$. If $K \subseteq \pi^{-1}(N) \subseteq G \times G$ is compact, then $\pi(K)$ is compact with $\pi(K) \subseteq N$, and is thus $(\mu * \nu)$-null. Then by Item 1 we have

$$0 \leqslant (\mu \times \nu)(K) \leqslant (\mu \times \nu)(\pi^{-1}(\pi(K))) \leqslant (\mu * \nu)(\pi(K)) = 0$$

   Since Radon measures are inner regular, on bounded sets, we get

$$(\mu \times \nu)(\pi^{-1}(N)) = \sup\{\, (\mu \times \nu)(K) : K \subseteq \pi^{-1}(N), K \text{ compact} \,\} = 0$$

   So $\pi^{-1}(N)$ is $(\mu \times \nu)$-null.

3. Suppose $U \subseteq G$ is open. For each $n \in \mathbb{N}$ we can find compact $K_n \subseteq U$ so $(\mu * \nu)(U) < (\mu \times \nu)(K_n) + n^{-1}$. Then find $f_n \in C_c^{[0,1]}(G)$ with $\operatorname{supp}(f_n) \subseteq U$ and $f_n \restriction K_n = 1$; let $g_n = \max\{\, f_1, \ldots, f_n \,\}$. Then $(\mu * \nu)$-almost-everywhere we have $g_n \nearrow 1_U$ as $n \to \infty$. (We let

$$F = \bigcup_{n=1}^{\infty} K_n$$

   so $U \backslash F$ is $(\mu * \nu)$-null, and $g_n \to 1_U$ on $F \cup (G \backslash U)$.)

   Hence by monotone convergence theorem, using the fact that $(\mu \times \nu)$-almost-everywhere we have $g_n \circ \pi \nearrow 1_U \circ \pi$ (by Item 2), we get that

$$\begin{aligned}
(\mu \times \nu)(\pi^{-1}(U)) &= \int_{G \times G} 1_U \circ \pi \mathrm{d}(\mu \times \nu) \\
&= \lim_{n \to \infty} \int_{G \times G} g_n \circ \pi \mathrm{d}(\mu \times \nu) \\
&= \lim_{n \to \infty} \int_G g_n \mathrm{d}(\mu * \nu) \\
&= \int_G 1_U \mathrm{d}(\mu * \nu) \\
&= (\mu * \nu)(U)
\end{aligned}$$

4. Now let $E \in \mathcal{B}(G)$, and find open $U_n vmE$ such that $(\mu * \nu)(U_n \backslash E) < n^{-1}$. Then let

$$V_n = \bigcap_{k=1}^{n} U_n$$

so we have $1_{V_n} \to 1_E$ on

$$G \backslash \bigcap_{n=1}^{\infty} V_n) \cup E$$

i.e. $(\mu * \nu)$-almost-everywhere. Hence by Item 2, we get $(\mu \times \nu)$-almost-everywhere that $1_{V_n} \circ \pi \to 1_E \circ \pi$. Thus by Lebesgue dominated convergence theorem we get that

$$\begin{aligned}(\mu \times \nu)(\pi^{-1}(E)) &= \lim_{n\to\infty} \int_{G\times G} 1_{V_n} \circ \pi \mathrm{d}(\mu * \nu) \\ &= \lim_{n\to\infty} \int_G 1_{V_n} \mathrm{d}(\mu * \nu) \\ &= (\mu * \nu)(E)\end{aligned}$$

□ Theorem 7.1

Remark 7.3. Some consequences:

1. For μ, ν, E as above we have

$$\begin{aligned}(\mu * \nu)(E) &= \int_G \int_G 1_E(xy) \mathrm{d}\mu(x) \mathrm{d}\nu(y) \\ &= \int_G \int_G 1_{Ey^{-1}}(x) \mathrm{d}\mu(x) \mathrm{d}\nu(y) \\ &= \int_G \mu(Ey^{-1}) \mathrm{d}\nu(y)\end{aligned}$$

and similarly

$$(\mu * \nu)(E) = \int_G \nu(x^{-1}E) \mathrm{d}\mu(x)$$

2. Let

$$B^\infty(G) = \overline{\mathrm{span}\{ 1_E : E \in \mathcal{B}(G) \}}^{\|\cdot\|_\infty} = \{ \varphi \colon G \to \mathbb{C} \mid \varphi \text{ bounded and Borel-measurable} \}$$

By LDCT we have for $\varphi \in B^\infty(G)$ that

$$\int_G \varphi \mathrm{d}(\mu * \nu) = \int_{G\times G} \varphi \circ \pi \mathrm{d}(\mu \times \nu) = \int_G \int_G \varphi(xy) \mathrm{d}\mu(x) \mathrm{d}\nu(y)$$

3. Let $L^\infty(G) = B^\infty(G)/\mathcal{N}_m$, where

$$\mathcal{N}_m = \{ f \in \mathcal{B}^\infty(G) : f = 0 \ m\text{-locally-almost-everywhere} \}$$

i.e. if $K \subseteq f^{-1}(\mathbb{C}\backslash\{ 0 \})$ is compact then $m(K) = 0$. Then a version of Riesz representation theorem tells us that $L^1(G)^* \cong L^\infty(G)$ via

$$\langle f, \varphi \rangle = \int_G f \varphi \mathrm{d}m$$

Corollary 7.4. *$M_c(G)$ and $M_a(G)$ are ideals in $M(G)$.*

Proof. If $N \in \mathcal{B}(G)$ an d$\mu, \nu \in M(G)$, we have

$$(\mu * \nu)(N) = \int_G \mu(Ny^{-1} \mathrm{d}\nu(y) = \int_G \nu(x^{-1}N) \mathrm{d}\mu(x)$$

Suppose one of μ, ν lies in $M_c(G)$ and $N = \{ x_0 \}$. Then clearly $(\mu * \nu)(\{ x_0 \}) = 0$. Thus $\mu * \nu \in M_c(G)$.

Likewise if N is m-(locally)-null and one of μ, ν lies in $M_a(G)$, then for $N' \subseteq N$ with $N' \in \mathcal{B}(G)$ we have for any $x \in G$ that $x^{-1}N', N'x^{-1}$ are also m-(locally)-null. Thus $(\mu * \nu)(N') = 0$. Thus $\mu * \nu \in M_a(G)$. □ Corollary 7.4

Remark 7.5. $M_{cs}(G)$ need not be a subalgebra of $M(G)$. Consider $G = K \times K$ for K an infinite compact grape, and m_K the normalized Haar measure on K. Then one can check that

$$(m_K \times \delta_e) * (\delta_e \times m_K) = m_K \times m_K = m_G \ll m_G$$

and $K \times \{ e \}, \{ e \} \times K$ are m_G-null. So $m_K \times \delta_e, \delta_e \times m_K \in M_{cs}(G)$.

Fact 7.6 (Hard)**.** *$M_{cs}(\mathbb{R})$ is not a subalgebra of $M(\mathbb{R})$. $M_{cs}(\mathbb{T})$ is not a subalgebra of $M(\mathbb{T})$.*

Theorem 7.7 (Bochner integral for bounded continuous functions)**.** *Suppose X is a locally compact space and $\mathcal{L}$ a Banach space, and let*

$$C_b(X, \mathcal{L}) = \left\{ F \colon X \to \mathcal{L} \mid F \text{ continuous}, \|f\|_\infty = \sup_{x \in X} \|f(x)\| < \infty \right\}$$

Then there is a bilinear map (integral)

$$\begin{aligned} C_b(X, G) \times M(X) &\to \mathcal{L} \\ (F, \mu) &\mapsto \int_X F \mathrm{d}\mu \end{aligned}$$

with

$$\left\| \int_X F \mathrm{d}\mu \right\| \leqslant \|F\|_\infty \|\mu\|_1$$

Furthermore if $T \in \mathcal{B}(\mathcal{L}, \mathcal{L}')$ (bounded linear operator), then

$$T\left(\int_X F \mathrm{d}\mu \right) = \int_X T \circ F \mathrm{d}\mu$$

Proof.

1. Let

   $$\mathcal{S} = \mathcal{S}(X, \mathcal{L}) = \operatorname{span}\{ 1_E(\cdot)\xi : E \in \mathcal{B}(G), \xi \in \mathcal{L} \}$$

   Each $\Phi \in \mathcal{S}$ admits a standard form

   $$\Phi = \sum_{j=1}^n q_{E_j}(\cdot)\xi_j$$

   where $\xi_1, \ldots, \xi_n \in \mathcal{L}$ and $E_1, \ldots, E_n \in \mathcal{B}(G)$ satisfy $E_i \cap E_j = \varnothing$ for $i \neq j$. Then $\mathcal{S}$ is a linear space of $\mathcal{L}$-valued functions.

   For $\mu \in M(X)$ and Φ as above, we let

   $$\int_X \Phi \mathrm{d}\mu = \sum_{j=1}^n \mu(E_j)\xi_j$$

   One checks that this is well-defined, that the map

   $$\begin{aligned} \mathcal{S} \times M(X) &\to \mathcal{L} \\ (\Phi, \mu) &\mapsto \int_X \Phi \mathrm{d}\mu \end{aligned}$$

   is bilinear, that

   $$\left\| \int_X \Phi \mathrm{d}\mu \right\| \leqslant \|\Phi\|_\infty \|\mu\|_1$$

   and that if $T \in \mathcal{B}(\mathcal{L}, \mathcal{L}')$ then

   $$T\left(\int_X \Phi \mathrm{d}\mu \right) = \int_X T \circ \Phi \mathrm{d}\mu$$

2. Let $\overline{\mathcal{S}} = \overline{\mathcal{S}(X, \mathcal{L})}^{\|\cdot\|_\infty}$. Hence if $\Psi \in \overline{S}$ then

$$\Psi = \lim_{n\to\infty} \Phi_n$$

for some $(\Phi_n)_{n=1}^\infty$ in $\mathcal{S}$. Then

$$\left(\int_X \Phi_n \mathrm{d}\mu\right)_{n=1}^\infty$$

is Cauchy in $\mathcal{L}$, and hence has a limit

$$\int_X \Psi \mathrm{d}\mu$$

This value is independent of the choice of Φ_n; thus the "usual" norm estimate and composition with bounded linear operators holds.

3. Let $K \subseteq X$ be compact. If $F \in C_b(X, \mathcal{L})$, then $F(K)$ is compact in $\mathcal{L}$, and hence is totally bounded. i.e. given $\varepsilon > 0$ we have

$$F(K) \subseteq \bigcup_{j=1}^n B(\xi_j, \varepsilon)$$

where $\xi_1, \ldots, \xi_n \in \mathcal{L}$. Let $E_1 = F^{-1}(B(\xi_1, \varepsilon)) \cap K$, and let

$$E_j = F^{-1}\left(B(\xi_j, \varepsilon) \setminus \bigcup_{i=1}^{j-1} B(\xi_i, \varepsilon)\right) \cap K$$

for $j \in \{2, \ldots, n\}$. Then

$$\Phi = \sum_{j=1}^n 1_{E_j}(\cdot)\xi_j$$

and we have

$$\max_{x\in K} \|F(x) - \Phi(x)\| - \|(F{\restriction}K) - \Phi\|_\infty < \varepsilon$$

Hence by Item 2 we have

$$\int_K F \mathrm{d}\mu$$

is "good".

4. Given $\mu \in M(X)$, find a sequence of compact sets for which

$$\lim_{n\to\infty} |\mu|(X \setminus K_n) = 0$$

Given $F \in C_b(X, \mathcal{L})$, let

$$\xi_n = \int_{K_n} F \mathrm{d}\mu = \int_X F \mathrm{d}\mu_{K_n}$$

(recall $\mu_K(E) = \mu(E \cap K)$). Then for $n, m \in \mathbb{N}$ we have

$$\begin{aligned}
\|\xi_n - \xi_m\| &= \left\|\int_X F \mathrm{d}(\mu_{K_n} - \mu_{K_m})\right\| \\
&\leqslant \|F\|\infty \|\mu_{K_n} - \mu_{K_m}\| \\
&\leqslant \|F\|_\infty |\mu|(K_n \triangle K_m) \\
&\leqslant \|F\|_\infty (|\mu|(G \setminus K_m) + |\mu|(G \setminus K_n))
\end{aligned}$$

So $(\xi_n)_{n=1}^\infty$ is Cauchy in $\mathcal{L}$. We call the limit

$$\int_X F \mathrm{d}\mu$$

one checks that this is independent of the sequence $(K_n)_{n=1}^\infty$. This integral is "good". □ Theorem 7.7

Definition 7.8. A Banach space $\mathcal{X}$ is a *Banach G-module* if there is an action

$$\begin{aligned} G \times \mathcal{X} &\to \mathcal{X} \\ (x, \xi) &\mapsto x \cdot \xi \end{aligned}$$

such that

- for a fixed x the map $\xi \mapsto x \cdot \xi$ is linear
- there is $C > 0$ such that $\|x \cdot \xi\| \leqslant C\|\xi\|$ for all x, ξ
- for any fixed $\xi \in \mathcal{X}$ the map $x \mapsto x \cdot \xi$ is a continuous map $G \to \mathcal{X}$. (Strong operator continuity.)

Theorem 7.9. *$\mathcal{X}$ is a Banach $M(G)$-module with the action $(\mu, \xi) \mapsto \mu \cdot \xi$ satisfying*

- *Bilinearity*
- $\|\mu \cdot \xi\| \leqslant C\|\mu\|_1\|\xi\|$
- $(\mu * \nu) \cdot \xi = \mu \cdot (\nu \cdot \xi)$.

Proof. Let

$$\mu \cdot \xi = \int_G x \cdot \xi \mathrm{d}\mu(x)$$

We us properties of the integral to check the last property. Let $\omega \in \mathcal{X}^*$ so $s \mapsto \langle \omega, s \cdot \xi \rangle$ is in $C_b(G) \subseteq B^\infty(G)$ and we have

$$\begin{aligned} \langle \omega, (\mu * \nu) \cdot \xi \rangle &= \int_G \int_G \langle \omega, (xy) \cdot \xi \rangle \mathrm{d}\nu(x) \mathrm{d}\mu(y) \\ &= \int_G \left\langle \omega, x \cdot \underbrace{\int_G y \cdot \xi \mathrm{d}\nu(y)}_{v \cdot \xi} \right\rangle \mathrm{d}\mu(x) \\ &= \int_G \langle \omega, x \cdot (\nu \cdot \xi) \rangle \mathrm{d}\mu(x) \\ &= \langle \omega, \mu \cdot (\nu \cdot \xi) \rangle \end{aligned}$$

(One should check the first equality.) So $(\mu * \nu) \cdot \xi = \mu \cdot (\nu \cdot \xi)$. ☐ Theorem 7.9

Recall our notation

$$\begin{aligned} (x * f)(y) &= f(x^{-1}y) \\ (f * x)(y) &= f(yx^{-1})(\Delta(x))^{-1} \end{aligned}$$

for m-almost-every y. These make $L^1(G)$ both a left and right contractive G-module; i.e. $\|x * f\|_1 = \|f\|_1 = \|f * x\|_1$. Thus we have that $L^1(G)$ is a contractive Banach $M(G)$-module with

$$\begin{aligned} \mu * f &= \int_G x * f \mathrm{d}\mu(x) \\ f * \mu &= \int_G f * x \mathrm{d}\mu(x) \end{aligned}$$

with $\|\mu * f\|_1 \leqslant \|\mu\|_1\|f\|_1$ and $\|f * \mu\|_1 \leqslant \|f\|_1\|\mu\|_1$.

Recall that $M_a(G) \cong L^1(G)$ by Radon-Nikodym theorem. (Recall $M_a(G)$ is the family of complex measures that are absolutely continuous with respect to m; recall further that this is an ideal of $M(G)$.) Thus if $\nu \in M_a(G)$ with $\nu \ll m$, say with $\frac{\mathrm{d}\nu}{\mathrm{d}m} = f \in L^1(G)$. We write $\nu = fm$; i.e.

$$(fm)(E) = \int_E f \mathrm{d}m$$

So for $h \in C_0(G)$ we get

$$\langle fm, h \rangle = \int_G hf \mathrm{d}m$$

Proposition 7.10.

1. *For $\mu \in M(G)$ and $f \in L^1(G)$ (so $fm \in M_a(G)$), we have*

$$\mu * (fm) = (\mu * f)m$$
$$(fm) * \mu = (f * \mu)m$$

2. *For $f, g \in L^1(G)$ we define*

$$f * g = (fm) * g = \int_G f(x)x * g\mathrm{d}x$$

(Bochner integral). Then

$$f * (gm) = f * g = \int_G f * yg(y)\mathrm{d}y$$

and

$$(f * g)m = (fm) * (gm)$$

Proof.

1. If $h \in C_0(G)$ we have

$$\begin{aligned}\int_G h\mathrm{d}(\mu * (fm)) &= \int_G \int_G h(xy)\mathrm{d}\mu(x)f(y)\mathrm{d}y \\ &= \int_G \int_G h(xy)f(y)\mathrm{d}y\mathrm{d}\mu(y) \text{ (Fubini)} \\ &= \int_G \int_G h(y)f(x^{-1}y)\mathrm{d}y\mathrm{d}\mu(y) \\ &= \int_G h(y)\int_G f(x^{-}y)\mathrm{d}\mu(x)\mathrm{d}y \text{ (Fubini)} \\ &= \int_G h\mu * f\mathrm{d}m\end{aligned}$$

and hence $\mu * (fm) = (\mu * f)m$. The rest is similar.

2. Similar. □ Proposition 7.10

So $(L^1(G), *)$ is a Banach algebra, canonically isomorphic to $M_a(G) \lhd M(G)$. We call this the *$(L^1$-)grape algebra*.

Theorem 7.11. *Let $\mathcal{X}$ be a non-degenerate Banach $L^1(G)$-module; i.e. there is a bilinear map $L^1(G) \times \mathcal{X} \to \mathcal{X}$ written $(f, \xi) \to f \cdot \xi$ such that*

- $\|f \cdot \xi\| \leqslant C\|f\|_1\|\xi\|$ *(where $C > 0$ is independent of f, ξ).*
- $(f * g) \cdot \xi = f \cdot (g \cdot \xi)$*.*
- $\mathcal{X}_0 = \mathrm{span}\{\, f \cdot \xi : f \in L^1(G), \xi \in \mathcal{X}\,\}$ *is dense in $\mathcal{X}$.*

Then $\mathcal{X}$ is a Banach G-module.

Proof. Let $(f_\alpha)_\alpha$ in $L^1(G)$ be a contractive summability kernel. (We'll see these on A2; in particular, we require $\|f_\alpha\|_1 \leqslant 1$ and

$$\lim_\alpha f_\alpha * f = f$$

for $f \in L^1(G)$.) Define an action $G \times \mathcal{X}_0 \to \mathcal{X}_0$ by

$$x \cdot \left(\sum_{j=1}^n f_j \cdot \xi\right) = \sum_{j=1}^n (x * f_j) \cdot \xi_j$$

We first check that this is well-defined. It is sufficient to check that if

$$\sum_{j=1}^{n} f_j \cdot \xi_j = 0$$

then

$$\sum_{j=1}^{n} (x * f_j) \cdot \xi_j = 0$$

Note, however, that

$$\begin{aligned} 0 &= \sum_{j=1}^{n} f_j \cdot \xi_j \\ &= \underbrace{x * f_\alpha}_{\in L^1(G)} \cdot \left(\sum_{j=1}^{n} f_j \cdot \xi_j \right) \\ &= \sum_{j=1}^{n} (x * \underbrace{f_\alpha * f_j}_{\xrightarrow{\alpha} f_j}) \\ &\xrightarrow{\alpha} \sum_{j=1}^{n} (x * f_j) \cdot \xi_j \\ &= x \cdot \left(\sum_{j=1}^{n} f_j \cdot \xi_j \right) \end{aligned}$$

i.e. $x \cdot 0 = 0$. Similarly, this action is linear on $\mathcal{X}_0$, and is thus well-defined.

Now if

$$\xi_0 = \sum_{j=1}^{n} f_j \cdot \xi_j \in \mathcal{X}_0$$

and $x \in G$ we have

$$\begin{aligned} \|x \cdot \xi_0\| &= \left\| \lim_\alpha \sum_{j=1}^{n} (x * f_\alpha * f_j) \cdot \xi_j \right\| \\ &= \lim_\alpha \|x * f_\alpha \cdot \xi_0\| \\ &\leqslant \limsup_\alpha C \underbrace{\|x * f_\alpha\|_1}_{\leqslant 1} \|\xi_0\| \\ &\leqslant C\|\xi_0\| \end{aligned}$$

Hence if we define $\pi_0(x) \in \mathcal{B}(\mathcal{X}_0)$ by $\pi_0(x)\xi_0 = x \cdot \xi_0$ for $\xi_0 \in \mathcal{X}_0$, then $\{ \pi_0(x) : x \in G \}$ is a uniformly bounded family of operators, and hence extends to a uniformly bounded family of operators $\{ \pi(x) : x \in G \} \subseteq \mathcal{B}(\mathcal{X})$. We let $x \cdot \xi = \pi(x)\xi$ and $\|x \cdot \xi\| \leqslant \|\pi(x)\|\|\xi\| \leqslant C\|\xi\|$.

It remains to check continuity in G. Suppose $\xi \in \mathcal{X}$ and $\varepsilon > 0$; pick

$$\xi_0 = \sum_{j=1}^{n} f_j \cdot \xi_j \in \mathcal{X}_0$$

with $\|\xi - \xi_0\| < \varepsilon$. Let V be a neighbourhood of e such that

$$\|x * f_j - f_j\| < \frac{\varepsilon}{n(\|\xi_j\| + 1)}$$

for $x \in V$. Then for $x \in V$ we have

$$\begin{aligned}\|\xi - x \cdot \xi\| &\leqslant \|\xi - \xi_0\| + \|\xi_0 - x \cdot \xi_0\| + \|x \cdot \xi_0 - x \cdot \xi\| \\ &< (1 + C)\varepsilon \sum_{j=1}^{n} C\|f_j - x * f_j\|_1 \|\xi_j\| \\ &< (1 + 2C)\varepsilon\end{aligned}$$

as desired. $\square$ Theorem 7.11

Our conclusion: there is a bijective correspondence between Banach G-modules and Banach $L^1(G)$-modules: given a Banach G-module, Theorem 7.9 gives rise to a Banach $M(G)$-module (non-degenerate for $L^1(G)$), which restricts to a Banach $L^1(G) \cong M_a(G)$-module, which by the last theorem gives rise to a G-module. (We will see on A2 that if $\mathcal{X}$ is a G-module then $f_\alpha \cdot \xi \xrightarrow{\alpha} \xi$ for $\xi \in \mathcal{X}$, which gives non-degeneracy.)

Example 7.12. Consider $M_c(G) \lhd M(G)$ a closed ideal, with

$$M(G) = \underbrace{M_d(G)}_{\cong \ell^1(G)} \oplus_{\ell^1} M_c(G)$$

Then $\ell^1(G) \cong M(G)/M_c(G)$ is a quotient algebra, and hence a Banach $M(G)$-module. Note that

$$\mu \cdot \delta_x = \sum_{y \in A(\mu)} \mu(\{ y \})\delta_{yx}$$

Since $\|\delta_x - \delta_{x'}\|_1 = 1$ for $x \neq x'$, this is *not* a continuous G-module.

Theorem 7.13 (Wendel)**.** *Suppose G and H are locally compact grapes. If there is an isometric isomorphism $\Phi\colon L^1(G) \to L^1(H)$, then there is a continuous isomorphism $\varphi\colon G \to H$ with continuous inverse.*

The requirement that Φ be isometric is important:

Example 7.14. Consider $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. It transpires that $\ell^1(\mathbb{Z}_4) \cong \ell^1(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong C(\{ 1, \dots, 4 \})$ via a non-isometric isomorphism.

Proof of Theorem 7.13. 1. Let

$$\mathcal{M}L^1(G) = \{ T \in \mathcal{B}(L^1(G)) : T(f * g) = T(f) * g \text{ for } f, g \in L^1(G) \}$$

(Here $\mathcal{B}(L^1(G))$ refers to bounded linear operators, not Borel sets.)

Claim 7.15. *Then $\mathcal{M}L^1(G) = \{ T_\mu : \mu \in M(G) \}$ where $T_\mu(f) = \mu * f$ and $\|T_\mu\| = \|\mu\|_1$.*

Proof. Suppose $T \in \mathcal{M}L^1(G)$, and let $(f_\alpha)_\alpha$ be a contractive summability kernel in $L^1(G)$. Then$(T(f_\alpha))_\alpha$ is a bounded net in $L^1(G) \hookrightarrow M(G)$, and hence admits a weak*-cluster-point by Banach-Alaoglu. By taking a subnet, we may assume that in the weak*topology we have

$$\mu = \lim_\alpha T(f_\alpha)$$

Hence in $M(G)$ we have

$$\begin{aligned}(\mu * f)m &= \mu * (fm) \\ &= \text{w*-}\lim_\alpha T(f_\alpha) * (fm) \\ &= \text{w*-}\lim_\alpha (T(f_\alpha) * f)m \\ &= \text{w*-}\lim_\alpha T(f_\alpha * f)m\end{aligned}$$

But since $f_\alpha * f \xrightarrow{\alpha} f$ in $L^1(G)$ and T is bounded (and hence continuous), we have that $T(f_\alpha * f) = T(f)$ in $L^1(G)$, so

$$\lim_\alpha T(f_\alpha * f)m = T(f)m$$

in norm, and in particular in the weak*topology.

TODO 4. *Typography*

so $\mu * f = T(f)$; i.e. $T = T_\mu$.

We have $\|T_\mu\| \leqslant \|\mu\|_1$ already. Conversely, we have

$$\begin{aligned}\|T_\mu\| &\geqslant \sup_\alpha \|T_\mu(f_\alpha)\|_1 \\ &= \sup_\alpha \|\mu * f_\alpha\|_1 \\ &= \sup_\alpha \sup_{\substack{h\in C_0(G)\\ \|h\|_\infty\leqslant 1}} |\langle \mu * f_\alpha, h\rangle| \\ &\geqslant \sup_{\|h\|_\infty\leqslant 1} \limsup_\alpha |\langle \mu, \underbrace{f_\alpha \cdot h}_{\xrightarrow{\alpha} h \text{ (A2)}} \rangle| \\ &= \sup_{\|h\|_\infty} |\langle \mu, h\rangle| \\ &= \|\mu\|_1\end{aligned}$$

as desired. $\square$ Claim 7.15

2. We define $\widetilde{\Phi}\colon M(G) \to M(H)$ by letting $T_{\widetilde{\Phi}(\mu)} = \Phi \circ T_\mu \circ \Phi^{-1}$. (Exercise, using Item 1.) Then $\widetilde{\Phi}$ is an isometric isomorphism which is *strictly continuous*: if $(\mu_\alpha)_\alpha$ is a net in $M(G)$ and $\mu \in M(G)$ has

$$\lim_\alpha \mu_\alpha * f = \mu * f$$

for any $f \in L^1(G)$, then

$$\lim_\alpha \widetilde{\Phi}(\mu_\alpha) * g = \widetilde{\Phi}(\mu) * g$$

for any $g \in L^1(H)$. Notice that $x_i \xrightarrow{i} x$ in G if and only if $\delta_{x_i} \xrightarrow{i,\text{strict}} \xi_x$ in $M(G)$. (Forward direction obvious, reverse an easy exercise.)

3. Let

$$\widetilde{G} = \operatorname{Ext} \underbrace{B(M(G))}_{\text{closed unit}} = \{\, z\delta_x : z \in \mathbb{T}, x \in G \,\}$$

Then $\widetilde{G} = \mathbb{T} \times G$ (as sets, and by a weak*-homeomorphism). Then $\widetilde{\Phi}$, being a surjective isometry, has

$$\widetilde{\Phi}(\widetilde{G}) = \widetilde{H} = \operatorname{Ext} B(M(H))$$

(Note that this together with linearity imply that φ is surjective.) We define $\zeta\colon G \to \mathbb{T}$ and $\varphi\colon G \to H$ by

$$\widetilde{\Phi}(\delta_x) = \zeta(x)\delta_{\varphi(x)}$$

Then

$$\zeta(xy)\delta_{\varphi(xy)} = \widetilde{\Phi}(\delta_{xy}) = \widetilde{\Phi}(\delta_x)\widetilde{\Phi}(\delta_y) = \zeta(x)\zeta(y)\delta_{\varphi(x)\varphi(y)}$$

So $\zeta(xy)\overline{\zeta(x)\zeta(y)}\delta_{e_H} = \delta_{\varphi(xy)^{-1}\varphi(x)\varphi(y)}$. But δ_{e_H} is supported on $\{\, e_H \,\}$, and $\delta_{\varphi(xy)^{-1}\varphi(x)\varphi(y)}$ is a probability measure. So φ and ζ are homomorphisms.

Now suppose $x_i \xrightarrow{i} x$ in G. So $\delta_{x_i} \xrightarrow{i,\text{strict}} \delta_x$ in $M(G)$. Then

$$\zeta(x_i)\delta_{\varphi(x)} = \widetilde{\Phi}(\delta_{x_i}) \xrightarrow{i,\text{strict}} \widetilde{\Phi}(\delta_x) = \zeta(x)\delta_{\varphi(x)}$$

So $\zeta(x_i x^{-1})\delta_{\varphi(x_i x^{-1})} \xrightarrow{i,\text{strict}} \delta_{e_H}$. We see by taking subsets if we must that 1 is the only cluster point of $\zeta(x_i x^{-1})$ in $\mathbb{T}$. It follows that ζ and φ are continuous.

4. We check that $\varphi^{-1}\colon H \to G$ is continuous. Note that $\Phi^{-1}\colon L^1(H) \to L^1(G)$ gives rise to a continuous homomorphism $\chi\colon H \to \mathbb{T}$ and a continuous isomorphism $\varphi\colon H \to G$. If $x \in G$ then

$$\begin{aligned} \delta_x &= \underbrace{\widetilde{\Phi^{-1}}}_{\widetilde{\Phi}^{-1}\text{ (check)}} \circ \widetilde{\Phi}(\delta_x) \\ &= \widetilde{\Phi}^{-1}(\zeta(x)\delta_{\varphi(x)}) \\ &= \zeta(x)\widetilde{\Phi^{-1}}(\delta_{\varphi(x)}) \\ &= \zeta(x)\chi(\varphi(x))\delta_{\psi(\varphi(x))} \end{aligned}$$

We deduce that $(\psi \circ \varphi)(x) = x$. So $\psi \circ \varphi = \mathrm{id}$, and $\psi = \varphi^{-1}$. □ Theorem 7.13

8 Unitary representations

Let $\mathcal{H}$ be a Hilbert space and $U(\mathcal{H}) = \{ U \in \mathcal{B}(\mathcal{H}) : U^*U = I = UU^* \}$.

Warning 8.1. In the infinite-dimensional setting, we must check both equalities $U^*U = I = UU^*$; it's possible for one to be satisfied but not the other.

Notation 8.2. For dual pairings, we will use $\langle \cdot, \cdot \rangle$. For sesquilinear forms, we will use $\langle \cdot \mid \cdot \rangle$. In this class we will use the physics convention: conjugate-linearity in the first argument, and linearity in the second argument.

On $\mathcal{B}(\mathcal{H})$ we consider, in addition to the norm topology, the *weak operator topology* and the *strong operator topology*:

$$\begin{aligned} \tau_{\mathrm{WO}} &= \sigma(\mathcal{B}(\mathcal{H}), \{ T \mapsto \langle \xi, T\eta \rangle \colon \mathcal{B}(\mathcal{H}) \to \mathbb{C}, \xi, \eta \in \mathcal{H} \}) \\ \tau_{\mathrm{SO}} &= \sigma(\mathcal{B}(\mathcal{H}), \{ T \mapsto T\xi \colon \mathcal{B}(\mathcal{H}) \to (H, \|\cdot\|), \xi \in \mathcal{H} \}) \end{aligned}$$

We have $\tau_{\mathrm{WO}} \subseteq \tau_{\mathrm{SO}}$; i.e. $T_\alpha \xrightarrow{\mathrm{SO},\alpha} T$ implies $T_\alpha \xrightarrow{\mathrm{WO},\alpha} T$.

Proposition 8.3.

1. *The map $B(\mathcal{B}(\mathcal{H})) \times B(\mathcal{B}(\mathcal{H})) \to B(\mathcal{B}(H))$ (closed unit balls) given by $(S,T) \mapsto ST$ is $\tau_{\mathrm{SO}} \times \tau_{\mathrm{SO}}$-$\tau_{\mathrm{SO}}$ continuous.*

2. *On $\mathcal{U}(\mathcal{H})$, the relativized topologies $\tau_{\mathrm{SO}} \restriction \mathcal{U}(\mathcal{H}) = \tau_{\mathrm{WO}} \restriction \mathcal{U}(\mathcal{H})$.*

Hence $(\mathcal{U}(\mathcal{H}), \tau_{\mathrm{WO}})$ is a topological grape.

Proof.

1. Suppose $S_\alpha \xrightarrow{\mathrm{SO},\alpha} S$ and $T_\alpha \xrightarrow{\mathrm{SO},\alpha} T$ in $B(\mathcal{B}(\mathcal{H}))$. Then for $\xi \in \mathcal{H}$ we have

$$\begin{aligned} \|S_\alpha T_\alpha \xi - ST\xi\| &\leqslant \|S_\alpha T_\alpha \xi - S_\alpha T\xi\| + \|S_\alpha T\xi - ST\xi\| \\ &\leqslant \|T_\alpha \xi - T\xi\| + \|S_\alpha T\xi - ST\xi\| \\ &\xrightarrow{\alpha} 0 \end{aligned}$$

2. Suppose $U_\alpha \xrightarrow{\mathrm{WO},\alpha} U$ in $\mathcal{U}(\mathcal{H})$. Then for $\xi \in \mathcal{H}$ we have

$$\begin{aligned} \|U_\alpha \xi - U\xi\|^2 &= \langle U_\alpha \xi - U\xi \mid U_\alpha \xi - U\xi \rangle \\ &= 2\|\xi\|^2 - 2\,\mathrm{Re}\langle U_\alpha \xi \mid U_\xi \rangle \\ &\xrightarrow{\alpha} 2\|\xi\|^2 - 2\,\mathrm{Re}\langle U\xi \mid U\xi \rangle \\ &= 0 \end{aligned}$$

as desired. □ Proposition 8.3

Remark 8.4.

1. The second item fails in $B(\mathcal{B}(\mathcal{H})$. Indeed, let $U\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the bilateral shift $U\delta_n = \delta_{n+1}$; so $U \in \mathcal{U}(\mathcal{H}) \subseteq B(\mathcal{B}(\mathcal{H}))$. One can check that $U^n \xrightarrow{\mathrm{WO},n} 0$ while $\|U^n\xi\| = \|\xi\|$ for $\xi \in \ell^2(\mathbb{Z})$.
2. The map $(S,T) \mapsto ST$ is not $(\tau_{\mathrm{WO}} \times \tau_{\mathrm{WO}})$-$\tau_{\mathrm{WO}}$ continuous. Let U be as above. So $U^n, U^{-n} \xrightarrow{\mathrm{WO},n} 0$ but $U^nU^{-n} = I \not\xrightarrow{\mathrm{WO},n} 0$.
3. For a fixed S the maps $T \mapsto TS$, $T \mapsto ST$, and $T \mapsto T^*$ are τ_{WO}-τ_{WO} continuous. (Check this.)
4. $T \mapsto T^*$ is not τ_{SO}-τ_{SO} continuous. (Consider the unilateral shift $S\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ so $S\delta_n = S\delta_{n+1}$ Then $(S^*)^n \to 0$ but S^n is always an isometry.

Proposition 8.5. *$\mathcal{U}(\mathcal{H})$ is the only subgrape of $B(\mathcal{B}(\mathcal{H}))$.*

Proof. If $U, U^{-1} \in B(\mathcal{B}(\mathcal{H}))$ then for $\xi \in \mathcal{H}$ we have

$$\|\xi\| = \|U^{-1}U\xi\| \leqslant \|U\xi\| \leqslant \|\xi\|$$

so $\|U\xi\| = \|\xi\|$. hence

$$\langle \xi \,|\, \xi \rangle = \|\xi\|^2 = \|U\xi\|^2 = \langle \xi \,|\, U^*U\xi \rangle$$

where $(U^*U)^* = U^*U$, so we can use the polarization identity: on any $\xi, \eta \in \mathcal{H}$ we have

$$4\langle \xi, \eta \rangle = \sum_{k=0}^{3} i^k \langle \xi + i^k\eta \,|\, \xi + i^k\eta \rangle = \sum_{k=0}^{3} i^k \langle \xi + i^k\eta \,|\, U^*U(\xi + i^k\eta) \rangle = 4\langle \xi \,|\, U^*U\eta \rangle$$

So $U^*U = I$, and $U^* = U^*UU^{-1} = U^{-1}$. □ Proposition 8.5

Definition 8.6. A *unitary representation* is a homomorphism $\pi\colon G \to \mathcal{U}(\mathcal{H})$, with $\mathcal{H}$ a Hilbert space, which is τ_G-τ_{SO} continuous. (If $x \cdot \xi = \pi(x)\xi$, we get a "unitary" Banach G-module.

Theorem 8.7. *There is a bijective correspondence between*

(i) Unitary representations $\pi\colon G \to \mathcal{U}(\mathcal{H})$ with $\mathcal{H}$ a Hilbert space.

(i′) Contractive (i.e. $C = 1$) Banach G-modules on a Hilbert space.

(ii) Non-degenerate $*$-representations $\pi_1\colon L^1(G) \to \mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ a Hilbert space.

(ii′) Contractive representations $\pi_1\colon L^1(G) \to \mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ a Hilbert space.

TODO 5. *typography*

Proof. For $(i) \iff (i')$ and $(ii) \iff (ii')$, we collect prior propositions on unitaries and the G-module to $L^1(G)$-module correspondence. It remains to check that $(i) \iff (ii)$.

If $\pi\colon G \to \mathcal{U}(\mathcal{H})$ is a unitary representation, then for $f \in L^1(G)$ we let $\pi_1(f) \in \mathcal{B}(\mathcal{H})$ be

$$\pi_1(f)\xi = \int_G f(x)\pi(x)\xi$$

(Bochner integral) for $\xi \in \mathcal{H}$. Then for $\xi, \eta \in \mathcal{H}$ we have

$$\begin{aligned}
\langle \pi_1(f)^*\xi \,|\, \eta \rangle &= \langle \xi \,|\, \pi_1(f)\eta \rangle \\
&= \int_G f(x)\langle \xi \,|\, \pi(x)\eta \rangle \mathrm{d}x \\
&= \int_G f(x)\langle \pi(x^{-1}\xi \,|\, \eta \rangle \mathrm{d}x \\
&= \int_G \underbrace{f(x^{-1})(\Delta(x))^{-1}}_{\overline{f^*(x)}} \langle \pi(x)\xi \,|\, \eta \rangle \mathrm{d}x \text{ (using } \pi(x^{-1}) = \pi(x)^*) \\
&= \int_G \langle f^*(x)\pi(x)\xi \,|\, \eta \rangle \mathrm{d}x \\
&= \langle \pi_1(f^*)\xi \,|\, \eta \rangle
\end{aligned}$$

So $\pi_1(f)^* = \pi_1(f^*)$. Conversely, if $\pi_1 \colon L^1(G) \to \mathcal{U}(\mathcal{H})$ is a $*$-homomorphism and $(f_\alpha)_\alpha$ is a summability kernel for $L^1(G)$, then $(f_\alpha^*)_\alpha$ is a summability kernel (check, might be useful on assignment), and we define

$$\pi(x)^* = \text{WO-}\lim_\alpha \pi_1(x * f_\alpha)^* = \text{WO-}\lim_\alpha \pi_1(f_\alpha^* * x^{-1}) = \pi(x^{-1})$$

One should check the first equality.

TODO 6. *What?*

☐ TODO 5

9 Gelfand theory for commutative Banach algebras

Let $\mathcal{A}$ be a commutative Banach algebra: so $\|ab\| \leqslant \|a\|\|b\|$ and $ab = ba$, etc.

Example 9.1.

1. Consider $C_0(X)$ where X is a locally compact Hausdorff space. This is unital if and only if X is compact.
2. Consider $(L^1(G), *)$ with G abelian. This is unital if and only if G is discrete (so $L^1(G) = \ell^1(G)$). (For the left-to-right implication, consider the multiplier $T_{fm-\delta_e}$ if f is the identity for $L^1(G)$. Then $\|T_{fm-\delta_e}\| = \|fm - \delta_e\|_1$, and the latter is $\geqslant 1 = \|\delta_e\|$ if G is non-discrete, while $T_{fm-\delta_e} = 0$ if $L^1(G)$ is unital.)
3. If S is an abelian semigrape, consider $(\ell^1(S), *)$ with

$$\sum_{s \in S} a(s)\delta_s * \sum_{t \in S} b(t)\delta_t = \sum_{u \in S}\left(\sum_{\substack{s,t \in S \times \\ st=u}} a(s)b(t)\right)\delta_u$$

   It is possible for $\ell^1(S)$ to be unital, with S being unital.
4. Consider $\mathbb{D} = \{\, z \in \mathbb{C} : |z| < 1 \,\}$ and

$$\mathcal{A}(\mathbb{D}) = \{\, f \in C(\overline{\mathbb{D}}) : f{\restriction}\mathbb{D} \text{ is holomorphic} \,\}$$

Definition 9.2. We let the *(Gelfand) spectrum* of $\mathcal{A}$ be

$$\widehat{\mathcal{A}} = \{\, \chi \colon \mathcal{A} \to \mathbb{C} \mid \chi \neq 0, \chi \text{ linear, } \mathbb{C}\text{-multiplicative} \,\}$$

We refer to the elements of $\widehat{\mathcal{A}}$ as *characters*.

We from now on assume that $\mathcal{A}$ is unital.

Proposition 9.3. *Let $\mathcal{A}$ be as above and $\chi \in \widehat{\mathcal{A}}$. Then*

1. $\chi(1_{\mathcal{A}}) = 1$.
2. *If $a \in \mathcal{A}^\times$ (i.e. a is invertible) then $\chi(a) \neq 0$.*
3. $|\chi(a)| \leqslant \|a\|$ *for $a \in \mathcal{A}$.*

Proof.

1. Since $\chi \neq 0$ we have a so $\chi(a) \neq 0$, and $\chi(1_A)\chi(a) = \chi(a)$.
2. We have $1 = \chi(1_A) = \chi(aa^{-1}) = \xi(a)\xi(a^{-1})$.

3. If $\lambda \in \mathbb{C}$ with $|\lambda| > \|a\|$ then $\|\lambda^{-1}a\| < 1$, and

$$(\lambda 1_{\mathcal{A}} - a)^{-1} = \lambda^{-1}(1_{\mathcal{A}} - \lambda^{-1}a)^{-1} = \lambda^{-1}\sum_{n=0}^{\infty} \lambda^{-n}a^n$$

(convergence in the Banach space $\mathcal{A}$), so $\chi(\lambda 1_{\mathcal{A}} - a) \neq 0$. i.e. $\lambda \neq \chi(a)$ if $|\lambda| > \|a\|$. The result follows. ☐ Proposition 9.3

Corollary 9.4. *With $\mathcal{A}$ as above we have that $\widehat{\mathcal{A}} \subseteq \mathcal{A}^*$ is* w*-*-compact.*

Proof. Since $\widehat{\mathcal{A}} \subseteq B(\mathcal{A}^*)$, it suffices to show that $\widehat{\mathcal{A}}$ is w*-closed (by Banach-Alaoglu). If $(\chi_\alpha)_\alpha$ is a net in $\widehat{\mathcal{A}}$ with $\chi_\alpha \xrightarrow{w^*,\alpha} \chi$, then for $a, b \in \mathcal{A}$ we have

$$\chi(ab) = \lim_\alpha \chi(ab) = \lim_\alpha \chi_\alpha(a)\chi_\alpha(b) = \chi(a)\chi(b)$$

and

$$1 = \lim_\alpha \chi_\alpha(1_{\mathcal{A}}) = \chi(1_{\mathcal{A}})$$

so $\chi \neq 0$. ☐ Corollary 9.4

Lemma 9.5. *Suppose $\mathcal{A}$ is as above and $\mathcal{I} \subsetneqq \mathcal{A}$ is an ideal. Then*

1. *$\mathcal{I} \cap \mathcal{A}^\times = \varnothing$.*
2. *$\overline{\mathcal{I}} \subsetneqq \mathcal{A}$ and is also an ideal.*
3. *$\mathcal{I}$ is contained in a maximal ideal $\mathcal{I} \subseteq \mathcal{M} \subsetneqq \mathcal{A}$.*
4. *If $\mathcal{I}$ is maximal then it is closed.*

Proof.

1. If $a \in \mathcal{A}^\times$ then $1_{\mathcal{A}} \in a\mathcal{A}$, so $a \notin \mathcal{I}$.
2. If $\|b\| < 1$ in $\mathcal{A}$ then $1 - b \in \mathcal{A}^*$. Indeed,

   $$(1-b)^{-1} = \sum_{n=0}^{\infty} b^n$$

   so the open set $U = \{\, a \in \mathcal{A} : \|a - 1_{\mathcal{A}}\| < 1 \,\} \subseteq \mathcal{A}^\times$. Then $\mathcal{I} \cap U = \varnothing$, hence $\overline{I} \cap U = \varnothing$, and $\overline{\mathcal{I}} \subsetneqq \mathcal{A}$. Also if

   $$a = \lim_{n\to\infty} a_n$$

   for $a_n \in \mathcal{I}$ and $b \in \mathcal{A}$ then

   $$ba = \lim_{n\to\infty} ba_n \in \overline{\mathcal{I}}$$

   So $\mathcal{I}$ is an ideal.
3. Let $\Xi = \{\, \mathcal{J} \subsetneqq \mathcal{A} : \mathcal{J} \text{ an ideal}, \mathcal{I} \subseteq \mathcal{J} \,\}$. Then Ξ is partially ordered by inclusion. If $\Gamma \subseteq \Xi$ is a chain then

   $$\mathcal{K} = \bigcup_{\mathcal{J}\in\Gamma} \mathcal{J} \in \Xi$$

   (using (1.)), and $\mathcal{K}$ is an upper bound for Γ. By Zorn's lemma we are done.
4. We use (2.) and maximality. ☐ Lemma 9.5

Theorem 9.6.

1. *If $a \in \mathcal{A}$ then $\sigma(a) = \{\, \lambda \in \mathbb{C} : \lambda 1 - a \notin \mathcal{A}^\times \,\} \neq \varnothing$.*

2. *(Gelfand-Mazur) If a (commutative, unital) Banach algebra is a division ring, then $\mathcal{A} = \mathbb{C}1_{\mathcal{A}}$.*

Proof.

1. This is done exactly as in the case $\mathcal{B}(\mathcal{X})$ (bounded operators on $\mathcal{X}$).

2. If there were $a \in \mathcal{A}\backslash\mathbb{C}1_{\mathcal{A}}$, then $\lambda 1 - a \notin \mathcal{A}^\times$ for all $\lambda \in \mathbb{C}$, contradicting the first point. ☐ Theorem 9.6

Theorem 9.7. *If $\mathcal{A}$ is a unital commutative Banach algebra, then its set of distinct maximal ideals is $\{\ker(\chi) : \chi \in \widehat{\mathcal{A}}\}$. (i.e. if $\chi_1 \neq \chi_2$ then $\ker(\chi_1) \neq \ker(\chi_2)$.)*

Proof. Since $\mathcal{A}/\ker(\chi) \cong \mathbb{C}$ is a field, each $\ker(\chi$ is a maximal ideal. If $\ker(\chi) = \ker(\chi')$ then for any $a \in \mathcal{A}$ we have

$$\chi(a)1_{\mathcal{A}} - a \in \ker(\chi) = \ker(\chi')$$

so

$$\chi'(a) = \chi'(\chi(a)1_{\mathcal{A}} - (\chi(a)_{\mathcal{A}} - a)) = \chi(a)$$

so $\chi = \chi'$.

If $\mathcal{M}$ is a maximal ideal of $\mathcal{A}$ then $\mathcal{A}/\mathcal{M}$ (with quotient norm

$$\|a + \mathcal{M}\| = \inf_{b \in \mathcal{M}} \|a - b\|$$

which one should check forms a Banach algebra) admits no proper ideals. Indeed, if $\mathcal{J} \subsetneqq \mathcal{A}/\mathcal{M}$ is an ideal, then $\mathcal{M} \subseteq q^{-1}(\mathcal{J}) \subsetneqq \mathcal{A}$ (where $q\colon \mathcal{A} \to \mathcal{A}/\mathcal{M}$ is the quotient map) and $q^{-1}(\mathcal{J})$ is an ideal, so $q^{-1}(\mathcal{J}) = \mathcal{M}$, and $\mathcal{J} = \{0 + \mathcal{M}\}$. Thus for $a \in \mathcal{A}\backslash\mathcal{M}$ we have

$$1_{\mathcal{A}} + \mathcal{M} \in \underbrace{(a + \mathcal{M}) \cdot (\mathcal{A}/\mathcal{M})}_{\text{principal ideal}}$$

and $a + \mathcal{M} \in (\mathcal{A}/\mathcal{M})^\times$. By the Gelfand-Mazur theorem, we have $\mathcal{A}/\mathcal{M} = \mathbb{C}(1_{\mathcal{A}} + \mathcal{M})$. Let $\chi\colon \mathcal{A} \to \mathbb{C}$ be given by $\chi(a)(1_{\mathcal{A}} + \mathcal{M}) = a + \mathcal{M}$. Then $\chi \in \widehat{\mathcal{A}}$ and $\mathcal{M} = \ker(\chi)$. ☐ Theorem 9.7

Corollary 9.8.

1. *We have*
$$\mathcal{A}\backslash\mathcal{A}^\times = \bigcup_{x \in \widehat{\mathcal{A}}} \ker\chi$$

2. *If $a \in \mathcal{A}$ then*
$$\sup_{\chi \in \widehat{\mathcal{A}}} |\chi(a)| = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$$

Proof.

1. If $a \in \mathcal{A}^\times$, we already saw that
$$a \in \mathcal{A}\backslash \bigcup_{x \in \widehat{\mathcal{A}}} \ker(\chi)$$
If $a \in \mathcal{A}\backslash\mathcal{A}^\times$ then $a\mathcal{A}$ is a proper ideal, and hence is contained in a maximal ideal $\ker(\chi)$.

2. Let $\lambda \in \mathbb{C}$ and $a \in \mathcal{A}$. Then
$$\begin{aligned} \lambda \in \sigma(a) &\iff \lambda 1_{\mathcal{A}} - a \in \mathcal{A}\backslash\mathcal{A}^\times \\ &\iff \lambda_{\mathcal{A}} - a \in \ker(\chi) \text{ for some } \chi \in \widehat{\mathcal{A}} \\ &\iff \lambda = \chi(a) \end{aligned}$$
Hence
$$\sup_{\chi \in \widehat{\mathcal{A}}} |\chi(a)| = \max_{\lambda \in \sigma(a)} |\lambda| = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$$
by Beurling's spectral radius formula. ☐ Corollary 9.8

10 Abelian harmonic analysis

Let G be a locally compact abelian grape.

Remark 10.1. Both $L^1(G)$ and $M(G)$ are abelian Banach algebras. (Indeed we have

$$\int_G h\mathrm{d}(\mu * \nu) = \int_G \int_G h(xy)\mathrm{d}\mu(x)\mathrm{d}\nu(y)$$

at which point we can apply Fubini-Tonelli.)

Proposition 10.2. *Suppose $\tau\colon G \to \mathbb{C}^\times$ is a continuous homomorphism. Then*

1. *$\tau = |\tau|\sigma$ where $\sigma\colon G \to \mathbb{T}$ is a continuous homomorphism.*
2. *τ is bounded if and only if $\tau(G) \subseteq \mathbb{T}$.*
3. *The set $\widehat{G} = \{\, \sigma\colon G \to \mathbb{T} \mid \sigma \text{ a continuous homomorphism} \,\}$ is a grape under pointwise operations.*

Proof.

1. We let
$$\sigma(x) = \frac{\tau(x)}{|\tau(x)|}$$
for $x \in G$.
2. We have $|\tau|(G) \subseteq (0,\infty)$. Then τ is bounded if and only if $|\tau|(G) = \{\, 1 \,\}$.
3. Obvious. Notice that $\sigma^{-1} = \overline{\sigma}$ (pointwise conjugation). $\square$ Proposition 10.2

Definition 10.3. We call $\widehat{G}$ the *dual grape* of G.

Theorem 10.4. *We have*

1. *$\widehat{L^1(G)} = \{\, \chi_\sigma : \sigma \in \widehat{G} \,\}$ where*
$$\chi_\sigma(f) = \int_G f\sigma\mathrm{d}m$$
(Recall $\widehat{L^1(G)}$ is the Gelfand spectrum.) Note that $\widehat{G} \subseteq C_b(G) \subseteq L^\infty(G)$.
2. *$\widehat{G} \cup \{\, 0 \,\}$ is a* w*- *compact set in $L^\infty(G)$, and hence $\widehat{G}$ is* w*- *locally compact.*
3. *$(\widehat{G}, \mathrm{w}^*)$ is a locally compact grape.*

Proof.

1. Let
$$\mathcal{A} = \begin{cases} L^1(G) = \ell^1(G) & \text{if } G \text{ discrete} \\ L^1(G) \oplus_{\ell^1} \mathbb{C}\delta_e \hookrightarrow \mathcal{M}(G) & \text{else} \end{cases}$$
If $\chi \in \widehat{L^1(G)}$, define $\widetilde{\chi}\colon \mathcal{A} \to \mathbb{C}$ by $\widetilde{\chi}(f + \lambda\delta_e) = \chi(f) + \lambda$ and $\widetilde{\chi} \in \widehat{\mathcal{A}}$. Hence $\|\widetilde{\chi}\| \leqslant 1$, so $\|\chi\| = \|\widetilde{\chi}{\restriction}L^1(G)\| \leqslant 1$, and in particular χ is bounded.

   We fix $\chi \in \widehat{L^(G)}$ and let $f, g \in L^1(G)$ with $\chi(f), \chi(g) \neq 0$. Then for $x \in G$ we have
$$\chi(x * f)\chi(g) = \chi(x * f * g) = \chi(x * g * f) = \chi(x * g)\chi(f)$$
Hence
$$\sigma(x) = \frac{\chi(x * f)}{\chi(f)}$$

is independent of $f \in L^1(G)\backslash\ker(\chi)$. Notice that σ is bounded in x:

$$|\sigma(x)| = \frac{|\chi(x * f)|}{|\chi(f)|} \leqslant \frac{\|x * f\|_1}{|\chi(f)|} = \frac{\|f\|_1}{|\chi(f)|}$$

and σ is continuous as the map $G \to L^1(G)$ given by $x \mapsto x * f$ is continuous.
If $x, y \in G$ and $f \in L^1(G)\backslash\ker(\chi)$ then $\chi(f * f) = \chi(f)^2 \neq 0$, so

$$\sigma(xy) = \frac{\chi(x * y * f * f)}{\chi(f * f)} = \frac{\chi(x * f * y * f)}{\chi(f)^2} = \sigma(x)\sigma(y)$$

so $\sigma\colon G \to \mathbb{C}^\times$ is a bounded homomorphism, and $\sigma \in \widehat{G}$.
Notice that if $\sigma \neq \tau$ in $\widehat{G}$ then $\{\, x \in G : \sigma(x) \neq \tau(x) \,\}$ is open in G, and hence not locally m-null, and $\chi_\sigma \neq \chi_\tau$.
Finally, notice that for $g \in L^1(G)$ we have

$$\chi_\sigma(g) = \int_G g\sigma \mathrm{d}m = \int_G g(x)\frac{\chi(x * f)}{\chi(f)}\mathrm{d}x = \frac{1}{\chi(f)}\chi\left(\underbrace{\int_G g(x) x * f \mathrm{d}y}_{g*f}\right) = \chi(g)$$

2. By Banach-Alaoglu it suffices to show that $\widehat{G} \cup \{\,0\,\} \subseteq B(L^\infty(G))$ is w*-closed. If $(\sigma_\alpha)_\alpha$ is a net in $\widehat{G} \cup \{\,0\,\}$ converging to $\sigma \in B(L^\infty(G))$, we can see for $f, g \in L^1(G)$ that

$$\langle f * g, \sigma\rangle = \lim_\alpha \langle f * g, \sigma_\alpha\rangle = \lim_\alpha \langle f, \sigma_\alpha\rangle\langle g, \sigma_\alpha\rangle = \langle f, \sigma\rangle\langle g, \sigma\rangle$$

so $\sigma \in \widehat{G} \cup \{\,0\,\}$. (Note that if $\tau \in \widehat{G}$ then

$$\langle f * g, \tau\rangle = \int_G\int_G f(x)g(x^{-1}y)\tau(y)\mathrm{d}x\mathrm{d}y = \int_g\int_G f(x)g(y)\tau(xy)\mathrm{d}x\mathrm{d}y = \langle f, \tau\rangle\langle g, \tau\rangle$$

which yields the desired result.)

If $\sigma \in \widehat{G}$ then since the weak*-topology is Hausdorff, there is a w*-openset W containing σ such that $0 \notin \overline{W}$. But $\overline{W} \cap \widehat{G} = \overline{W} \cap (\widehat{G} \cup \{\,0\,\})$ is compact.

3. Let $M\colon L^\infty(G) \to \mathcal{B}(L^2(G))$ (bounded linear operators) be given by $M(\varphi)\xi = \varphi{\cdot}\xi$ (m-almost-everywhere pointwise multiplication). Then for$\xi, \eta \in L^2(G)$ we have

$$\langle \xi \mid M(\varphi)\eta\rangle = \int_G \varphi \underbrace{\bar{\xi}\eta}_{\substack{\in L^1(G),\\ \text{Cauchy-Schwarz}}} \mathrm{d}m$$

Also, if $f \in L^1(G)$, then

$$\langle \varphi, f\rangle = \int_G \varphi f \mathrm{d}m = \langle \overline{\operatorname{sgn}} f \cdot |f|^{\frac{1}{2}} \mid M(\varphi)|f|^{\frac{1}{2}}\rangle$$

Hence M is a w*-WO homeomorphism onto its range; i.e. $\varphi_\alpha \xrightarrow{\mathrm{w}^*,\alpha}$ in $L^\infty(G)$ if and only if $M(\varphi_\alpha) \xrightarrow{\mathrm{WO},\alpha} M(\varphi)$ in $M(L^\infty(G))$. Now, since for $\sigma \in \widehat{G}$ we have $\sigma(G) \subseteq \mathbb{T}$ we see that $M(\sigma) \in U(L^2(G))$. (One checks that $M(\overline{\varphi}) = M(\varphi)^*$. Hence $M{\restriction}\widehat{G}\colon \widehat{G} \to M(\widehat{G}) \subseteq U(L^2(G))$ is a w*-WO homeomorphism. The result then follows. □ Theorem 10.4

Proposition 10.5.

1. *If G is discrete, then $\widehat{G}$ is compact.*

2. *If G is compact, then $\widehat{G}$ is discrete.*

Proof.

1. $L^1(G) = \ell^1(G)$ is unital, so $\widehat{G} \cong \widehat{\ell^1(G)}$ is compact.

2. We normalize m so $m(G) = 1$. if$\sigma \in \widehat{G}\backslash\{ 1 \}$, then there is $y \in G$ with $\sigma(y) \neq 1$. hence

$$\int_G \sigma(x)\mathrm{d}x = \int_G \sigma(yx)\mathrm{d}x = \sigma(y)\int_G \sigma(x)\mathrm{d}x$$

   and hence

$$\int_G \sigma(x)\mathrm{d}x = 0$$

   Clearly

$$\int_G 1(x)\mathrm{d}x = 1$$

   Hence

$$\left\{ \tau \in \widehat{G} : |\langle \tau, 1\rangle - \underbrace{\langle 1, 1\rangle}_{1}| < \frac{1}{2} \right\}$$

   is a w*-open neighbourhood of 1 and equals 1. Thus $\widehat{G}$ is discrete. □ Proposition 10.5

Example 10.6.

1. Consider $G = \mathbb{Z}$; we use additive notation. if $\sigma \in \widehat{\mathbb{Z}}$, let $z = \sigma(1)$ (where 1 is the generator of $\mathbb{Z}$, not its identity). Then for $n \in \mathbb{Z}$ we have $\sigma(n) = z^n$. Write $\sigma = \sigma_z$. Clearly for any $z \in \mathbb{T}$ we have σ_z defines an element of $\widehat{\mathbb{Z}}$. Thus $\widehat{\mathbb{Z}} = \{ \sigma_z : z \in \mathbb{T} \}$, and if $z \neq z'$ then $\sigma_z \neq \sigma_{z'}$.

   Let us consider a w*-open neighbourhood of $1 = \sigma_1 \in \widehat{\mathbb{Z}}$

$$U = \bigcap_{k=-n}^{n} \{ \sigma_z \in \widehat{\mathbb{Z}} : |\langle \sigma_z, \delta_k\rangle - \langle \sigma_z, \delta_0\rangle| < 1 \} = \bigcap_{k=-n}^{n} \{ \sigma \in \widehat{\mathbb{Z}} : |z^k - 1| < 1 \}$$

   Write $z = \exp(it)$ for $-\pi < t \leqslant \pi$. For $k \in \{ -n, \ldots, n \}$ we have

$$1 > |z^k - 1|^2 = |\exp(ikt) - 1|^2 = 2 - 2\cos(kt)$$

   So $\cos(kt) > \frac{1}{2}$ and $kt \in (-\frac{\pi}{3}, \frac{\pi}{3})$ (modulo 2π). Hence $U = \{ \exp(it) : t \in (-\frac{\pi}{3n}, \frac{\pi}{3n}) \}$. Hence a w*-neighbourhood of σ_1 in $\widehat{\mathbb{Z}}$ is a neighbourhood base of 1 in $\mathbb{T}$. Thus $\mathbb{T} \cong \{ \sigma_z : z \in \mathbb{T} \}$ has an induced w*-topology finer than the ambient topology. On sets, comparable compact Hausdorff topologies coincide.

2. Consider $G = \mathbb{R}$. Suppose $\sigma \in \widehat{\mathbb{R}}$. Then σ is continuous with $\sigma(0) = 1$, so there is $\alpha > 0$ so

$$\int_0^\alpha \sigma(x)\mathrm{d}x \neq 0$$

   Now if $y \in \mathbb{R}$ then

$$\sigma(y)\int_0^\alpha \sigma(x)\mathrm{d}x = \int_0^\alpha \sigma(y + x)\mathrm{d}x = \int_{-y}^{\alpha - y} \sigma(x)\mathrm{d}x$$

   The fundamental theorem of calculus then tells us that σ is differentiable. Now, for $x \in \mathbb{R}$ we have

$$\sigma'(x) = \lim_{h\to 0} \frac{\sigma(x + h) - \sigma(x)}{h} = \sigma(x)\lim_{h\to 0}\frac{\sigma(h) - \sigma(0)}{h} = \sigma(x)\sigma'(0)$$

Let $f(x) = \exp(-\sigma'(0)x)\sigma(x)$. Then $f(0) = 1$ and $f'(x) = 0$ (product rule) so by the mean value theorem we have $f(x) = 1$ for all x; i.e. $\sigma(x) = \exp(zx)$ (where $z \in \mathbb{C}$). Moreover $\sigma(\mathbb{R}) \subseteq \mathbb{T}$, so $a = is$ for $s \in \mathbb{R}$. Let $\sigma = \sigma_s$, where $\sigma_s(x) = \exp(isx)$. Clearly $s \neq t$ in $\mathbb{R}$, so $\sigma_s \neq \sigma_t$, and $\sigma_s \in \widehat{\mathbb{R}}$.

Consider a w*-open neighbourhood of σ_0:

$$\begin{aligned} U_{a,\varepsilon} &= \left\{ \sigma \in \widehat{\mathbb{R}} : |\langle \sigma_s, 1_{[-a,a]}\rangle - \langle \sigma_0, 1_{[-a,a]}\rangle| < \varepsilon \right\} \\ &= \left\{ \sigma_s \in \widehat{\mathbb{R}} : \left| \int_{-a}^{a} (\exp(isx) - 1)\mathrm{d}x \right| < \varepsilon \right\} \\ &= \left\{ \sigma_s \in \widehat{\mathbb{R}} : 2\left| \underbrace{\frac{\sin(as)}{s} - a}_{\psi_a(s)} \right| < \varepsilon \right\} \end{aligned}$$

where ψ_a is an analytic and hence continuous function. Also

$$\lim_{s \to \pm\infty} |\psi_a(s)| = |a|$$

and

$$\lim_{a \to \infty} \psi_a(s) = \infty$$

We conclude that $\{ U_{a,\varepsilon} : a > 0, \varepsilon > 0 \}$ is a usual neighbourhood basis of 0 in $\mathbb{R}$. Hence the weak* topology is finer than the ambient topology. But

$$\text{w*-}\lim_{s \to t} \sigma_s = \sigma_t$$

(easy exercise). So the weak* topology is coarser than the ambient topology. So

$$\widehat{\mathbb{R}} = \{ \sigma_s : s \in \mathbb{R} \} \cong \mathbb{R}$$

as locally compact grapes.

3. Consdier $G = \mathbb{T}$. Consider $\sigma_1 : \mathbb{R} \to \mathbb{T}$ with $\sigma_1(t) = \exp(it)$; so $\ker(\sigma_1) = 2\pi\mathbb{Z}$. If $\tau \in \widehat{\mathbb{T}}$ then $\tau \circ \sigma_1 \in \widehat{\mathbb{R}}$ so $\tau \circ \sigma_1(x) = \exp(isx)$ for some $s \in \mathbb{R}$, with $1 = \tau \circ \sigma_1(2\pi) = \exp(i2\pi s)$, so $s = n \in \mathbb{Z}$. Hence $\tau \circ \sigma_1(x) = \exp(ixn) = \sigma_1(x)^n$ for $x \in \mathbb{R}$. Hence $\widehat{\mathbb{T}} = \{ z \mapsto z^n : n \in \mathbb{Z} \}$. The topology is discrete.

Suppose $\mathcal{A}$ is a commutative unital Banach algebra; e.g. $\mathcal{A} = L^1(G) + \mathbb{C}\delta_e \subseteq M(G)$. Recall Beurling's spectral radius formula:

$$\sup_{\chi \in \hat{\mathcal{A}}} \|\chi(a)\| = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}} \leqslant \|a\|$$

Definition 10.7. For $f \in L^1(G)$ we define the *Fourier transform* of f to be $\widehat{f} : \widehat{G} \to \mathbb{C}$ given by

$$\widehat{f}(\sigma) = \int_G f\overline{\sigma}\mathrm{d}m$$

Theorem 10.8 (Riemann-Lebesgue, Gelfand)**.** *The map $L^1(G) \to C_0(\widehat{G})$ given by $f \mapsto \widehat{f}$ is a homomorphism with*

1. $\|\widehat{f}\|_\infty = \lim_{n \to \infty} \|f^{*n}\|_1^{\frac{1}{n}} \leqslant \|f\|_1$.
2. *$A(\widehat{G}) = \{ \widehat{f} : f \in L^1(G) \}$ is dense in $C_0(\widehat{G})$.*

Proof. We recall that $\widehat{G} \cup \{ 0 \}$ is compact. We have that $\widehat{f}(\sigma) = \chi_{\overline{\sigma}}(f)$ is continuous in σ as $\widehat{G}$ has the weak* topology. If we let $\widehat{f}(0) = 0$, then $\widehat{f}$ is continuous on $\widehat{G} \cup \{ 0 \}$ (from the proof of a previous theorem)

TODO 7. *which*

Hence $\widehat{f} \in C_0(\widehat{G})$. We now verify the required conditions.

1. This is simply Beurling's spectral radius formula.

2. We notice that $A(\widehat{G})$ is point-separating on $\widehat{G}$. (If $\sigma \neq \tau$ in $\widehat{G}$ then $\chi_{\overline{\sigma}} \neq \chi_{\overline{\tau}}$, so there is $f \in L^1(G)$ with

$$\widehat{f}(\sigma) = \chi_{\overline{\sigma}}(f) \neq \chi_{\overline{\tau}}(f) = \widehat{f}(\tau)$$

Since $f \mapsto \widehat{f}$ is (almost) the Gelfand transform, we get that $f \mapsto \widehat{f}$ is multiplicative, so $A(\widehat{G})$ is a subalgebra. We also have for $f \in L^1(G)$ and $\sigma \in \widehat{G}$ that

$$\widehat{f^*}(\sigma) = \int_G f^*(x)\overline{\sigma(x)}\mathrm{d}x = \int_G \overline{f(x^{-1})\sigma(x)}\mathrm{d}x = \int_G \overline{f(x)}\sigma(x)\mathrm{d}x = \overline{\widehat{f}(\sigma)}$$

So $\widehat{f^*} = \overline{\widehat{f}}$ (pointwise conjugate). So by Stone-Weierstrass theorem, we're done. ☐ Theorem 10.8

Lemma 10.9. *The map $G \times \widehat{G} \to \mathbb{T}$ given by $(x, \sigma) \mapsto \sigma(x)$ is continuous.*

Proof. Fix $\sigma \in \widehat{G}$ and $x \in G$. Let $f \in L^1(G)$ have $\widehat{f}(\sigma) \neq 0$. Then

$$\widehat{f}(\sigma)\sigma(x) = \int_G f(x)\overline{\sigma(yx^{-1})}\mathrm{d}y = \int_G f(xy)\overline{\sigma(y)}\mathrm{d}y = \widehat{f \cdot x}(\sigma)$$

Now if also $\tau \in \widehat{G}$ and $y \in G$ then

$$\begin{aligned}
\left|\widehat{f}(\sigma)\sigma(x) - \widehat{f}(\tau)\tau(y)\right| &= \left|\widehat{f \cdot x}(\sigma) - \widehat{f \cdot y}(\tau)\right| \\
&\leqslant \left|\widehat{f \cdot x}(\sigma) - \widehat{f \cdot x}(\tau)\right| + \left|\widehat{f \cdot x}(\tau) - \widehat{f \cdot y}(\tau)\right| \\
&\leqslant \left|\widehat{f \cdot y}(\sigma) - \widehat{f \cdot y}(\tau)\right| + \|f \cdot x - f \cdot y\|_1 \\
&\xrightarrow{y \to x, \tau \to \sigma} 0
\end{aligned}$$

Since $\widehat{f}$ is continuous, this shows that $\tau(y) \xrightarrow{y \to x, \tau \to \sigma} \sigma(x)$. ☐ Lemma 10.9

Definition 10.10. A function $u \colon G \to \mathbb{C}$ is called *positive-definite* if for each $x_1, \ldots, x_n \in G$ and $n \in \mathbb{N}$ the matrix $[u(x_j^{-1}x_i)]$ is positive semidefinite; i.e. if for $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ we have

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_i \overline{\lambda_j} u(x_j^{-1}x_i) = \left\langle \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \middle| [u(x_j^{-1}x_i)] \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \right\rangle \geqslant 0$$

Proposition 10.11. *A positive-definite function $u \colon G \to \mathbb{C}$ satisfies*

1. *$u(x^{-1}) = \overline{u(x)}$ for $x \in G$*

2. *$|u(x)| \leqslant u(e)$ for $x \in G$.*

Proof. Let $u = 2$, $x_1 = e$, and $x_2 = x$. Then

$$\begin{pmatrix} u(e) & u(x^{-1}) \\ u(x) & u(e) \end{pmatrix}$$

is positive semidefinite. Then the claims are just exercises in linear algebra. ☐ Proposition 10.11

Notation 10.12. We let $B^+(G)$ denote the space of continuous positive definite functions on G.

So $B^+(G) \subseteq C_b(G)$.

Example 10.13.

1. Note that $\widehat{G} \subseteq B^+(G)$. Indeed, if $x_1, \ldots, x_n \in G$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ then

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_i \overline{\lambda_j} \underbrace{\sigma(x_j^{-1} x_i)}_{\overline{\sigma(x_j)}\sigma(x_i)} = \left| \sum_{j=1}^n \lambda_j \sigma(x_j) \right|^2 \geqslant 0$$

2. (Reverse Fourier-Stieltjes transform) If $\mu \in M(\widehat{G})$, we let $\check{\mu} \colon G \to \mathbb{C}$ be

$$\check{\mu}(x) = \int_{\widehat{G}} \sigma(x) \mathrm{d}\mu(\sigma)$$

   If $\mu \in M_+(G)$ then $\check{\mu}$ is positive definite. Indeed, suppose $x_1, \ldots, x_n \in G$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Then

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_i \overline{\lambda_j} \underbrace{\check{\mu}(x_j^{-1} x_i)}_{\int_{\widehat{G}} \overline{\sigma(x_j)}\sigma(x_i) \mathrm{d}\mu(\sigma)} = \int_{\widehat{G}} \left| \sum_{j=1}^n \lambda_j \sigma(x_j) \right|^2 \mathrm{d}\mu(\sigma) \geqslant 0$$

Proposition 10.14. *If $\mu \in M(\widehat{G})$ then $\check{\mu}$ is uniformly continuous.*

Proof. First, suppose $K = \mathrm{supp}(\mu)$ is compact in $\widehat{G}$. Suppose $\varepsilon > 0$, and for each $\sigma \in K$ let

- U_σ be a neighbourhood of e in G such that $x \in U_\sigma$ implies $|\sigma(x) - 1| < \varepsilon$
- W_σ be a neighbourhood of σ in $\widehat{G}$, $V_\sigma \subseteq U_\sigma$ be such that

$$\tau \in W_\sigma, x \in V_\sigma \implies |\tau(x) - 1| < \varepsilon$$

(by joint continuity of $G \times \widehat{G} \to \mathbb{T}$). We have that

$$K \subseteq \bigcup_{i=1}^n W_{\sigma_i}$$

for some $\sigma_1, \ldots, \sigma_n \in K$, and we let

$$V = \bigcap_{i=1}^n V_{\sigma_i} \subseteq G$$

Hence if $x \in V$ and $\tau \in K$ then $|\tau(x) - 1| < \varepsilon$. Now, if $x, y \in G$ with $xy^{-1} \in V$ then

$$|\check{\mu}(x) - \check{\mu}(y)| \leqslant \int_{\widehat{G}} |\sigma(x) - \sigma(y)| \mathrm{d}|\mu|(\sigma) = \int_{\widehat{G}} \underbrace{|\sigma(xy^{-1}) - 1|}_{<\varepsilon} \mathrm{d}|\mu|(G) \leqslant \varepsilon |\mu|(G)$$

Now if $\mu \in M(\widehat{G})$, we can find compact $K \subseteq \widehat{G}$ so $\|\mu - \mu_K\|_1 < \varepsilon$. The usual approximation of $\check{\mu}$ by $\widecheck{\mu_K}$ applies $\square$ Proposition 10.14

Corollary 10.15. *If $\mu \in M_+(\widehat{G})$, then $\check{\mu} \in B^+(G)$.*

A problem: we don't yet know that $f \neq 0$ in $L^1(G)$ implies $\widehat{f} \neq 0$ in $C_0(\widehat{G})$.

Proposition 10.16 (Injectivity of the reverse Fourier-Stieltjes transform)**.** *If $\mu \neq \nu$ in $M(\widehat{G})$ then $\check{\mu} \neq \check{\nu}$ in $C_b(G)$.*

Proof. If $f \in L^1(G)$, we have for $\mu \in M(G)$ that

$$\int_{\widehat{G}} \widehat{f} \mathrm{d}\mu = \int_{\widehat{G}} \int_G f(x) \overline{\sigma(x)} \mathrm{d}x \mathrm{d}\mu(\sigma) = \int_G f(x) \int_{\widehat{G}} \sigma(x^{-1}) \mathrm{d}\mu(\sigma) \mathrm{d}x = \int_G f(x) \check{\mu}(x^{-1}) \mathrm{d}x \tag{3}$$

Let $\nu(E) = \mu(E^{-1})$ for $E \in \mathcal{B}(G)$. One can check that $\check{\nu}(x) = \check{\mu}(x^{-1})$. Hence if $\check{\mu} = 0$, then since $A(\widehat{G})$ is dense in $C_0(\widehat{G})$, we see that for $h \in C_0(\widehat{G})$ we have

$$\int_{\widehat{G}} h \mathrm{d}\mu = 0$$

and thus $\mu = 0$. It is evident that $\mu \mapsto \check{\mu}$ is linear. □ Proposition 10.16

Theorem 10.17 (Bochner's theorem)**.** $B^+(G) = \{ \check{\mu} : \mu \in M_+(G) \}$. *Hence the map* $M_+(G) \to B^+(G)$ *given by* $\mu \mapsto \check{\mu}$ *is a bijection.*

Proof. Suppose $u \in B^+(G) \backslash \{ 0 \}$. We normalize so $u(e) = \|u\|_\infty = 1$. Define a sesquilinear form on $L^1(G) \times L^1(G)$ by

$$[f \mid g] = \int_G f^* * g u \mathrm{d}m$$

Notice that

$$|[f \mid g]| \leqslant \|f^* * g\|_1 \|u\|_\infty \leqslant \|f\|_1 \|g\|_1$$

so $[\cdot \mid \cdot]$ is continuous on $L^1(G) \times L^1(G)$. Now

$$\begin{aligned}
[f \mid g] &= \int_G \int_G \overline{f(x^{-1})} g(x^{-1}y) u(y) \mathrm{d}x \mathrm{d}y \\
&= \int_G \int_G \overline{f(x^{-1})} g(y) u(xy) \mathrm{d}x \mathrm{d}y \\
&= \int_G \int_G \overline{f(x)} g(y) u(x^{-1}y) \mathrm{d}x \mathrm{d}y
\end{aligned}$$

(since G is unimodular). Suppose

$$\varphi = \sum_{i=1}^n a_i 1_{E_i} \in S^1(G)$$

(i.e. simple, integrable, $E_i \in \mathcal{B}(G)$, $m(E_i) < \infty$, and $E_i \cap E_j = \varnothing$ for $i \neq j$). (Assume also that $\operatorname{supp}(\varphi)$ is compact.)

Suppose $\varepsilon > 0$. We can assume by taking Borel decompositions of each E_i that there are $x_i \in E_i$ for each i such that

$$|u(x^{-1}y) - u(x_j^{-1}x_i)| m(E_j) m(E_i) < \frac{\varepsilon}{\sum_{i,j=1}^n |a_i||a_j| + 1}$$

by continuity of u. Then

$$S = \sum_{i=1}^n \sum_{j=1}^n \overline{a_j} a_i u(x_j^{-1}x_i) m(E_j) m(E_j) \geqslant 0$$

and

$$\begin{aligned}
|[\varphi \mid \varphi] - S| &= \left| \sum_{i=1}^n \sum_{j=1}^n \overline{a_j} a_i \int_{E_i} \int_{E_j} (u(x^{-1}y) - u(x_j^{-1}x_i)) \mathrm{d}x \mathrm{d}y \right| \\
&\leqslant \sum_{i=1}^n \sum_{j=1}^n |a_j||a_i| \sup_{(x,y) \in E_j \times E_i} |u(x^{-1}y) - u(x_j^{-1}x_i)| m(E_j) m(E_i) \\
&< \varepsilon
\end{aligned}$$

Hence $[\varphi \mid \varphi] > -\varepsilon$. The decomposition above can be done for any $\varepsilon > 0$; hence $[\varphi \mid \varphi] \geqslant 0$. Approximating f in $L^1(G)$ by elements φ as above, and using continuity of $[\cdot \mid \cdot]$ we get that $[f \mid f] \geqslant 0$.

We may apply Cauchy-Schwarz inequality to see that

$$|[f \mid g]|^2 \leqslant [f \mid f][g \mid g]$$

We let $\mathcal{V}$ denote a base at e in relatively compact symmetric neighbourhoods. If $V \in \mathcal{V}$, we let $k_V = (m(V))^{-1}1_V$. Notice that $k_V^* = k_V$ by unimodularity. Also $(k_V * k_V)_{V \in \mathcal{V}}$ is a summability kernel; i.e. $\|k_V * k_V\|_1 \leqslant$, $\operatorname{supp}(k_V * k_V) \subseteq V^2$, and

$$\int_G k_V * k_V \mathrm{d}m = \chi_1(k_V * k_V) = 1$$

In particular, we have

$$\lim_V [k_V \mid k_V] = \lim_V \int_G k_V * k_V u \mathrm{d}m = u(e) = 1$$

and

$$[k_V \mid f] = \int_G k_V * f u \mathrm{d}m \xrightarrow{V \searrow \{e\}} \int_G f u \mathrm{d}m$$

Hence

$$\left|\int_G f u \mathrm{d}m\right|^2 = \lim_V |[k_v \mid f]|^2 \leqslant \limsup_V [k_V \mid k_V][f \mid f] = [f \mid f]$$

Let $h = f^* * f$, so $h^* = h$. (One should check this.) Let $h^{*2} = h * h$, $h^{*4} = h^{*2} * h^{*2}$, etc. Then

$$\begin{aligned} \left|\int_G f u \mathrm{d}m\right|^2 \leqslant [f \mid f] &= \int_G h u \mathrm{d}m \\ &\leqslant [h \mid h]^{\frac{1}{2}} \\ &= \left(\int_G h^{*2} u \mathrm{d}m\right)^{\frac{1}{2}} \\ &\leqslant [h^{*2} \mid h^{*2}]^{\frac{1}{4}} \\ &\leqslant [h^{*4} \mid h^{*4}]^{\frac{1}{8}} \\ &\leqslant \cdots \\ &\leqslant [h^{*2^n} \mid h^{*2^n}]^{2^{-(n+1)}} \\ &= \left(\int_G h^{*2^{n+1}} u \mathrm{d}m\right)^{2^{-(n+1)}} \\ &\leqslant \left\|h^{*2^{n+1}}\right\|_1^{\frac{1}{2^{n+1}}} \\ &\xrightarrow{n \to \infty} \left\|\widehat{h}\right\|_\infty \end{aligned}$$

Thus

$$\left|\int f u \mathrm{d}m\right|^2 \leqslant \left\|\widehat{h}\right\|_\infty = \left\|\widehat{f^* f}\right\|_\infty = \left\|\left|\widehat{f}\right|^2\right\|_\infty = \left\|\widehat{f}\right\|_\infty$$

Since $A(\widehat{G})$ is dense in $C_0(\widehat{G})$ we have that

$$\widehat{f} \mapsto \int_G f u \mathrm{d}m$$

extends to a continuous linear functional on $C_0(\widehat{G})$. So, by the Riesz representation theorem, there is $\mu \in M(\widehat{G})$ with

$$\int_G f u \mathrm{d}m = \int_{\widehat{G}} \widehat{f} \mathrm{d}\mu$$

By Equation (3), we have

$$\int_{\widehat{G}} \widehat{f} \mathrm{d}\mu = \int_G f(x) \check{\mu}(x^{-1}) \mathrm{d}x = \int_G f(x) \check{\nu}(x) \mathrm{d}x$$

for some ν. Hence $u = \check{\nu}$. If $\varphi \in C_0(\widehat{G})$ then we may write

$$\varphi = \lim_{n \to \infty} \widehat{f_n}$$

by density of $A(\widehat{G})$. Then

$$\int_{\widehat{G}} |\varphi|^2 \mathrm{d}\mu = \lim_{n\to\infty} \int_{\widehat{G}} \overline{\widehat{f_n}}\widehat{f_n} \mathrm{d}\mu = \lim_{n\to\infty} \int_G f_n^* * f_n u \mathrm{d}m \geqslant 0$$

so $\mu \in M_+(G)$. ☐ Theorem 10.17

Proposition 10.18 (Another class of positive definite functions)**.** *Suppose $f \in L^1 \cap L^2(G)$. Then $f^* * f \in B^+ \cap L^1(G)$.*

Proof. That $f^* * f \in L^1(G)$ follows from the closure of $L^1(G)$ under convolution. We compute, for almost every $x \in G$,

$$\begin{aligned}(f^* * f)(x) &= \int_G \overline{f(y^{-1})} f(y^{-1}x)\mathrm{d}x \\ &= \int_G \overline{\tilde{f}(y)}\tilde{f}(x^{-1}y)\mathrm{d}y \\ &= \left\langle \tilde{f} \middle| x * \tilde{f} \right\rangle \\ &= \left\langle x^{-1} * \tilde{f} \middle| \tilde{f} \right\rangle \text{ (inner product on } L^2(G))\end{aligned}$$

where $\tilde{f}(y) = f(y^{-1})$ for almost every y; note that $\tilde{f} \in L^1 \cap L^2(G)$ by unimodularity. Since $C_c(G)$ is dense in $L^2(G)$, we get that $L^2(G)$ has continuity of translation (same proof as for $L^1(G)$). Hence $x \mapsto \langle \tilde{f}, x * \tilde{f} \rangle$ is continuous, so $f^* * f$ may be taken to be continuous. Now let $x_1, \ldots, x_n \in G$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. Then

$$\begin{aligned}&\sum_{j=1}^n \sum_{i=1}^n \overline{\lambda_j}\lambda_i f^* * f(x_j^{-1}x_i) \\ &= \sum_{j=1}^n \sum_{i=1}^n \overline{\lambda_j}\lambda_i \left\langle x_j * \tilde{f} \middle| x_i * \tilde{f} \right\rangle \\ &= \left\| \sum_{i=1}^n \lambda_i x_i * \tilde{f} \right\|_2^2 \\ &\geqslant 0\end{aligned}$$

as desired. ☐ Proposition 10.18

Corollary 10.19. *If $f \in C_c(G)$ then $f^* * f \in B^+ \cap L^1(G)$.*

We let $B(G) = \{\check{\mu} : \mu \in M(\widehat{G})\}$. Since the map $M(\widehat{G}) \to V(G) \subseteq C_{ub}(G)$ (where the latter is the collection of uniformly continuous bounded functions on G) given by $\mu \mapsto \check{\mu}$ is linear (easily seen). The Hahn-Jordan decomposition of measures then shows that $B(G) = \operatorname{span} B^+(G)$.

Exercise 10.20 (Probably on A3). Show that the map $G \to B^1(G)$ given by $x \mapsto x * f$ is continuous in G and isometric in the norm on $B^1(G)$ given by $\|f\|_{B^1(G)} = \|f\|_1 + \|\mu\|_1$ where $f = \check{\mu}$ by Bochner's theorem.

Theorem 10.21 (Inversion theorem)**.** *Let $B^1(G) = B \cap L^1(G)$.*

1. *If $f \in B^1(G)$ then $\hat{f} \in L^1(\widehat{G})$.*
2. *For a suitable normalization of the Haar measures m_G and $m_{\widehat{G}}$ we have for $f \in V^1(G)$ that*

$$f(x) = \int_{\widehat{G}} \hat{f}(\sigma)\sigma(x)\mathrm{d}\sigma$$

i.e. $f = \check{\hat{f}}$.

Proof. We proceed in stages.

(I) If $h \in L^1(G)$ and $f = \check{\mu} \in B^1(G)$, then

$$(h * \check{\mu})(e) = \int_G h(x)\check{\mu}(x^{1-}e)\mathrm{d}x = \int_G \int_{\widehat{H}} h(x)\overline{\sigma(x)}\mathrm{d}\mu(\sigma)\mathrm{d}x = \int_{\widehat{G}} \widehat{h}\mathrm{d}\mu$$

If also $g = \check{\nu} \in B(G)$ then

$$\int_{\widehat{G}} \widehat{h}\widehat{\check{\nu}}\mathrm{d}\mu = \int_{\widehat{G}} \widehat{h * \check{\nu}}\mathrm{d}\mu = (h * \check{\nu} * \check{\mu})(e) = (h * \check{\mu} * \check{\nu}) = \int_{\widehat{G}} \widehat{h}\widehat{\check{\mu}}\mathrm{d}\nu$$

Since $A(\widehat{G}) = \{\, \widehat{f} : f \in L^1(G) \,\}$ is dense in $C_0(G)$, we have

$$\widehat{\check{\nu}}\mathrm{d}\mu = \widehat{\check{\mu}}\mathrm{d}\nu \tag{4}$$

i.e.

$$\text{“}\frac{\mathrm{d}\mu}{\mathrm{d}\nu} = \frac{\widehat{\check{\mu}}}{\widehat{\check{\nu}}}\text{”}$$

almost everywhere on $\widehat{G}$.

(II) We will define a functional J on $C_c(\widehat{G})$, which will give (1). Fix $\psi \in C_c(\widehat{G})$. For each $\sigma \in \mathrm{supp}(\psi)$ there is $u \in C_c(G)$ with $\widehat{u}(\sigma) \neq 0$ (since $C_c(G)$ is dense in $L^1(G)$). Then

$$\widehat{u^* * u}(\sigma) = \overline{\widehat{u}(\sigma)}\widehat{u}(\sigma) > 0$$

and hence, by compactness, we may find $u_1, \ldots, u_n \in C_c(G)$ such that

$$g = \sum_{i=1}^n u_i^* * u_i$$

- $\mathrm{supp}(\psi) \subseteq \mathrm{supp}^\circ(\widehat{g}) = \{\, \sigma \in \widehat{G} : \widehat{g}(\sigma) \neq 0 \,\}$
- $g \in B^+ \cap L^1(G) \subseteq B^1(G)$ (by the previous corollary), and hence $g = \check{\nu}_0$ for some $\nu_0 \in M_+(\widehat{G})$ (by Bochner's theorem).

We let

$$J(\psi) = \int_{\widehat{G}} \frac{\psi}{\widehat{\check{\nu}}_0}\mathrm{d}\nu_0$$

If $f = \check{\mu} \in B^1(G)$ then we use Equation (4):

$$\begin{aligned} J(\psi) &= \int_{\widehat{G}} \frac{\psi}{\widehat{\check{\nu}}_0\widehat{\check{\mu}}}\widehat{\check{\mu}}\mathrm{d}\nu_0 \\ &= \int_{\widehat{G}} \frac{\psi}{\widehat{\check{\nu}}_0\widehat{\check{\mu}}}\widehat{\check{\nu}}_0\mathrm{d}\mu \\ &= \int_{\widehat{G}} \frac{\psi}{\widehat{\check{\mu}}}\mathrm{d}\mu \end{aligned}$$

where

$$\psi\frac{\widehat{f}}{\widehat{f}} = \psi 1_{\mathrm{supp}^\circ(\widehat{f})}$$

Again, Equation (4) tells us that this is independent of the choice of $\mu \in M(\widehat{G})$ with $\check{\mu} \in B^1(G)$. Notice that since $\widehat{g} = \widehat{\check{\nu}}_0 \geqslant 0$, we see that $J(\psi) > 0$ if $\psi \in C_c^+(G)$. Also

$$J(\psi\widehat{\check{\mu}}) = \int_{\widehat{G}} \psi\mathrm{d}\mu \tag{5}$$

for appropriate μ. Now let $\psi \in C_c(G)$ and $\tau \in \widehat{G}$; then for suitable $\nu \in M(\widehat{G})$ we have

$$J(\psi \cdot \tau) = \int_{\widehat{G}} \frac{\psi(\tau\sigma)}{\widehat{\widehat{\nu}}(\sigma)} \mathrm{d}\nu(\sigma) = \int_{\widehat{G}} \frac{\psi(\sigma)}{\widehat{\widehat{\nu}}(\overline{\tau}\sigma)} \mathrm{d}\nu(\overline{\tau}\sigma)$$

(Recall the change-of-variables formula

$$\int_X f \circ T \mathrm{d}\nu = \int_X f \mathrm{d}(\nu \circ T^{-1})$$

for integration with respect to pushforward measures.)

Exercise 10.22 (Probably A3). Show that

$$\begin{aligned} \check{\mu}(x) &= \tau(x)\check{\mu}(x) \\ \widehat{\widehat{\mu}}(\sigma) &= \widehat{\widehat{\nu}}(\overline{\tau}\sigma) \end{aligned}$$

In particular, the first equation shows that $\check{\mu} \in B^1(G)$.

We hence see, using Equation (4), that

$$J(\psi \cdot \tau) = \int_{\widehat{G}} \frac{\psi(\sigma)}{\widehat{\widehat{\mu}}(\sigma)} \mathrm{d}\mu(\sigma) = J(\psi)$$

So J is the Haar integral. Furthermore, Equation (5) yields for suitable μ and $\psi \in C_c(G)$ that

$$\int_{\widehat{G}} \psi \mathrm{d}\mu = J(\psi\widehat{\widehat{\mu}}) \tag{6}$$

i.e. $\mathrm{d}\mu(\sigma) = \widehat{\widehat{\mu}}(\sigma)\mathrm{d}\sigma$. Hence $\mu \in M_a(\widehat{G})$; i.e. $\mathrm{d}\mu = \widehat{\widehat{\mu}}\mathrm{d}m_{\widehat{G}}$ with $\widehat{\widehat{\mu}} \in L^1(G)$ (by Radon-Nikodym). This proves (1).

To see (2), note that Equation (6) yields for $x \in G$ and suitable μ that

$$\check{\mu}(x) = \int_{\widehat{G}} \sigma(x)\mathrm{d}\mu(\sigma) = \int_{\widehat{G}} \sigma(x)\widehat{\widehat{\mu}}(\sigma)\mathrm{d}\sigma$$

Writing $f = \check{\mu}$, we are done. ☐ Theorem 10.21

We consider what constitutes "suitable" normalizations of m_G and $m_{\widehat{G}}$, as in the statement of the previous theorem.

1. Suppose G is compact and $m_G(G) = 1$. Then for $\sigma \in \widehat{G}$ we have, as in the proof of discreteness of $\widehat{G}$, that

$$\widehat{1}(\sigma) = \begin{cases} 1 & \text{if } \sigma = 1 \\ 0 & \text{else} \end{cases}$$

Since $1 \in B^+ \cap L^1(G) \subseteq B^1(G)$. Hence by the inversion theorem we have

$$1 = 1(e) = \int_{\widehat{G}} \widehat{1}(\sigma) \underbrace{\sigma(e)}_{=1} \mathrm{d}\sigma = m_{\widehat{G}}(\{1\})$$

So $m_{\widehat{G}}$ is the counting measure.

2. Suppose G is discrete. Let $m_G(\{e\}) = 1$; i.e. that m_G is the counting measure. Let $f = 1_{\{e\}} = 1^*_{\{e\}} * 1_{\{e\}} \in B^+ \cap L^1(G) \subseteq B^1(G)$. Then

$$\widehat{f}(\sigma) = \sum_{x \in G} \overline{\sigma(x)} 1_{\{e\}}(x) = 1$$

and the inversion theorem yields that

$$m_{\widehat{G}}(\widehat{G}) = \int_{\widehat{G}} 1 \mathrm{d}m_{\widehat{G}} = \int_G \widehat{f}(\sigma)\mathrm{d}\sigma = f(e) = 1$$

3. Let $G = \mathbb{R}$. Let $m_{\mathbb{R}}$ satisfy $m_{\mathbb{R}}([0,1]) = 1$. We shall choose $\alpha, \beta > 0$ such that $\alpha m_{\mathbb{R}}$ and $\beta m_{\mathbb{R}}$ (also normalized as above) satisfy the inversion theorem. Since $\exp(-|x|) \geqslant 0$ for $x \in \mathbb{R}$, we get on $\mathbb{R} \cong \widehat{\mathbb{R}}$ that
$$s \mapsto \alpha \int_{\mathbb{R}} \exp(-isx)\exp(-|x|)\mathrm{d}x = 2\alpha \int_0^\infty = \frac{2\alpha}{1+s^2}$$
is positive-definite. Hence by the inversion theorem we have that
$$\exp(-|x|) = 2\alpha \int_{\mathbb{R}} \frac{\exp(isx)}{1+s^2}\beta \mathrm{d}s$$
for $x \in \mathbb{R}$. In particular, letting $x = 0$, we get that
$$1 = 2\alpha\beta \int_{\mathbb{R}} \frac{1}{1+s^2}\mathrm{d}s = 2\alpha\beta\pi$$
i.e. $\alpha\beta = \frac{1}{2\pi}$. Typical choices are $\alpha = 1$ and $\beta = \frac{1}{2\pi}$ or $\alpha = \beta = \frac{1}{\sqrt{2\pi}}$.

Remark 10.23.

1. If $\mu, \nu \in M(\widehat{G})$, then $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$ (pointwise product), so $B(G) = \{\, \check{\mu} : \mu \in M(\widehat{G}) \,\}$ is a subalgebra of $C_b(G)$.
2. Let $B^2(G) = B \cap L^2(G)$. If $f \in B^1(G)$, then
$$\int_G |f|^2 \mathrm{d}m \leqslant \|f\|_1 \|f\|_\infty < \infty$$
so $B^1(G) \subseteq B^2(G)$.

Theorem 10.24 (Plancherel theorem)**.** *If $f \in L^1 \cap L^2(G)$, then $\|\widehat{f}\|_{L^2(\widehat{G})} = \|f\|_{L^2(G)}$ (provided the measures are normalized as in the inversion theorem). Furthermore, there is a unitary $U\colon L^2(G) \to L^2(\widehat{G})$ such that $Uf = \widehat{f}$ for $f \in L^1 \cap L^2(G)$.*

Proof. We have by a previous proposition

TODO 8. *ref*

that $f^* * f \in B^+ \cap L^1(G) \subseteq B^1(G)$, so the inversion theorem applies. Thus, using unimodularity of G and the inversion theorem, we have
$$\begin{aligned}
\int_G |\widehat{f}|^2 \mathrm{d}m_G &= \int_G f^*(x^{-1}f(x)\mathrm{d}x \\
&= \int_G f^*(x) f(x^{-1}e)\mathrm{d}x \\
&= (f^* * f)(e) \\
&= \int_{\widehat{G}} \widehat{f^* * f}(\sigma) \underbrace{\sigma(e)}_{=1} \mathrm{d}\sigma \\
&= \int_{\widehat{G}} \overline{\widehat{f}(\sigma)}\widehat{f}(\sigma)\mathrm{d}\sigma \\
&= \int_{\widehat{G}} |\widehat{f}|^2 \mathrm{d}m_{\widehat{G}}
\end{aligned}$$
so we get the first statement.

We have that $L^1 \cap L^2(G)$ is dense in $L^2(G)$. Let $\mathcal{K} = \{\, \widehat{f} : f \in L^1 \cap L^2(G) \,\} \subseteq L^2(\widehat{G})$. It remains to show that $\mathcal{K}$ is dense in $L^2(\widehat{G})$.

Note that $\mathcal{K}$ is invariant under translation: we have $\sigma * \widehat{f} = \widehat{\sigma \cdot f}$ for $\sigma \in \widehat{G}$ and $f \in L^1 \cap L^2(G)$. Furthermore, $\mathcal{K}$ is invariant under multiplication by $\{\, \widehat{x} : x \in G \,\}$: we have $\widehat{x}\widehat{f} = \widehat{x * f}$ for $x \in G$ and $f \in L^1 \cap L^2(G)$. We shall use this to show that $\mathcal{K}^\perp = \{\, 0 \,\}$, which in a Hilbert space suffices to show density.

Suppose then that $\psi \in \mathcal{K}^\perp$. Then for $\varphi \in \mathcal{K}$ we have

$$0 = \langle \psi \mid \widehat{x}\varphi \rangle = \int_{\widehat{G}} \overline{\psi(\sigma)}\varphi(\sigma)\sigma(x)\mathrm{d}\sigma$$

So $\overline{\psi}\varphi = 0$ by the uniqueness proposition for inverse transform.

TODO 9. *ref*

Fix $f \in C_c^+(G)$ with

$$\int_G f\mathrm{d}m = 1$$

Then $\varphi_0 = \widehat{f} \in \mathcal{K}$ has

$$\varphi_0(1) = \int_G f\mathrm{d}m_G = 1$$

so there is a neighbourhood U of 1 with $\varphi_0(\tau) > 0$ for $\tau \in U$. In particular, for ψ as above we have

$$0 = \overline{\psi}(\overline{\sigma} * \varphi_0) = \sigma * (\overline{\psi}(\overline{\sigma} * \varphi_0) = \sigma * \overline{\psi}\varphi_0$$

(One should check this.) Hene $\sigma * \overline{\psi}(\tau) = 0$ for almost every $\tau \in U$; i.e. $\overline{\psi}(\overline{\sigma}\tau) = 0$ for such τ. Thus $m_{\widehat{G}}$-almost-everywhere we have $\overline{\psi} = 0$. □ Theorem 10.24

Remark 10.25. If $f \in L^1 \cap L^2(\widehat{G})$ (with $\mathcal{K}$ as above), then $U^*f = \check{f}$,

$$\check{f}(x) = \int_G f(\sigma)\sigma(x)\mathrm{d}\sigma$$

TODO 10. *Conjunction?*

We do this using the first computation in the proof of the Plancherel theorem.

Lemma 10.26.

1. *If $\varphi, \psi \in C_c(\widehat{G})$, then $\varphi * \psi = \widehat{h}$ for some $h \in B^1(G)$.*
2. *Let $A^p(\widehat{G}) = \{\, \widehat{f} : f \in B^p(G) \,\}$ for $p \in \{1, 2\}$. Then $A^p(\widehat{G})$ is dense in $L^p(\widehat{G})$.*

Proof.

1. $C_c(\widehat{G}) \subseteq L^2(\widehat{G})$, so $\check{\varphi} = U^*\varphi, \check{\psi} = U^*\psi \in L^2(G)$, and $\overbrace{\varphi * \psi} = \check{\varphi}\check{\psi} \in L^1(G)$. But $\check{\omega} \in B(G)$ for any $\omega \in L^1(\widehat{G})$; so $\underbrace{\varphi * \psi} \in B^1(G)$. Let $h = \underbrace{\varphi * \psi}$, and apply the inversion theorem.
2. Suppose $f \in L^p(\widehat{G})$ and $\varepsilon > 0$. Let $(k_i)_i$ be a contractive summability kernel for $L^1(\widehat{G})$. Then for some i we have $\|f - k_i * f\|_p < \varepsilon$ (A2Q1). Let $\varphi, \psi \in C_c(\widehat{G})$ satisfy

$$\begin{aligned} \|k_i - \varphi\|_1 &< \varepsilon \\ \|\psi - f\|_p &< \varepsilon \end{aligned}$$

Then

$$\begin{aligned} \|f - \varphi * \psi\|_p &\leqslant \|f - k_i * f\|_p + \|k_i * f - k_i * \psi\|_p + \|k_i * \psi - \varphi * \psi\|_p \\ &< \varepsilon + \varepsilon + \varepsilon \underbrace{\|\psi\|_p}_{\leqslant \varepsilon + \|f\|_p} \end{aligned}$$

Thus by the first item, we have $\varphi * \psi \in A^1(\widehat{G}) \subseteq A^2(\widehat{G})$, so we are done. □ Lemma 10.26

Our goal now is Pontryagin duality. If $x \in G$, we let $\widehat{x} \in \widehat{\widehat{G}}$ be $\widehat{x}(\sigma) = \sigma(x)$. We wish to show that the map $G \to \widehat{\widehat{G}}$ given by $x \mapsto \widehat{x}$ is a surjective homeomorphism.

Remark 10.27. It is evident that $x \mapsto \widehat{x}$ is a homomorphism.

Given a symmetric relatively compact neighbourhood $V \subseteq G$ of e, we let $h_V = \frac{1}{m(V)} 1_V * 1_V$. Then

1. Since $1_V^* = 1_V$ (using unimodularity), we have that $h_V \in B^+ \cap L^1(G) \subseteq B^1(G)$.
2. $\operatorname{supp}(h_V) \subseteq V^2$.
3. The value at e is given by
$$h_V(e) = \frac{1}{m(V)} \int_V 1_V(x) 1_V(x^{-1}e) \mathrm{d}x = 1$$

Warning 10.28. $(h_V)_{V \in \mathcal{V}}$ (where $\mathcal{V}$ is the class of symmetric neighbourhoods of e) is *not* a summability kernel.

Proposition 10.29. *The map $G \to \widehat{\widehat{G}}$ given by $x \mapsto \widehat{x}$ is injective.*

Proof. For h_V as above, the inversion theorem yields that

$$h_V(x) = \int_{\widehat{G}} \widehat{h_V}(\sigma)\sigma(x)\mathrm{d}\sigma = \int_G \widehat{h_V}\widehat{x}\mathrm{d}m_{\widehat{G}}$$

If $x \neq e$, find V so $x \notin V^2$; then

$$\int_G \widehat{h}\widehat{x}\mathrm{d}m_{\widehat{G}} = h_V(x) = 0 \neq 1 = h_V(1) = \int_G \widehat{h} \underbrace{\widehat{e}}_{1} \mathrm{d}m_{\widehat{G}}$$

So $\widehat{x} \neq 1 = \widehat{e}$. ☐ Proposition 10.29

Theorem 10.30 (Pontryagin duality theorem)**.** *The map $G \to \widehat{\widehat{G}}$ given by $x \mapsto \widehat{x}$ is a surjective homeomorphism.*

Proof. Let $\Gamma = \{ \widehat{x} : x \in G \} \subseteq \widehat{\widehat{G}}$.

(I) We show that the map $G \to \Gamma$ given by $x \mapsto \widehat{x}$ is a homeomorphism onto its image. Suppose $(x_\alpha)_\alpha$ is a net in G and $x_0 \in G$. Consider the following convergences:

1. $x_\alpha \xrightarrow{\alpha} x_0$ in G.
2. $f(x_\alpha) \xrightarrow{\alpha} f(x_0)$ for all $f \in B^1(G)$. (This is $\sigma(G, B^1(G))$-convergence.)
3. $\widehat{x_\alpha} \xrightarrow{\alpha} \widehat{x_0}$ in $\widehat{\widehat{G}}$.

We will show that these are equivalent.

Since $B^1(G) \subseteq C_b(G)$, we get (1) implies (2). For h_V as above we have $x_0 * h_V \in B^1(G)$. If (2) holds, then
$$h_V(x_0^{-1}x_\alpha) = (x_0 * h_V)(x_\alpha) \xrightarrow{\alpha} (x_0 * h_V)(x_0) = h_V(e) = 1$$
Hence by construction of h_V we see that $x_0^{-1}x_\alpha$ is eventually inside V^2. Thus (2) implies (1).

On $\widehat{G}$ the topology $w^* = \sigma(L^\infty(\widehat{G}), L^1(\widehat{G}))$ coincides with $\tau = \sigma(L^\infty(\widehat{G}), A^1(\widehat{G}))$. Indeed, $\tau \subseteq w^*$, and since $A^1(\widehat{G})$ is dense in $L^1(\widehat{G})$, we get that $\tau\restriction \operatorname{ball}(L^\infty(\widehat{G}))$ (closed unit ball) is Hausdorff. Two comparable compact Hausdorff topologies on $\operatorname{ball}(L^\infty(\widehat{G}))$ must coincide. Now we use the inversion theorem: if $f \in B^1(G)$ and $x \in G$ then

$$f(x) = \int_G \widehat{f}(\sigma)\sigma(x)\mathrm{d}\sigma = \int_G \widehat{f}\widehat{x}\mathrm{d}m_{\widehat{G}}$$

It is then immediate that (2) and (3) are equivalent.

(II) Γ is closed in $\widehat{\widehat{G}}$. By A1Q1, since Γ is homeomorphic to G, we get that Γ is complete, and thus closed.

(III) We show that $\Gamma = \hat{\hat{G}}$. If $\Gamma \subsetneq \hat{\hat{G}}$, then there is $\chi \in \hat{\hat{G}}$ and a neighbourhood U of $1_{\hat{G}}$ such that $U^2\chi \cap \Gamma = \varnothing$. Hence if $\varphi, \psi \in C_c^+(\hat{\hat{G}})$ with $\operatorname{supp}\varphi \subseteq U$ and $\operatorname{supp}\psi \subseteq U\chi$, then $\varphi * \psi \neq 0$ but $(\varphi * \psi)(\hat{x}) = 0$ for each $\hat{x} \in \Gamma$. By lemma

TODO 11. *ref*

there is $h \in B^1(\hat{G})$ such that $\hat{h} = \varphi * \psi$; so, by inversion theorem, we have

$$0 = \hat{h}(\hat{x}) = \int_{\hat{G}} h(\sigma)\overline{\hat{x}(\sigma)}\mathrm{d}\sigma = \int_{\hat{G}} h(\sigma)\sigma(x^{-1})\mathrm{d}\sigma = \check{h}(x^{-1})$$

(Recall if $h \in L^1(\hat{G})$ then $\hat{h} \in A(\hat{\hat{G}})$.) Hence $h = 0$ on $\hat{G}$ by uniqueness proposition

TODO 12. *ref*

This contradicts our construction, so $\Gamma = \hat{\hat{G}}$. ☐ Theorem 10.30

Definition 10.31. If $\mu \in M(G)$, we let the *Fourier-Stieltjes transform* of μ be

$$\hat{\mu}(\sigma) = \int_G \overline{\sigma(x)}\mathrm{d}\mu(x)$$

for $\sigma \in \hat{G}$. We let $B(\hat{G}) = \{ \hat{\mu} : \mu \in M(G) \} \subseteq C_b(\hat{G})$.

Theorem 10.32 (Uniqueness theorem)**.** *The Fourier-Stieltjes transform $M(G) \to B(\hat{G})$ is injective. Hence the Fourier transform $L^1(G) \to A(\hat{G})$ given by $f \mapsto \hat{f}$ is injective.*

Proof. Let $\iota\colon G \to \hat{\hat{G}}$ be $\iota(x) = \hat{x}$. Given $\mu \in M(G)$, we have $\mu \circ \iota^{-1} \in M(\hat{\hat{G}})$. Then for $\sigma \in \hat{G}$ we have

$$\hat{\mu}(\sigma) = \int_G \underbrace{\overline{\sigma(x)}}_{\hat{x}(\overline{\sigma})} \mathrm{d}\mu(x) = \int_{\hat{\hat{G}}} \hat{x}(\overline{\sigma})\mathrm{d}(\mu \circ \iota^{-1})(x) = \widehat{\mu \circ \iota^{-1}}(\overline{\sigma})$$

Hence if $\mu \neq 0$ then $\mu \circ \iota^{-1} \neq 0$; by the uniqueness proposition

TODO 13. *ref*

we then have that $\widehat{\mu \circ \iota^{-1}} \neq 0$, and $\hat{\mu} \neq 0$. (It is clear that $\mu \mapsto \hat{\mu}$ is linear.) ☐ Theorem 10.32

11 Harmonic analysis on compact grapes

Let G be a compact grape. We *always* assume $m(G) = 1$.

Fact 11.1.

1. *If $\pi\colon G \to \mathcal{B}(\mathcal{H})^\times$ is a representation, then there is $S \in \mathcal{B}(\mathcal{H})^\times$ such that $S\pi(G)S^{-1} \subseteq U(\mathcal{H})$.*
2. *If $\pi\colon G \to \mathcal{B}(\mathcal{X})^\times$ where $\mathcal{X}$ is a finite-dimensional Banach space, then there is invertible $S\colon \mathcal{X} \to \mathcal{H}$ such that $S\pi(G)S^{-1} \subseteq U(\mathcal{H})$. (For us $\mathcal{H}$ always means a Hilbert space.)*

The moral is that for us it suffices to consider unitary representations of G.

Fact 11.2 (Projections on Hilbert spaces)**.**

(i) *If $\mathcal{L} \subseteq \mathcal{H}$ is a closed subspace, then there is a unique orthogonal projection $P_\mathcal{L} \in \mathcal{B}(\mathcal{H})$ with $P_\mathcal{L}^2 = P_\mathcal{L}^* = P_\mathcal{L}$ and $\operatorname{Ran} P_\mathcal{L} = \mathcal{L}$.*

(ii) *If $P = P^2 = P^*$ in $\mathcal{B}(\mathcal{H})$, then $P = P_\mathcal{L}$ with $\mathcal{L} = \operatorname{Ran}(P)$ (automatically closed).*

(iii) *If $\xi \in \mathcal{H}$ has $\|\xi\| = 1$ then $P_\xi = P_{\mathbb{C}\xi} = \xi\langle \xi \mid \cdot \rangle$. (i.e. $P_\xi(\eta) = \xi\langle \xi \mid \eta \rangle = \langle \xi \mid \eta \rangle \xi$.)*

(iii') *If $\xi, \eta \in \mathcal{H}$ with $\|\xi\| = \|\eta\|$, then*

$$\|P_\xi - P_\eta\| \leqslant \|\xi\langle \xi \mid \cdot \rangle - \xi\langle \eta \mid \cdot \rangle\| + \|\xi\langle \eta \mid \cdot \rangle - \eta\langle \eta \mid \cdot \rangle\| \leqslant 2\|\xi - \eta\|$$

Hence the map $\xi \mapsto P_\xi$ is continuous.

Definition 11.3. Suppose $\pi \colon G \to U(\mathcal{H})$ be a unitary.

- A closed subspace $\mathcal{L}$ of $\mathcal{H}$ is *π-invariant* if $\pi(x)\mathcal{L} \subseteq \mathcal{L}$ for each $x \in G$.
- We say π is *irreducible* if the only non-zero closed π-invariant subspace is $\mathcal{H}$.

Lemma 11.4.

1. *A closed subspace $\mathcal{L} \subseteq \mathcal{H}$ is π-invariant if and only if $\pi(x)P_{\mathcal{L}} = P_{\mathcal{L}}\pi(x)$ for each $x \in G$.*
2. *A closed subspace $\mathcal{L} \subseteq \mathcal{H}$ is π-invariant if and only if $\mathcal{L}^\perp$ is π-invariant.*

Proof.

1. ($\implies$) For $x \in G$ we have $\pi(x)P_{\mathcal{L}} = P_{\mathcal{L}}\pi(x)P_{\mathcal{L}}$. Hence

$$P_{\mathcal{L}}\pi(x) = (\pi(x^{-1})P_{\mathcal{L}})^* = (P_{\mathcal{L}}\pi(x^{-1})P_{\mathcal{L}})^* = P_{\mathcal{L}}\pi(x)P_{\mathcal{L}} = \pi(x)P_{\mathcal{L}}$$

   (since $\pi(x^{-1}) = (\pi(x))^{-1} = (\pi(x))^*$).

   ($\impliedby$) Obvious.

2. We have $P_{\mathcal{L}^\perp} = I - P_{\mathcal{L}}$ commutes with each $\pi(x)$ exactly when $P_{\mathcal{L}}$ does. □ Lemma 11.4

Proposition 11.5. *If $\mathcal{H}$ is finite-dimensional then it admits an irreducible π-invariant subspace.*

Proof. Let $\mathcal{L} \neq \{ 0 \}$ be a π-invariant subspace of minimal dimension. □ Proposition 11.5

Theorem 11.6. *Suppose G is a compact grape and $\pi \colon G \to U(\mathcal{H})$ a unitary representation. Then*

1. *π admits a non-zero, finite-dimensional π-invariant subspace.*
2. *If π is irreducible, then it is finite-dimensional.*
3. *Generally (without assuming irreducibility), π is* completely reducible*: there is a family $\{ \mathcal{L}_\alpha \}_{\alpha \in A}$ of closed subspaces such that*
   (a) *Each $\mathcal{L}_\alpha$ is π-invariant.*
   (b) *Each $\mathcal{L}_\alpha$ is irreducible for π.*
   (c) *$\mathcal{L}_\alpha \perp \mathcal{L}_\beta$ for $\alpha \neq \beta$ in A.*
   (d) *The internal direct sum*

$$\bigoplus_{\alpha \in A} \mathcal{L}_\alpha = \left\{ \sum_{i=1}^{n} \xi_{\alpha_i} : n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \text{ distinct in } A, \xi_{\alpha_i} \in \mathcal{L}_{\alpha_i} \right\}$$

   is dense in $\mathcal{H}$.

   (Note that these conditions together with the assumption that the $\mathcal{L}_\alpha$ are closed will imply that the $\mathcal{L}_\alpha$ are finite-dimensional.) We write

$$\pi = \bigoplus_{\alpha \in A} \pi(\cdot)\restriction\mathcal{L}_\alpha$$

   on

$$\mathcal{H} = \ell\text{-}\bigoplus_{\alpha \in A} \mathcal{L}_\alpha$$

Note that by Pythagoras' theorem every $\xi \in \mathcal{H}$ can be written uniquely in the form

$$\xi = \sum_{\alpha \in A} \xi_\alpha$$

with each $\xi_\alpha \in \mathcal{L}_\alpha$ and

$$\|\xi\|^2 = \sum_{\alpha \in A} \|\xi_\alpha\|^2$$

Proof.

1. Fix $\xi \in \mathcal{H}$ with $\|\xi\| = 1$. Consider the operator

$$K_\xi = \int_G P_{\pi(x)\xi} \mathrm{d}x$$

(Bochner integral, since $x \mapsto P_{\pi(x)\xi}$ is continuous). Each of these is rank 1 and thus a compact operator; so $K_\xi \in \mathcal{K}(\mathcal{H})$ (the Banach space of compact operators on $\mathcal{H}$). Also if $\eta, \zeta \in \mathcal{H}$ then

$$\begin{aligned}
\langle K_\xi \eta \mid \zeta \rangle &= \int_G \big\langle \pi(x)\xi \langle \pi(x)\xi \mid \eta \rangle \,\big|\, \zeta \big\rangle \mathrm{d}x \\
&= \int_G \langle \pi(x)\xi \mid \zeta \rangle \langle \eta \mid \pi(x)\xi \rangle \mathrm{d}x \\
&= \int_G \big\langle \eta \,\big|\, \pi(x)\xi \langle \pi(x)\xi \mid \zeta \rangle \big\rangle \mathrm{d}x \\
&= \langle \eta \mid K_\xi \zeta \rangle
\end{aligned}$$

so $K_\xi^* = K_\xi$. If we let $\eta = \xi = \zeta$, then we get

$$\langle \xi \mid K_\xi \xi \rangle = \int_G |\langle \xi \mid \pi(x)\xi \rangle|^2 \mathrm{d}x$$

where $\langle \xi \mid \pi(e)\xi \rangle = 1 > 0$; hence $\langle \xi \mid K_\xi \xi \rangle > 0$, and $K_\xi \neq 0$. Also,if $y \in G$ and $\eta \in \mathcal{H}$ then

$$\begin{aligned}
\pi(y) K_\xi \eta &= \int_G \pi(yx) \langle \pi(x)\xi \mid \eta \rangle \mathrm{d}x \\
&= \int_G \pi(x) \langle \pi(x)\xi \mid \pi(y)\eta \rangle \mathrm{d}x \\
&= K_\xi \pi(y) \eta
\end{aligned}$$

Thus $\pi(y)K_\xi = K_\xi \pi(y)$. We now apply the spectral theorem to K_ξ to get a sequence of orthogonal projections $\{ P_1, P_2, \dots \}$ (perhaps finite) and $\lambda_1, \lambda_2, \dots \in \mathbb{R} \backslash \{ 0 \}$ such that

$$\lim_{n \to \infty} \lambda_n = 0$$

and

- $K_\xi = \sum_{n=1,2,\dots} \lambda_n P_n$ (converges in norm, if the sequence is infinite).
- Each $1 \leqslant \dim(P_n(\mathcal{H})) < \infty$.
- $P_n P_m = 0$ if $n \neq m$.
- For $T \in \mathcal{B}(\mathcal{H})$ we have $TK_\xi = K_\xi T$ if and only if $TP_n = P_n T$ for each n.

We thus have $\pi(x)P_n = P_n \pi(x)$ for each $x \in G$; so $\mathcal{L}_n = \operatorname{Ran} P_n$ is π-invariant.

2. By (1) and the last proposition, if π is infinite dimensional, then it admits an (irreducible) π-invariant subspace.

3. We let
$$\Lambda = \{ \lambda = \{ \mathcal{L}_\alpha \}_{\alpha \in A_\lambda} : \lambda \text{ satisfies (a)-(c) above} \}$$
By (1) and the last proposition we get $\Lambda \neq \varnothing$ and Λ is partially ordered by $\subseteq$. Let $\Gamma \subseteq \Lambda$ be a chain; so $\{ \mathcal{L} : \mathcal{L} = \mathcal{L}_\alpha \text{ for some } \alpha \in A_\lambda, \lambda \in \Gamma \} \in \Lambda$ is an upper bound for Λ. By Zorn's lemma, there is a maximal element $\mu = \{ \mathcal{L}_\alpha \}_{\alpha \in A_\mu} \in \Lambda$. Let
$$\mathcal{M} = \overline{\bigoplus_{\alpha \in A_\mu} \mathcal{L}_\alpha}$$
Then $\mathcal{M}$ is π-invariant by continuity of each $\pi(x)$. If $\mathcal{M}^\perp \neq \{ 0 \}$, then (1) and the last proposition yield an irreducible π-invariant subspace $\mathcal{L} \subseteq \mathcal{M}^\perp$. Then $\mu \cup \{ \mathcal{L} \} \in \Lambda$ violates maximality of μ, a contradiction. ☐ Theorem 11.6

Lemma 11.7 (Schur's lemma)**.** *Suppose $\pi \colon G \to U(\mathcal{H})$ is a finite-dimensional unitary representation. Then*

1. *π is irreducible if and only if $(\pi(G))' = \{ T \in \mathcal{B}(\mathcal{H}) : T\pi(x) = \pi(x)T$ for all $x \in G \}$ is $\mathbb{C}I$.*
2. *If $\pi' \colon G \to U(\mathcal{H}')$ is another unitary representation and π and π' are irreducible, then if $A \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ satisfies $A\pi(x) = \pi'(x)(A)$ for each $x \in G$, then $A = cU$ for some $c \in \mathbb{C}$ and unitary U. (In particular, if $c \neq 0$ we get $\dim(\mathcal{H}) = \dim(\mathcal{H}')$.*

We sometimes call elements of $(\pi(G))'$ *intertwiners*. The finite dimensional assumption is actually superfluous, once we know the spectral theorem for von Neumann algebras.

Proof.

1. If $T \in (\pi(G))'$ then so too is T^*. Indeed, for $x \in G$ we have
$$T^*\pi(x) = (\pi(x^{-1}T)^* = (T\pi(x^{-1}))^* = \pi(x)T^*$$
Hence each $\mathrm{Re}(T) = \frac{1}{2}(T + T^*), \mathrm{Im}(T) = \frac{1}{2i}(T - T^*) \in (\pi(G))'$. If $A = A^* \in (\pi(G))'$, we can use spectral theorem to write
$$A = \sum_{k=1}^n \lambda_k P_k$$
Then each P_k has $P_k\pi(x) = \pi(x)P_k$ for all $x \in G$; so $\mathrm{Ran}(P_k)$ is π-invariant.

   ($\Longrightarrow$) If π is irreducible, then $A = A^* \in (\pi(G))'$ implies $A = cI$ for $c \in \mathbb{R}$.

   ($\Longleftarrow$) The only orthogonal projections in $(\pi(G))'$ are 0 and I; we then use the previous lemma.

   TODO 14. *Ref?*

2. If $A\pi(x) = \pi'(x)A$ then
   - $\ker(A)$ is π-invariant, and hence either$\{ 0 \}$ or $\mathcal{H}$.
   - $\mathrm{Ran}(A)$ is π'-invariant, and hence respectively either $\mathcal{H}$ or $\{ 0 \}$.

   So A is either 0 or invertible. In the latter case we hvave
$$A^*A\pi(x) = A^*\pi'(x)A = \pi(x)A^*A$$
(where the last equality follows as in (1)). So $A^*A = cI$ for some $c > 0$. Let $U = \frac{1}{\sqrt{c}}A$. ☐ Lemma 11.7

Corollary 11.8. *If G is a compact abelian grape, then each irreducible representation is multiplication by a character $\sigma \in \widehat{G}$ on $\mathbb{C}$.*

Again, had we more spectral theory, we could dispense with the compactness hypothesis.

Proof. If $\pi\colon G \to U(\mathcal{H})$ is an irreducible representation, then for $x \in G$ we have $\pi(x) \in (\pi(G))' = \mathbb{C}I$. Hence we can write $\pi(x) = \sigma(x)I$ for $\sigma(x) \in \mathbb{T}$ (since π is unitary). Moreover we have

$$\sigma(xy)I = \pi(xy) = \pi(x)\pi(y) = (\sigma(x)I)(\sigma(y)I) = \sigma(x)\sigma(y)I$$

Clearly $x \mapsto \sigma(x)$ is continuous, as π is. By irreducibility, we get $\dim(\mathcal{H}_\pi) = 1$. □ Corollary 11.8

Definition 11.9. If $\pi\colon G \to U(\mathcal{H})$ and $\pi'\colon G \to U(\mathcal{H}')$ are unitary representations (not necessarily irreducible or finite dimensional), then we say π is *unitarily equivalent* to π' if there is a unitary $U \in \mathcal{B}(\mathcal{H}', \mathcal{H})$ such that $U\pi'(x) = \pi(x)U$ for $x \in G$; i.e. $\pi'(x) = U^*\pi(x)U$. We then set

$$\mathrm{Irr}(G) = \{\, \pi\colon G \to U(d) : \pi \text{ a continuous homomorphism}, (\pi(G))' = \mathbb{C}I_d \text{ (in } M_d(\mathbb{C})) \,\}$$

where $U(d)$ is the $d \times d$ unitary grape. We let $\widehat{G} = \mathrm{Irr}(G)/\!\approx$ where $\pi \approx \pi'$ if π and π' are unitarily equivalent. "Properly" speaking, we have

$$\widehat{G} = \{\, [\pi] \mid \pi\colon G \to U(\mathcal{H}_\pi) \text{ (finite dimensional irreducible unitary representation)} \,\}$$

We have a "standard abuse of notation": we consider $\widehat{G}$ as a full set of representation of its equivalence classes; i.e. we write "$\pi \in \widehat{G}$" rather than $[\pi] \in \widehat{G}$. We have the convention that $\pi \neq \pi'$ in $\widehat{G}$ means that $\pi \not\approx \pi'$.

11.1 Matrix coefficient functions

Given $\pi \in \widehat{G}$, we let

$$\mathcal{T}_\pi = \mathrm{span}\{\, \langle \xi \,|\, \pi(\cdot)\eta \rangle : \xi, \eta \in \mathcal{H}_\pi \,\} \subseteq C(G) \subseteq L^2(G)$$

since $m(G) = 1$. (Note that if $U \in U(H_\pi)$ then $\langle U\xi \,|\, \pi(\cdot)U\eta\rangle = \langle \xi \,|\, U^*\pi(\cdot)U\eta\rangle$; so $\pi \mapsto \mathcal{T}_\pi$ is independent of equivalence class.)

Let $d_\pi = \dim(\mathcal{H}_\pi)$ and $\{\, e_1, \ldots, e_d \,\}$ be an orthonormal basis for $\mathcal{H}_\pi$. Then for $\xi, \eta \in H_\pi$ we have

$$\langle \xi \,|\, \pi(\cdot)\eta \rangle = \left\langle \sum_{j=1}^{d_\pi} \langle e_j \,|\, \xi\rangle e_j \,\middle|\, \pi(\cdot) \sum_{i=1}^{d_\pi} \langle e_i \,|\, \eta\rangle e_i \right\rangle = \sum_{j=1}^{d_\pi}\sum_{i=1}^{d_\pi} \langle \xi \,|\, e_j\rangle\langle e_i \,|\, \eta\rangle \underbrace{\langle e_j \,|\, \pi(\cdot) e_i\rangle}_{\pi_{ij}}$$

Then with respect to the basis $\{\, e_1, \ldots, e_{d_\pi} \,\}$ we have that $\pi(x) = [\pi_{ij}(x)]$, and $\mathcal{T}_\pi = \mathrm{span}\{\, \pi_{ij} : i, j \in \{\, 1, \ldots, d_\pi \,\}\,\}$. This leads to:

Theorem 11.10 (Schur's orthogonality relations)**.** *Suppose $\pi, \pi' \in \widehat{G}$. Then*

1. *If $\pi \neq \pi'$ (i.e. they aren't unitarily equivalent) then $\mathcal{T}_\pi \perp \mathcal{T}_{\pi'}$ in $L^2(G)$.*
2. *If $\xi, \eta, \zeta, \omega \in \mathcal{H}_\pi$, then*
$$\int_G \overline{\langle \xi \,|\, \pi(x)\eta\rangle}\langle \zeta \,|\, \pi(x)\omega\rangle \mathrm{d}x = \frac{1}{d_\pi}\langle \zeta \,|\, \xi\rangle\langle \eta \,|\, \omega\rangle$$

In particular, with the notation as above, we get that $\{\, \sqrt{d_\pi}\pi_{ij} : i, j \in \{\, 1, \ldots, d_\pi \,\}\,\}$ is an orthonormal basis for $\mathcal{T}_\pi$.

Proof. Suppose $A \in \mathcal{B}(\mathcal{H}_{\pi'}, \mathcal{H}_\pi)$, and let

$$\widetilde{A} = \int_G \pi(x)A\pi'(x^{-1})\mathrm{d}x$$

(Bochner integral in a finite-dimensional Banach space). Then for $y \in G$ we have

$$\widetilde{A}\pi'(y) = \int_G \pi(x)A\pi'(\underbrace{x^{-1}y}_{(y^{-1}x)^{-1}})\mathrm{d}x = \int_G \pi(yx)A\pi'(x^{-1}\mathrm{d}x = \pi(x)\widetilde{A}$$

Hence, by Schur's lemma, we have

$$\widetilde{A} = \begin{cases} 0 & \text{if } \pi \neq \pi' \\ cI & \text{else} \end{cases}$$

where $c \neq 0$. Now suppose $\xi, \eta \in \mathcal{H}_{\pi'}$, $\zeta, \omega \in \mathcal{H}_\pi$, and $A = \omega\langle \eta \,|\, \cdot \rangle \in \mathcal{B}(\mathcal{H}_{\pi'}, \mathcal{H}_\pi)$. Then

$$\begin{aligned} \widetilde{A} &= \int_G \pi(x)\omega\langle \pi'(x)\eta \,|\, \cdot \rangle \mathrm{d}x \\ \langle \zeta \,|\, \widetilde{A}\xi \rangle &= \int_G \langle \zeta \,|\, \pi(x)\omega \rangle \langle \pi'(x)\eta \,|\, \xi \rangle \mathrm{d}x \\ &= \int_G \overline{\langle \xi \,|\, \pi'(x)\eta \rangle} \langle \zeta \,|\, \pi(x)\omega \rangle \mathrm{d}x \end{aligned}$$

Hence if $\pi \neq \pi'$, we get the first result. If $\pi = \pi'$, then $\widetilde{A} = cI$ for some $c \in \mathbb{C}$; we compute

$$\begin{aligned} c &= \frac{1}{d_\pi} \operatorname{Tr}(\widetilde{A}) \\ &= \frac{1}{d_\pi} \int_G \operatorname{Tr}(\pi(x) A \pi(x^{-1})) \mathrm{d}x \\ &= \frac{1}{d_\pi} \int_G \operatorname{Tr}(A) \mathrm{d}x \\ &= \frac{1}{d_\pi} \operatorname{Tr}(A) \\ &= \frac{1}{d_\pi} \sum_{i=1}^{d_\pi} \langle e_i \,|\, A e_i \rangle \\ &= \frac{1}{d_\pi} \sum_{i=1}^{d_\pi} \langle e_i \,|\, \omega \rangle \langle \eta \,|\, e_i \rangle \\ &= \frac{1}{d_\pi} \langle \eta \,|\, \omega \rangle \end{aligned}$$

(where the last equality follows from Parseval). ☐ Theorem 11.10

Definition 11.11. We set

$$\mathcal{T}(G) = \bigoplus_{\pi \in \widehat{G}} \mathcal{T}_\pi \subseteq C(G) \subseteq L^2(G)$$

We look to defining the tensor product of representations. If $\mathcal{H}, \mathcal{H}'$ are finite dimensional Hilbert spaces, then on $\mathcal{H} \otimes \mathcal{H}'$, the quantity

$$\left\langle \sum_{i=1}^{n} \xi_i \otimes \xi_i' \,\middle|\, \sum_{j=1}^{n'} \eta_j \otimes \eta_j' \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n'} \langle \xi_i \,|\, \eta_j \rangle_{\mathcal{H}} \langle \xi_i' \,|\, \eta_j' \rangle_{\mathcal{H}'}$$

is well-defined and sesquilinear. (To check this, one fixes $\eta \otimes \eta'$ and checks that $(\xi, \xi') \mapsto \langle \xi \otimes \xi' \,|\, \eta \otimes \eta' \rangle$ is bilinear on $\overline{\mathcal{H}} \times \overline{\mathcal{H}'}$ (where $\overline{\mathcal{H}}$ has the same addition and conjugated scalar multiplication; i.e. $a \cdot \xi = \overline{a}\xi$). One then does the same on the right.) If $\mathcal{H}, \mathcal{H}'$ have orthonormal bases $\{ e_1, \ldots, e_d \}$ and $\{ e_1', \ldots, e_{d'}' \}$, then $\{ e_i \otimes e_j' : i \in \{ 1, \ldots, d \}, j \in \{ 1, \ldots, d' \} \}$ is a basis for $\mathcal{H} \otimes \mathcal{H}'$ with $\langle e_i \otimes e_j' \,|\, e_k \otimes e_\ell' \rangle = \delta_{ij}\delta_{j\ell}$ (Kronecker δ). So $\{ e_i \otimes e_j' : i \in \{ 1, \ldots, d \}, j \in \{ 1, \ldots, d' \} \}$ is an orthonormal basis for $\mathcal{H} \otimes \mathcal{H}'$. If $\omega \in \mathcal{H} \otimes \mathcal{H}'$, we write

$$\omega = \sum_{i=1}^{d} \sum_{j=1}^{d'} \omega_{ij} e_i \otimes e_j'$$

and

$$\langle \omega \,|\, \omega \rangle = \sum_{i=1}^{d} \sum_{j=1}^{d'} |\omega_{ij}|^2 \geqslant 0$$

is non-zero if $\omega \neq 0$. So $\langle \cdot \mid \cdot \rangle$ is an inner product on $\mathcal{H} \otimes \mathcal{H}'$.

If $U \in U(\mathcal{H})$ and $U' \in U(\mathcal{H}')$, then

$$(U \otimes U') \sum_{i=1}^{n} \xi_i \otimes \xi_i' = \sum_{i=1}^{n} U\xi_i \otimes U'\xi_i'$$

is a well-defined unitary operator. Given $\pi, \pi' \in \widehat{G}$, the map

$$\begin{aligned} \pi \otimes \pi' \colon G &\to U(\mathcal{H}_\pi \otimes \mathcal{H}_{\pi'}) \\ x &\mapsto \pi(x) \otimes \pi'(x) \end{aligned}$$

defines a unitary representation of G that is independent of unitary equivalence class up to unitary equivalence.

Warning 11.12. There is no reason to expect that $\pi \otimes \pi'$ be irreducible.

By complete reducibility, we have

$$\pi \otimes \pi' = \bigoplus_{i=1}^{n} \pi_i^{(m_i)}$$

for $\pi_1, \ldots, \pi_n \in \widehat{G}$ and $m_i \in \mathbb{N}$ the "multiplicity". So $\mathcal{T}(G)$ is an algebra of functions. Indeed, given $\pi, \pi' \in \widehat{G}$ and $\xi, \eta \in \mathcal{H}_\pi$, $\zeta, \omega \in \mathcal{H}_{\pi'}$, we have

$$\begin{aligned} \langle \xi \mid \pi(\cdot)\eta \rangle \langle \zeta \mid \pi(\cdot)\omega \rangle &= \langle \xi \otimes \zeta \mid \pi \otimes \pi'(\cdot)\eta \otimes \omega \rangle \\ &= \left\langle \xi \otimes \zeta \,\middle|\, \left(\bigoplus_{i=1}^{n} \pi_i^{(m_i)} \right) \eta \otimes \omega \right\rangle \\ &= \left\langle \xi \otimes \zeta \,\middle|\, \sum_{i=1}^{n} \sum_{j=1}^{m_i} P_{ij}\pi_i(\cdot)P_{ij}\eta \otimes \omega \right\rangle \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m_i} \langle P_{ij}(\xi \otimes \zeta) \mid \pi_i(\cdot)P_{ij}(\eta \otimes \omega) \rangle \\ &\in \mathcal{T}(G) \end{aligned}$$

where P_{ij} are orthogonal projections.

Definition 11.13 (Conjugate representation)**.** Suppose $\pi \in \widehat{G}$ and $\{ e_1, \ldots, e_{d_\pi} \}$ an orthonormal basis for $\mathcal{H}_\pi$ with $\pi_{ij}(\cdot) = \langle e_j \mid \pi(\cdot)e_i \rangle$. We define $\overline{\pi} \colon G \to U(\mathcal{H}_\pi)$ by $\overline{\pi}(x) = [\overline{\pi_{ij}(x)}]$ (with respect to the chosen orthonormal basis).

Suppose $\pi = U^*\pi'(\cdot)U$ for unitary U. Then $(U^*)_{ik} = \overline{U_{ki}}$. Then

$$\pi = U^*\pi'(\cdot)U = \left[\sum_{k,\ell=1}^{d_\pi} \overline{U_{ik}}\pi'_{k\ell}(\cdot)U_{\ell j} \right]$$

So

$$\overline{\pi} = \left[\sum_{k,\ell=1}^{d_\pi} U_{ik}\overline{\pi'_{k\ell}(\cdot)U_{\ell j}} \right] = (\overline{U})^* \overline{\pi'}(\cdot)\overline{U}$$

(where $\overline{U} = [\overline{U_{ij}}]$). Thus $\pi \approx \pi'$ implies $\overline{\pi} \approx \overline{\pi'}$.

Note also that $\mathcal{T}(G)$ is conjugate-closed: we have $\overline{\langle \xi \mid \pi(\cdot)\eta \rangle} = \langle \overline{\xi} \mid \overline{\pi}(\cdot)\overline{\eta} \rangle$ where $\overline{\xi}$ and $\overline{\eta}$ are pointwise conjugated with respect to some orthonormal basis.

Remark 11.14. If G is abelian then for $\sigma, \sigma' \in \widehat{G}$ we have $\sigma \otimes \sigma' \cong \sigma\sigma'$ as $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$; hence $\overline{\sigma} = \sigma^{-1}$.

Notation 11.15. We let $\lambda \colon G \to U(L^2(G))$ be the left regular representation, so $\lambda(x)f(y) = f(x^{-1}y)$ for almost every y. Note that $C(G) \subseteq L^2(G)$ is a dense (hence not closed) λ-invariant subspace.

Theorem 11.16 (Peter-Weyl)**.**

1. *For $\pi \in \widehat{G}$ let $\{ e_1^\pi, \dots, e_{d_\pi}^\pi \}$ be an orthonormal basis for $\mathcal{H}_\pi$, and let*

$$\mathcal{T}_{\pi,j} = \operatorname{span}\{ \pi_{ij} : i \in \{ 1, \dots, d_\pi \} \} \subseteq \mathcal{T}_\pi \subseteq C(G) \subseteq L^2(G)$$

   Then $\mathcal{T}_{\pi,j}$ is λ-invariant, and $\lambda_{\pi,j} = P_{\pi,j}\lambda(\cdot)\restriction\mathcal{T}_{\pi,j} \approx \overline{\pi}$ (where $P_{\pi,j}$ is the orthogonal projection onto $\mathcal{T}_{\pi,j}$).

2. *We have*

$$\mathcal{T}(G) = \bigoplus_{\pi \in \widehat{G}} \mathcal{T}_\pi$$

   is uniformly dense in $C(G)$, and hence L^2-dense in $L^2(G)$.

3. *We have*

$$\lambda = \bigoplus_{\pi \in \widehat{G}} \pi^{(d_\pi)}$$

   on

$$L^2(G) = \ell^2\text{-}\bigoplus_{\pi \in \widehat{G}} \bigoplus_{j=1}^{d_\pi} \mathcal{T}_{\overline{\pi},j} \cong \ell^2\text{-}\bigoplus_{\pi \in \widehat{G}} \mathcal{H}_\pi^{(d_\pi)}$$

   and in particular $\{ \sqrt{d_\pi}\pi_{ij} : i, j \in \{ 1, \dots, d_\pi \}, \pi \in \widehat{G} \}$ is an orthonormal basis for $L^2(G)$.

Proof.

1. If $x, y \in G$ then using the matrix product we have

$$\lambda(x)\pi_{ij}(y) = \pi_{ij}(x^{-1}y) = \sum_{k=1}^{d_\pi} \underbrace{\pi_{ik}(x^{-1})}_{\overline{\pi_{ki}}(x)} \pi_{kj}(y)$$

   i.e.

$$\lambda(x)\pi_{ij} = \sum_{k=1}^{d_\pi} \overline{\pi_{ki}}(x)\pi_{kj}$$

   Let $U_j : \mathcal{H}_\pi \to \mathcal{T}_{\pi,j}$ be given by $U_j e_i^\pi = \sqrt{d_\pi}\pi_{ij}$. Then for $x \in G$ we have

$$\begin{aligned} U_j^* \lambda_{\pi,j}(x) U_j e_i^\pi &= U_j^* \lambda_{\pi,j}(x) \sqrt{d_\pi}\pi_{ij} \\ &= U_j^* \sqrt{d_\pi} \sum_{k=1}^{d_\pi} \overline{\pi_{ki}}(x)\pi_{kj} \\ &= \sum_{k=1}^{d_\pi} \overline{\pi_{ki}(x)} e_k^\pi \\ &= \overline{\pi}(x) e_i^\pi \end{aligned}$$

   so $U_j^* \lambda_{\pi,j}(\cdot) U_j = \overline{\pi}$.

2. Let us see that $\mathcal{T}(G)$ is point separating. Notice that if $x \neq e$ in G and V is a symmetric relatively compact neighbourhood of e with $x \in V^2$ then $\lambda(x)1_V = 1_{xV}$ and $1_{xV} \neq 1_V = \lambda(e)1_V$ so $\lambda(x) \neq \lambda(e)$. Hence if $x \neq y$ in G then $\lambda(x) \neq \lambda(y)$ (as $\lambda(x^{-1}y) = \lambda(e)$). By complete reducibility there is a finite-dimensional λ-invariant λ-irreducible subspace $\mathcal{L} \subseteq L^2(G)$ such that $\lambda(x)\restriction\mathcal{L} \neq \lambda(y)\restriction\mathcal{L}$. Then there are $\xi, \eta \in \mathcal{L}$ such that$\pi = \lambda(\cdot)\restriction\mathcal{L}$ satisfies $\langle \xi \,|\, \pi(x)\eta \rangle \neq \langle \xi \,|\, \pi(y)\eta \rangle$. Hence, by Stone-Weierstrass we have $\mathcal{T}(G)$ is uniformly dense in $C(G)$.

3. We simply use (1), and use (2) to see that $\{ \sqrt{d_\pi}\pi_{ij}(\cdot) : i, j \in \{ 1, \dots, d_\pi \}, \pi \in \widehat{G} \}$ is a maximal orthonormal set in $L^2(G)$.

□ Theorem 11.16

11.2 Fourier analysis on compact grapes

Definition 11.17 (Fourier transform). If $f \in L^1(G)$ and $\pi \in \widehat{G}$ we let

$$\widehat{f}(\pi) = \int_G f(x)\pi(x^{-1}\mathrm{d}x \in \mathcal{B}(H_\pi)$$

(Bochner integral). This is also

$$\left[\int_G f(x)\underbrace{\pi_{ij}(x^{-1})}_{\overline{\pi_{ji}(x)}}\mathrm{d}x\right]$$

where we've chosen an orthonormal basis for H_π.

If $f \in L^2(G) \subseteq L^1(G)$ (by the last result of Hölder/Cauchy-Schwarz inequality), then by the results on orthonormal bases in Hilbert spaces we get L^2-convergence

$$\begin{aligned}
f &= \sum_{\pi\in\widehat{G}}\sum_{i,j=1}^{d_\pi}\langle\sqrt{d_\pi}\pi_{ij} \mid f\rangle\sqrt{d_\pi}\pi_{ij} \\
&= \sum_{\pi\in\widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi}\underbrace{\left(\int_G f(x)\overline{\pi_{ij}}(x)\mathrm{d}x\right)}_{\int_G f(x)\pi_{ji}(x^{-1})}\mathrm{d}x\pi_{ij} \\
&= \vdots \\
&= \sum_{\pi\in\widehat{G}} d_\pi \operatorname{Tr}((\widehat{f}(\pi))\pi(\cdot))
\end{aligned}$$

where there may be an arithmetic error in the last formula. This leads to:

Theorem 11.18 (Inversion theorem). *If $f \in \mathcal{T}(G)$ then for $x \in G$ we have*

$$f(x) = \sum_{\pi\in\widehat{G}} d_\pi \operatorname{Tr}(\widehat{f}(\pi)\pi(x))$$

Proof. The right hand side (call it $\widetilde{f}$) is in $\mathcal{T}(G)$, and $\|f-\widetilde{f}\|_2 = 0$, so $f = \widetilde{f}$ on G as each is continuous. ☐ Theorem 11.18

Theorem 11.19 (Plancherel/Riesz-Fischer). *If $f \in L^1(G)$ then*

$$f \in L^2(G) \iff \sum_{\pi\in\widehat{G}} d_\pi\|\widehat{f}(\pi)\|^2_{\mathrm{HS}(H_\pi)}$$

where

$$\|A\|^2_{\mathrm{HS}(H_\pi)} = \sum_{i,j=1}^{d_\pi}|\langle e_j^\pi \mid Ae_i^\pi\rangle|^2$$

is the Hilbert-Schmidt norm*. Furthermore we have*

$$\|f\|_2 = \left(\sum_{\pi\in\widehat{G}} d_\pi\|\widehat{f}(\pi)\|^2_{\mathrm{HS}(H_\pi)}\right)^{\frac{1}{2}}$$

i.e.

$$L^2(G) = \ell^2\text{-}\bigoplus_{\pi\in\widehat{G}}\sqrt{d_\pi}\,\mathrm{HS}(H_\pi)$$

Proof. Riesz-Fischer theorem. □ Theorem 11.19

Theorem 11.20 (Parseval's formula)**.** *If $f, g \in L^2(G)$ then*

$$\int_G \overline{f} g \mathrm{d}m = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{Tr}((\widehat{f}(\pi))^* \widehat{g}(\pi))$$

Proposition 11.21 (Uniqueness)**.** *If $\mu \in M(G)$ then the Fourier-Stieltjes transform is given on π in $\widehat{G}$ by*

$$\widehat{\mu}(\pi) = \int_G \pi(x^{-1}) \mathrm{d}\mu(x)$$

Then if $\widehat{\mu}(\pi) = 0$ for every $\pi \in \widehat{G}$ we must have $\mu = 0$.

Proof. If $\widehat{\mu}(\pi) = 0$ for all π then

$$\int_G f \mathrm{d}\mu = 0$$

for all $f \in \mathcal{T}(G)$. So

$$\int_G f \mathrm{d}\mu = 0$$

for all $f \in C(G)$, since $\overline{\mathcal{T}(G)}^{\|\cdot\|_\infty} = C(G)$ by Peter-Weyl. Hence $\mu = 0$ (by Riesz representation theorem). □ Proposition 11.21

11.3 Character theory

If $\rho\colon G \to U(\mathcal{H})$ is a finite-dimensional unitary representation, we define its character to be $\chi_\rho = \operatorname{Tr} \circ \rho\colon G \to \mathbb{C}$.

Proposition 11.22. *Suppose $\pi, \pi' \in \widehat{G}$ and $\rho\colon G \to U(\mathcal{H})$ is a finite dimensional representation. Then*

1. $\chi_\pi \chi_{\pi'} = \chi_{\pi \otimes \pi'} = \sum_{i=1}^n m_i \chi_{\pi_i}$, *where*

$$\pi \otimes \pi' = \bigoplus_{i=1}^n \pi_i^{(m_i)}$$

   with $\pi_i \in \widehat{G}$.

2. $\int_G \overline{\chi_\pi} \chi_\rho \mathrm{d}m = m(\pi, \rho) := \max\{ m \in \{0\} \cup \mathbb{N} : \pi^{(m)}$ *is equivalent to a subring of* $\rho \}$.

3. $\rho \in \widehat{G} \iff \int_G |\chi_\rho|^2 \mathrm{d}m = 1$

4. *If we let 1 be the trivial representation then*

$$m(1, \pi \otimes \pi') = \begin{cases} 1 & \text{if } \pi' = \pi \\ 0 & \text{else} \end{cases}$$

Proof.

1. Suppose $x \in G$. Then

$$\begin{aligned} \chi_\pi(x) \chi_{\pi'}(x) &= \operatorname{Tr}(\pi(x)) \operatorname{Tr}(\pi'(x)) \\ &= \operatorname{Tr}(\pi(x) \otimes \pi'(x)) \text{ (check, linear algebra)} \\ &= \operatorname{Tr}\left(\bigoplus_{i=1}^n \pi_i^{(m_i)}(x) \right) \\ &= \sum_{i=1}^n m_i \operatorname{Tr}(\pi_i(x)) \\ &= \sum_{i=1}^n m_i \chi_{\pi_i} \end{aligned}$$

2. Suppose
$$\rho = \bigoplus_{i=1}^{n} \pi_i'^{(m_i)}$$
then
$$\pi \otimes \rho = \bigoplus_{i=1}^{n} (\pi \otimes \pi_i')^{(m_i)}$$
We then use the first item and the Schur orthogonality relations.

3. If
$$\rho = \bigoplus_{i=1}^{n} \pi_i^{(m_i)}$$
then as above we have
$$\chi_\rho = \sum_{i=1}^{n} m_i \chi_{\pi_i}$$
So
$$\overline{\chi_\rho}\chi_\rho = \sum_{i,j=1}^{n} m_i m_j \overline{\chi_{\pi_j}}\chi_{\pi_i}$$
So
$$\int_G |\chi_\rho|^2 \mathrm{d}m = \sum_{i,j=1}^{n} m_i m_j \underbrace{\int_G \chi_{\overline{\pi_j}}\chi_{\pi_i} \mathrm{d}m}_{\delta_{ij}} \mathrm{d}m = \sum_{k=1}^{n} m_k^2$$
This is > 1 unless ρ is irreducible.

4. Combine the second and third items. □ Proposition 11.22

Definition 11.23 (Normalized characters)**.** If $\pi \in \widehat{G}$ we let $\psi_\pi = \frac{1}{d_\pi}\chi_\pi$.

Then if
$$\pi \otimes \pi' = \bigoplus_{i=1}^{n} \pi_i^{(m_i)}$$
for distinct $\pi_1, \ldots, \pi_n \in \widehat{G}$, then
$$\psi_\pi \psi_{\pi'} = \sum_{i=1}^{n} \frac{m_i}{d_\pi d_{\pi'}} \chi_{\pi_i} = \underbrace{\sum_{i=1}^{n} \frac{m_i d_{\pi_i}}{d_\pi d_{\pi'}} \psi\psi_{\pi_i}}_{\text{convex combination}}$$
This motivates the following:

Definition 11.24. A *discrete hypergrape* is a set Γ such that $\ell^1(\Gamma)$ admits a product which satisfies

1. $\delta_\gamma \cdot \delta_{\gamma'} \in \mathrm{Prob}(\Gamma) = \left\{ (p_\gamma)_{\gamma\in\Gamma} : \sum_{\gamma\in\Gamma} p_\gamma = 1, p_\gamma \geqslant 0 \right\}$.
2. There is an identity for $\cdot$, call it δ_1
3. There is an involution $\gamma \mapsto \overline{\gamma}$ (i.e. with $\gamma = \overline{\overline{\gamma}}$) such that $\delta_1 \in \mathrm{supp}(\delta_\gamma \cdot \delta_{\gamma'})$ if and only if $\gamma' = \overline{\gamma}$.

12 Amenability

Definition 12.1 (von Neumann)**.** A discrete grape G is called *amenable* (Day) provided there is a finitely additive probability measure $\mu\colon \mathcal{P}(G) \to [0,1]$ satisfying

- $\mu(\varnothing) = 0$
- $\mu(A \cup B) = \mu(A) + \mu(B)$ when $A \cap B = \varnothing$.
- $\mu(G) = 1$.
- $\mu(xE) = \mu(E)$ for $x \in G$ and $E \in \mathcal{P}(G)$.

Proposition 12.2. *There is a bijective correspondence between finitely additive probability measures on a set X and*

$$\mathcal{M}\ell^\infty(X) = \{\, M \in \ell^\infty(X)^* : M(\varphi) \geqslant 0 \text{ if } \varphi \geqslant 0 \text{ in } \ell^\infty(X), M(1) = 1 \,\}$$

(These are called means*.)*

Proof. Given $M \in \mathcal{M}\ell^\infty(X)$, let $\mu(E) = M(1_E)$. Conversely, given a finitely additive probability measure μ consider $S(X) = \operatorname{span}\{\, 1_E : E \in \mathcal{P}(X) \,\}$. Then check that

- $S(X)$ is dense in $\ell^\infty(X)$.
- Each $\psi \in S(X)$ can be uniquely represented in the form

  $$\psi = \sum_{i=1}^n a_i 1_{E_i}$$

  with the a_i distinct elements of $\mathbb{C}$ and $E_i \cap E_j = \varnothing$ for $i \neq j$.

Define $M_0\colon S(X) \to \mathbb{C}$ by

$$M_0(\psi) = \sum_{i=1}^n a_i \mu(E_i)$$

Then this is a bounded linear functional on $S(X)$, and hence extends uniquely to $\ell^\infty(X)$. □ Proposition 12.2

Example 12.3 (Ultrafilter limits). Let $\mathcal{U}$ be an ultrafilter on X; i.e. $\mathcal{U} \subseteq \mathcal{P}(X) \backslash \{\varnothing\}$ with $A, B \in \mathcal{U} \implies A \cap B \in \mathcal{U}$, and if $E \in \mathcal{P}(X)$ then exactly one of E and $X \backslash E$ lies in $\mathcal{U}$.

Define $\delta_{\mathcal{U}}\colon \mathcal{P}(X) \to [0,1]$ by

$$\delta_{\mathcal{U}}(E) = \begin{cases} 1 & \text{if } E \in \mathcal{U} \\ 0 & \text{else} \end{cases}$$

The associated mean on $\ell^\infty(X)$ will be denoted $L_{\mathcal{U}}$ (ultrafilter limit).

Definition 12.4. We say a discrete grape is *amenable* if there is $M \in \mathcal{M}\ell^\infty(G)$ such that $M(\varphi \cdot x) = M(\varphi)$ for $\varphi \in \ell^\infty(G)$ and $x \in G$.

Question 12.5. Now let G be a (not necessarily discrete) locally compact grape. What space replaces $\ell^\infty(G)$? $L^\infty(G)$? $C_b(G)$? $C_{\mathrm{lu}}(G) = \{\, \varphi \in C_b(G) : x \mapsto \varphi \cdot x\colon G \to C_b(G) \text{ is continuous} \,\}$? (One should check that $C_{\mathrm{lu}}(G)$ is closed in $C_b(G)$.)

Definition 12.6. Let $\mathcal{E}$ be any of $L^\infty(G), C_b(G), C_{\mathrm{lu}}(G)$. We let $\mathcal{M}\mathcal{E} = \{\, M \in \mathcal{E}^* : M(\varphi) \geqslant 0 \text{ if } \varphi \geqslant 0, M(1) = 1 \,\}$ denote the *means* on $\mathcal{E}$. We call $M \in \mathcal{M}\mathcal{E}$ *left-invariant* if $M(\varphi \cdot x) = M(\varphi)$ for $\varphi \in \mathcal{E}$ and $x \in G$.

We will tend to prefer $L^\infty(G)$ and $C_{\mathrm{lu}}(G)$.

Remark 12.7. Since the map $C_{\mathrm{lu}}(G) \times G \to C_{\mathrm{lu}}(G)$ given by $(\varphi, x) \mapsto \varphi \cdot x$ is continuous, we may define an action of $L^1(G)$ on $C_{\mathrm{lu}}(G)$

$$\varphi \cdot f = \int_G (\varphi \cdot x) f(x) \mathrm{d}x$$

(Bochner integral) for $\varphi \in L^1(G)$ and $f \in C_{\mathrm{lu}}(G)$.

Notation 12.8. Let

$$P^1(G) = \left\{ f \in L^1(G) : f \geqslant 0 \text{ almost everywhere}, \int_G f \mathrm{d}m = 1 \right\}$$

Proposition 12.9. *Suppose $M \in \mathcal{M}C_{\mathrm{lu}}(G)$. Then M is left-invariant if and only if $M(\varphi \cdot f) = M(\varphi)$ for all $\varphi \in C_{\mathrm{lu}}(G)$ and $f \in P^1(G)$.*

Proof.

$(\implies)$ Note that

$$M(\varphi \cdot f) = \int_G \underbrace{M(\varphi \cdot x)}_{=M(\varphi)} f(x) \mathrm{d}x = M(\varphi)$$

$(\impliedby)$ If $x \in G$ and $f \in P^1(G)$, then $x * f \in P^1(G)$. Then for $\varphi \in C_{\mathrm{lu}}(G)$, $x \in G$, and $f \in P^1(G)$ we have

$$M(\varphi \cdot x) = M((\varphi \cdot x) \cdot f) = M(\varphi \cdot (x * f)) = M(\varphi)$$

(One should check the second equality.) ☐ Proposition 12.9

Notation 12.10. We run into a problem: for $\varphi \in L^\infty(G)$ the map $x \mapsto \varphi \cdot x$ may not be norm continuous. For $f \in L^1(G)$ and $\varphi \in L^\infty(G)$, we define $\varphi \cdot f$ by

$$\langle \varphi \cdot f, g \rangle = \int_G \varphi \cdot f = \int_G \varphi f * g \mathrm{d}m$$

i.e. if $L_f : L^1(G) \to L^1(G)$ is convolution on the left by f, then we set $\varphi \cdot f = L_f^* \varphi$ (adjoint operator).

Remark 12.11. Notice that if $f, f' \in L^1(G)$ and $\varphi \in L^\infty(G)$, then

$$\varphi \cdot (f * f') = L_{f*f'}^*(\varphi) = (L_f L_{f'})^* \varphi = L_{f'}^* L_f^* \varphi = (\varphi \cdot f) \cdot f'$$

Likewise we have $(\varphi \cdot f) \cdot x = \varphi \cdot (f * x)$ for $x \in G$. (One should check this.) Finally, note that

$$\|\varphi \cdot f\|_\infty = \|L_f^* \varphi\|_\infty \leqslant \|L_f\| \|\varphi\|_\infty \leqslant \|f\|_1 \|\varphi\|_\infty$$

Proposition 12.12. *If $\varphi \in L^\infty(G)$ and $f \in L^1(G)$, then $\varphi \cdot f \in C_{\mathrm{lu}}(G)$.*

Proof. First note that for $x, y \in G$ we have

$$\|(\varphi \cdot f) \cdot x) - (\varphi \cdot f) \cdot y\|_\infty \leqslant \|\varphi\|_\infty \|f * x - f * y\|_1 \xrightarrow{x \to y} 0$$

One checks that this implies that $\varphi \cdot f$ is equal almost everywhere to an element of $C_{\mathrm{lu}}(G)$. ☐ Proposition 12.12

Theorem 12.13. *The following are equivalent:*

1. *$L^\infty(G)$ admits a left-invariant mean.*
2. *$C_c(G)$ admits a left-invariant mean.*
3. *$C_{\mathrm{lu}}(G)$ admits a left-invariant mean.*

Proof.

(1) $\implies$ (2) Restriction.

(2) $\implies$ (3) Restriction.

(3) $\implies$ (1) Let $(k_\alpha)_{\alpha\in A} \subseteq P^1(G)$ be a summability kernel. If $\varphi \in L^\infty(G)$ then $\varphi \cdot k_\alpha \in C_{\mathrm{lu}}(G)$ for each α by previous lemma. Let $\mathcal{U}$ be an ultrafilter on A containing all cofinal subsets. If $a, b \in \ell^\infty(A)$ with $\lim_{\alpha\in A}(a_\alpha - b_\alpha) = 0$, then $L_{\mathcal{U}}(a) = L_{\mathcal{U}}(b)$. (Recall that $L_{\mathcal{U}}$ denotes the ultrafilter limit mean.) Given left-invariant $M \in \mathcal{M}C_{\mathrm{lu}}(G)$, we let

$$\begin{aligned} M_{\mathcal{U}} \colon L^\infty(G) &\to \mathbb{C} \\ \varphi &\mapsto L_{\mathcal{U}}((M(\varphi \cdot k_\alpha))_{\alpha\in A}) \end{aligned}$$

It is now straightforward to check that

- $M_{\mathcal{U}}$ is linear and bounded with $\|M_{\mathcal{U}}\| \leqslant \|M\|$.
- $M_{\mathcal{U}}(\varphi) \geqslant 0$ if $\varphi \geqslant 0$ in $L^\infty(G)$.
- $M_{\mathcal{U}}(1) = 1$.

So $M \in \mathcal{M}L^\infty(G)$. Now if $f \in P^1(G)$ then

$$\lim_{\alpha\in A} k_\alpha * f = f = \lim_{\alpha\in A} f * k_\alpha$$

(by A2). Hence for $\varphi \in L^\infty(G)$ we have

$$\begin{aligned} M_{\mathcal{U}}(\varphi \cdot f) &= L_{\mathcal{U}}\big((M(\varphi \cdot (f * k_\alpha)))_{\alpha\in A}\big) \\ &= L_{\mathcal{U}}\big((M(\underbrace{\varphi \cdot (k_\alpha * f)}_{(\varphi\cdot k_\alpha)\cdot f}))_{\alpha\in A}\big) \\ &= L_{\mathcal{U}}\big((M(\varphi \cdot k_\alpha))_{\alpha\in A}\big) \\ &= M_{\mathcal{U}}(\varphi) \end{aligned}$$

☐ Theorem 12.13

Corollary 12.14. *G is amenable if and only if there is $M \in \mathcal{M}L^\infty(G)$ such that $M(\varphi \cdot f) = M(\varphi)$ for $\varphi \in L^\infty(G)$ and $f \in P^1(G)$.*

Proof. Built into the proof of the previous theorem. ☐ Corollary 12.14

Notation 12.15. Since $(L^1(G))^* = L^\infty(G)$, we regard $L^1(G) \subseteq (L^\infty(G))^*$.

Lemma 12.16.

1. *$\mathcal{M}L^\infty(G)$ is w*-compact and convex.*
2. *$\overline{P^1(G)}^{\mathrm{w}*} = \mathcal{M}L^\infty(G)$.*

Proof.

1. It is straightforward that $\mathcal{M}L^\infty(G)$ is convex and w*-closed. Moreover, $\mathcal{M}L^\infty(G) \subseteq \mathrm{ball}((L^\infty(G))^*)$ (closed unit ball); hence by Banach-Alaoglu it follows that $\mathcal{M}L^\infty(G)$ is w*-compact. Indeed, note that since $\|\varphi\|_\infty 1 - |\varphi| \geqslant 0$, we have $M(|\varphi|) \leqslant \|\varphi\|_\infty$. Next, by Cauchy-Schwarz inequality, we have

$$|M(\overline{\varphi}\psi)| \leqslant (M(\overline{\varphi}\varphi))^{\frac{1}{2}}(M(\psi\overline{\psi}))^{\frac{1}{2}} \leqslant \left\||\varphi|^2\right\|_\infty^{\frac{1}{2}}\left\||\psi|^2\right\|_\infty^{\frac{1}{2}} = \|\varphi\|_\infty\|\psi\|_\infty$$

(note that Cauchy-Schwarz applies since $M(\overline{\varphi}\psi)$ is a Hermitian bilinear form). So

$$|M(\varphi)| = |M(1\varphi)| \leqslant \|1\|_\infty\|\varphi\|_\infty = \|\varphi\|_\infty$$

2. Since $P^1(G) \subseteq \mathcal{M}L^\infty(G)$, we get $\overline{P^1(G)}^{w^*} \subseteq \mathcal{M}L^\infty(G)$ by (1). Let $M \in \mathcal{M}L^\infty(G) \subseteq \mathrm{ball}((L^\infty(G))^*)$ (by proof of (1)). Then by Goldstine's theorem we have a net $(f_\alpha)_\alpha$ in $\mathrm{ball}(L^1(G)$ such that

$$M = \text{w*-}\lim_\alpha f_\alpha$$

Write each

$$f_\alpha = \sum_{k=0}^{3} i^k f_{\alpha,k}$$

with each $f_{\alpha,k} \geqslant 0$ and $f_{\alpha,k} \leqslant |f_\alpha|$; so $\|f_{\alpha,k}\|_1 \leqslant \|f\|_1$. If $\varphi \geqslant 0$ in $L^\infty(G)$ then

$$0 \leqslant M(\varphi) = \lim_\alpha i^k \underbrace{\int_G f_{\alpha,k}\varphi \,\mathrm{d}m}_{\geqslant 0}$$

So since positives span $L^\infty(G)$ we see that

$$\begin{aligned} M &= \text{w*-}\lim_\alpha (f_{\alpha,0} - f_{\alpha,2}) \\ 0 &= \text{w*-}\lim_\alpha (f_{\alpha,1} - f_{\alpha,3}) \end{aligned}$$

But also

$$1 = M(1) = \lim_\alpha \int_G (f_{\alpha,0} - f_{\alpha,2}) \mathrm{d}m = \lim_\alpha (\|f_{\alpha,0}\|_1 - \|f_{\alpha,2}\|_1)$$

But each of $\|f_{\alpha,0}\|_1, \|f_{\alpha,2}\|_1$ lies in $[0,1]$. So

$$\begin{aligned} \lim_\alpha \|f_{\alpha,0}\|_1 &= 1 \\ \lim_\alpha \|f_{\alpha,2}\|_1 &= 0 \end{aligned}$$

We conclude that

$$M = \text{w*-}\lim_\alpha \frac{1}{\|f_{\alpha,0}\|_1} f_{\alpha,0} \in \overline{P^1(G)}^{w^*}$$

as desired. □ Lemma 12.16

Theorem 12.17 (Reiter)**.** *The following are equivalent:*

1. *G is amenable.*
2. *There is a net $(f_\alpha)_\alpha$ in $P^1(G)$ such that*

$$\lim_\alpha \|f * f_\alpha - f_\alpha\|_1 = 0$$

for $f \in P^1(G)$.

3. *Given $\varepsilon > 0$ and $K \subseteq G$ compact there is $r \in P^1(G)$ such that $\|x * r - r\|_1 < \varepsilon$ for $x \in K$.*
4. *There is a net (r_α) in $P^1(G)$ such that for $K \subseteq G$ compact we have*

$$\lim_\alpha \sup_{x \in K} \|x * r_\alpha - r_\alpha\|_1 = 0$$

(We call such a net a Reiter net*.)*

5. *There is a net (r_α) in $P^1(G)$ such that*

$$\lim_\alpha \|x * r_\alpha - r_\alpha\| = 0$$

for $x \in G$. (We call such a net an asymptotically invariant net*.)*

Proof.

(1) $\implies$ (2) Let $M \in \mathcal{M}L^\infty(G)$ satisfy that$M(\varphi \cdot f) = M(\varphi)$ for $\varphi \in L^\infty(G)$ and $f \in P^1(G)$ (by last corollary).

TODO 15. *ref*

Let $(g_\alpha)_{\alpha\in A}$ in $P^1(G)$ satisfy
$$M = w^* \lim_{\alpha\in A} g_\alpha$$
(by lemma). Then for $\varphi \in L^\infty(G)$ and $f \in P^1(G)$ we have
$$0 = M(\varphi - \varphi \cdot f) = \lim_{\alpha\in A} \int_G g_\alpha(\varphi - \varphi \cdot f)\mathrm{d}m = \lim_{\alpha\in A} \int_G (f * g_\alpha - g_\alpha)\varphi \mathrm{d}m$$
So
$$\text{w-}\lim_{\alpha\in A}(f * g_\alpha - g_\alpha) = 0$$
(weak limit). If $F \subseteq P^1(G)$ is finite, we let
$$C_F = \operatorname{conv}\{\, (f * g_\alpha - g_\alpha)_{f\in F} : \alpha \in A \,\} \subseteq (L^1(G))^F$$
(finite product of Banach spaces). By the Hahn-Banach theorem we have $\overline{C_F}^{\mathrm{w}} = \overline{C_F}^{\|\cdot\|}$ (where $\|\cdot\|$ is any "natural" norm on $(L^1(G))^F$). So $0 \in \overline{C_F}^{\mathrm{w}} = \overline{C_F}^{\|\cdot\|}$. Now let
$$C_{P^1(G)} = \operatorname{conv}\{\, (f * g_\alpha - g_\alpha)_{f\in P^1(G)} : \alpha \in A \,\} \subseteq (L^1(G), \|\cdot\|_1)^{P^1(G)}$$
Since $0 \in \overline{C_F}^{\|\cdot\|}$ for each F, we have that $0 \in \overline{C_{P^1(G)}}^{\mathrm{prod}}$. Hence there is a net (f_β) in $\operatorname{conv}\{\, g_\alpha : \alpha \in A \,\}$ such that
$$0 = \text{prod-}\lim_\beta (f * f_\beta - f_\beta)$$
for $f \in P^1(G)$. So
$$0 = \lim_\beta \|f * f_\beta - f_\beta\|_1$$
for each $f \in P^1(G)$.

(2) $\implies$ (3) Fix $\varepsilon > 0$, $f \in P^1(G)$, and $K \subseteq G$ compact. Let U be a relatively compact neighbourhood of e such that $\|x * f - f\|_1 < \varepsilon$ for $x \in U$. Then
$$\left\| \frac{1}{m(U)} 1_U * f - f \right\|_1 \leqslant \frac{1}{m(U)} \int_U \|x * f - f\|_1 \mathrm{d}x \leqslant \varepsilon$$
Let $x_1, \ldots, x_n \in G$ be such that
$$K \subseteq \bigcup_{k=1}^n x_k U$$
Use the hypothesis to find α_0 such that
$$\Bigg\| \underbrace{\frac{1}{m(U)} 1_{x_k U}}_{\in P^1(G)} * f * f_{\alpha_0} - f_{\alpha_0} \Bigg\|_1 < \varepsilon$$
for $k \in \{\, 1, \ldots, n \,\}$. So $\|f * f_{\alpha_0} - f_{\alpha_0}\|_1 < \varepsilon$. We let $r = f * f_{\alpha_0}$. Then for $x \in U$ and $k \in \{\, 1, \ldots, n \,\}$ we have
$$\begin{aligned}
\|(x_k x) * r - r\|_1 &\leqslant \left\| (x_k x) * r - \frac{1}{m(U)} 1_{x_k U} * r \right\|_1 + \left\| \frac{1}{m(U)} 1_{x_k U} * r - f_{\alpha_0} \right\|_1 + \|f_{\alpha_0} - r\|_1 \\
&\leqslant \left\| x_k * \left(x * f x * f - \frac{1}{m(U)} 1_U * f \right) * f_{\alpha_0} \right\|_1 + 2\varepsilon \\
&\leqslant \|x * f - f\|_1 + \left\| f - \frac{1}{m(U)} 1_U * f \right\|_1 + 2\varepsilon \\
&< 4\varepsilon
\end{aligned}$$

Thus

$$\sup_{x \in K} \|x * r - r\|_1 \leqslant 4\varepsilon$$

(3) $\Longrightarrow$ (4) Let $A = \{ (K, \varepsilon) : K \subseteq G \text{ compact}, \varepsilon > 0 \}$, preordered by $(K, \varepsilon) \leqslant (K', \varepsilon')$ if $K \subseteq K'$ and $\varepsilon > \varepsilon'$. For each $\alpha = (K, \varepsilon) \in A$ we let r_α satisfy (3).

(4) $\Longrightarrow$ (5) Clear.

(5) $\Longrightarrow$ (1) Any w*-cluster point of an asymptotically invariant net is left-invariant. ☐ Theorem 12.17

Corollary 12.18. *The following are equivalent:*

1. *G is amenable.*
2. *$L^\infty(G)$ admits a right-invariant mean.*
3. *$L^\infty(G)$ admits a two-sided invariant mean.*

Note: we are not suggesting that any left-invariant mean is also right-invariant; just that such means exist.

Proof.

(1) $\Longrightarrow$ (2) Let $M \in \mathcal{M}C_b(G)$ be a left-invariant mean. Consider the map $\varphi \mapsto \breve{\varphi}$ for $\varphi \in C_b(G)$ give by $\breve{\varphi}(x) = \varphi(x^{-1})$. This is an isomorphism of the algebra $C_b(G)$ with $\breve{1} = 1$ and $\breve{\varphi} \geqslant 0$ if $\varphi \geqslant 0$. Let $\breve{M}$ be given by $\breve{M}(\varphi) = M(\breve{\varphi})$. Then $\breve{M}$ is right-invariant. Hence there is a right-invariant mean on C_{ru}, and hence on $L^\infty(G)$.

(1) $\Longrightarrow$ (3) Let (f_α) be an asymptotically left-invariant net in $P^1(G)$. Then (f_α^*) is an asymptotically right invariant net. Consider the net $(f_\alpha * f_\alpha^*)$ in $P^1(G)$. (Recall that $P^1(G)$ is closed under convolution.) Now if $x, y \in G$ we have

$$\begin{aligned} \|x * f_\alpha * f_\alpha^* * y - f_\alpha * f^*\|_1 &\leqslant \|x * f_\alpha * f_\alpha^* * y - x * f_\alpha * f_\alpha^*\|_1 + \|x * f_\alpha * f_\alpha^* - f_\alpha * f_\alpha^*\|_1 \\ &\leqslant \|f_\alpha^* * y - f_\alpha^*\|_1 + \|x * f_\alpha - f_\alpha\|_1 \\ &\xrightarrow{\alpha} 0 \end{aligned}$$

Anny w*-cluster point of this last net in $\mathcal{M}L^\infty(G)$ is thus a two-sided invariant mean. ☐ Corollary 12.18

13 Extent of amenable grapes

Remark 13.1. If G is compact then G is amenable.

Proposition 13.2. *If G is abelian then G is amenable.*

Proof. For $x \in G$ we let $L_x \in \mathcal{B}(L^1(G))$ be $L_x(f) = x * f$. Then $L_x^*(\varphi) = \varphi \cdot x$ for $\varphi \in L^\infty(G)$. We recall that $\mathcal{M}L^\infty(G)$ is w*-compact and convex, and each $L_x^*(\mathcal{M}L^\infty(G)) \subseteq \mathcal{M}L^\infty(G)$. Since G is abelian we get that $\{ L_x^* : x \in G \}$ is a commuting (semi)grape of affine maps in $\mathcal{M}L^\infty(G)$. We then apply Markov-Kakutani; any fixed point is then a left-invariant mean. ☐ Proposition 13.2

Remark 13.3. Suppose $\beta\colon G \to H$ is a continuous homomorphism with dense range. Then the map $C_{\mathrm{lu}}(H) \to C_{\mathrm{lu}}(G)$ given by $\varphi \mapsto \varphi \circ \beta$ satisfies:

- It is a linear isometry (dense range)

  TODO 16. *conjunction?*

- $1_H \circ \beta = 1_G$
- $\varphi \circ \beta \geqslant 0$ if $\varphi \geqslant 0$.

Note that $(\varphi \circ \beta) \cdot x = (\varphi \cdot \beta(x)) \circ \beta$, which is why each $\varphi \circ \beta \in C_{\mathrm{lu}}(G)$.

Proposition 13.4. *If $\beta\colon G \to H$ is a continuous homomorphism with dense range and G is amenable, then H is amenable.*

Proof. Let M_G be a left-invariant mean on $C_{\mathrm{lu}}(G)$. Define M_H on $C_{\mathrm{lu}}(H)$ by $M_H(\varphi) = M_G(\varphi \circ \beta)$. Then M_H is a left-invariant mean on $C_{\mathrm{lu}}(H)$. □ Proposition 13.4

Remark 13.5. Some consequences:

1. Let G_d be G with the discrete topology. If G_d is amenable, then so is G. Indeed, we just consider the identity map $\beta\colon G_d \to G$. (In this case we say that G is *discretely amenable.*)
2. If N is a closed normal subgrape of G and G is amenable then so too is G/N. Indeed, we just consider the quotient map $\beta\colon G \to G/N$.

Proposition 13.6. *Suppose G admits an amenable closed normal subgrape N for which G/N is amenable. Then G is amenable.*

Proof. (The philosophy is to use Weil's "integral" formula.) Let $q\colon G \to G/N$ denote the quotient map. Then $\varphi \mapsto \varphi \circ q$ is a map

$$C_{\mathrm{lu}}(G/N) \to C_{\mathrm{lu}}(G:N) = \{\, \varphi \in C_{\mathrm{lu}}(G) : \varphi = n \cdot \varphi \text{ for } n \in N \,\}$$

that is surjective. Indeed, if $\varphi \in C_{\mathrm{lu}}(G:N)$, we let $\tilde{\varphi}(xN) = \varphi(x)$. Since q is an open map it follows that $\tilde{\varphi} \in C_{\mathrm{lu}}(G/N)$, and $\tilde{\varphi} \circ q = \varphi$.

Let $M_N \in \mathcal{M}C_b(N)$ be left-invariant. Let $T_{M_N} : C_{\mathrm{lu}}(G) \to C_{\mathrm{lu}}(G:N)$ be given by

$$T_{M_N}\varphi(x) = M_N(\varphi \cdot x{\restriction}N) = M_N(n \mapsto \varphi(xn))$$

Then

- $|T_{M_N}\varphi(x)| \leqslant \|\varphi \cdot x\|_\infty = \|\varphi\|_\infty$, and T_{M_N} is linear.
- $|T_{M_N}\varphi(x) - T_{M_N}\varphi(x)| \leqslant \|\varphi \cdot x - \varphi \cdot y\|_\infty$; so $T_{M_N}\varphi$ is continuous and $(T_{M_N}\varphi) \cdot z = T_{M_N}(\varphi \cdot z)$, so $T_{M_N}\varphi \in C_{\mathrm{lu}}(G)$.
- $T_{M_N}(C_{\mathrm{lu}}(G)) \subseteq C_{\mathrm{lu}}(G:N)$ since for $x \in G$ and $n \in N$ we have

$$T_{M_N}\varphi(xn) = M_N(\varphi \cdot (xn){\restriction}N) = M_N(n' \mapsto \varphi(xnn')) = M_N(\varphi \cdot x{\restriction}N) = T_{M_N}\varphi(x)$$

Let $\widetilde{T_{M_N}}\varphi \in C_{\mathrm{lu}}(G/N)$ be the associated element, as above. We have left-invariant $M_{G/N} \in \mathcal{M}C_{\mathrm{lu}}(G/N)$. Let $M_G\colon C_{\mathrm{lu}}(G) \to \mathbb{C}$ be given by $M_G(\varphi) = M_{G/N}(\widetilde{T_{M_N}}\varphi)$. One checks that $\widetilde{T_{M_N}}(\varphi \cdot x) = \widetilde{T_{M_N}}\varphi \cdot xN$; it then follows that M_G is a left-invariant mean. □ Proposition 13.6

Corollary 13.7. *Solvable grapes are amenable.*

Proof. Evident induction. (Recall here that $G^{(n)} = \overline{[G^{(n-1)}, G^{(n-1)}]}$ (closure) with $G^{(0)} = G$.) □ Corollary 13.7

Example 13.8. Euclidean motion $\mathbb{R}^n \rtimes \mathrm{SO}(n)$.

Remark 13.9 (Tits). If $\mathbb{K}$ is a field and $G \leqslant \mathrm{GL}_n(\mathbb{K})$ (discrete) then either

- $G \supseteq F$ with $F \cong F_2$ (free grape on two generators)
- $G \supseteq G_1$ with $[G : G_1] < \infty$ and G_1 is solvable.

Proposition 13.10. *If G is amenable and H is an open subgrape, then H is amenable.*

Proof. Let T be a transversal for right cosets of H in G. We define $S_T\colon C_b(H) \to C_b(G)$ by $S_T\varphi(ht) = \varphi(h)$ with $h \in H$ and $t \in T$. Then S_T is a linear isometry with $S_T 1_H = 1_G$ and $S_T\varphi \geqslant 0$ if $\varphi \geqslant 0$. Let $M_H \in \mathcal{M}C_b(H)$ be given by $M_H(\varphi) = M_G(S_T\varphi)$ (where M_G is a left-invariant mean in $\mathcal{M}C_B(G)$). ☐ Proposition 13.10

Proposition 13.11. *Suppose there is a family $(G_\alpha)_{\alpha\in A}$ of open subgrapes indexed over a directed set A with $G_\alpha \subseteq G_{\alpha'}$ if $\alpha \leqslant \alpha'$; suppose each G_α is amenable, and that*

$$G = \bigcup_{\alpha\in A} G_\alpha$$

Then G is amenable.

Proof. For each α let M_α be a left-invariant mean in $\mathcal{M}C_B(G_\alpha)$. Let $\widetilde{M_\alpha} \in \mathcal{M}C_b(G)$ be given by $\widetilde{M_\alpha}(\varphi) = M_\alpha(\varphi 1_{G_\alpha})$. Then $(\widetilde{M_\alpha})_{\alpha\in A}$ lies in $\mathcal{M}C_b(G)$, and hence has a cluster point M. If $x \in G$, say $x \in G_{\alpha_0}$, and $\varphi \in C_b(G)$, then for $\alpha \geqslant \alpha_0$, we have

$$\widetilde{M_\alpha}(\varphi \cdot x) = M_\alpha((\varphi 1_{G_\alpha}) \cdot x) = M_\alpha(\varphi 1_{G_\alpha}) = \widetilde{M_\alpha}(\varphi)$$

It follows that M is left-invariant. ☐ Proposition 13.11

Remark 13.12. If we do not have an increasing family of open amenable subgrapes, then we can't conclude that G is amenable. Consider for example

$$F_2 = \bigcup_{x\in F_2} \langle x\rangle$$

Theorem 13.13 (Følner). *The following are equivalent:*

1. *G is amenable.*
2. *Given $\varepsilon, \delta > 0$ and $K \subseteq G$ compact, there are $E \subseteq G$ compact and Borel $N \subseteq K$ such that $m(N) < \delta$ and*
$$\frac{m(xE \triangle E)}{m(E)} < \varepsilon$$
for $x \in K \backslash N$. (Here $\triangle$ denotes the symmetric difference.)
3. *Given $\varepsilon > 0$ and $K \subseteq G$ compact, there is compact $F \subseteq G$ such that*
$$\frac{m(xF \triangle F)}{m(F)} < \varepsilon$$
for $x \in K$. (This is the Følner condition*.)*
4. *There is a net (F_α) of compact subsets of G such that for any compact $K \subseteq G$ we have*
$$\lim_\alpha \sup_{x\in K} \frac{m(xF_\alpha \triangle F_\alpha)}{m(F_\alpha)} = 0$$
(We call this a Følner net*.)*

Before the proof, some consequences:

Example 13.14 (Discrete abelian grapes are amenable). Suppose G is an abelian grape; then

$$G = \bigcup_{F\subseteq G \text{ finite}} \langle F\rangle$$

By the previous proposition

TODO 17. *ref*

it suffices to consider a finitely generated grape. There is an obvious quotient map $q_F : \mathbb{Z}^F \to \langle F \rangle$. Hence it suffices to see that any $\mathbb{Z}^k$ (for $k \in \mathbb{N}$) is amenable. Consider the sequence $F_n = \{ -n, -(n-1), \dots, n-1, n \}^k$. One checks that this is a Følner sequence. In fact $\frac{1}{(2n+1)^k} 1_{F_n}$ is a Reiter sequence.

Example 13.15. Consider $F_2 = \langle a, b \rangle$. If $K \subseteq F_2$ is finite, we let

$$\partial K = \{ x \in K : \{ ax, bx, a^{-1}x, b^{-1}x \} \nsubseteq K \}$$

Then an inequality something like $|K| \leqslant 2|\partial K|$ holds (see Cayley graph), which implies that the Følner condition must fail.

Proof of Theorem 13.13.

(1) $\implies$ (2)

(I) Given $\varepsilon' > 0$, let us find

- compact $E_1 \supseteq E_2 \supseteq \dots \supseteq E_n$ with $m(E_n) > 0$ and
- $\lambda 1, \dots, \lambda_n > 0$ such that

$$\sum_{j=1}^{n} \lambda_j = 1$$

such that

$$\psi = \sum_{j=1}^{n} \frac{\lambda_j}{m(E_j)} 1_{E_j}$$

satisfies

$$\|x * \psi - \psi\|_1 < \varepsilon' \text{ for } x \in K \tag{7}$$

First, Retier's, theorem gives $r \in P^1(G)$ such that $\|x * r - r\|_1 < \varepsilon'$ for $x \in K$. There is a sequence $(f_n')_{n=1}^{\infty}$ in $C_c(G)$ such that

$$\lim_{n\to\infty} \|f_n' - r\|_1 = 0$$

Then let

$$f_n = \frac{1}{\|f_n'\|_1} |f_n'| \in P^1(G)$$

and check that

$$\lim_{n\to\infty} \|f_n - r\|_1 = 0$$

Hence there is $f \in C_c(G)$ such that $\|x * f - f\|_1 < \varepsilon'$.

Now we perform a "layer cake" construction. Fix $n \in \mathbb{N}$. For $j \in \{ 1, \dots, n \}$, let

$$E_j = f^{-1}\left(\left[\frac{j}{n+1}\|f\|_\infty, \infty\right)\right)$$

So $\mathrm{supp}(f) \supseteq E_1 \supseteq \dots \supseteq E_n$ with $m(E_n)$. We then define

$$\psi_n' = \sum_{j=1}^{n} \frac{\|f\|_\infty}{n+1} 1_{E_j}$$

This then satisfies

$$\psi_n' \leqslant f \leqslant \psi_n' + \frac{1}{n+1} 1_{\mathrm{supp}(f)}$$

It follows that

$$0 < \int_G \psi_n' \mathrm{d}m = \underbrace{\sum_{j=1}^{n} \frac{\|f\|_\infty m(E_j)}{n+1}}_{\|\psi_n'\|_1} \leqslant \int_G f \mathrm{d}m = 1 \leqslant \int_G \psi_n' \mathrm{d}m + \frac{m(\mathrm{supp}(f))}{n+1}$$

Let

$$\psi_n = \frac{1}{\|\psi'_n\|_1}\psi'_n = \sum_{j=1}^{n} \underbrace{\frac{\|f\|_\infty m(E_j)}{(n+1)\|\psi'_n\|_1}}_{\lambda_j > 0} \frac{1}{m(E_j)} 1_{E_j}$$

and observe that $\psi_n = \sum_{j=1}^{n} \frac{\lambda_j}{m(E_j)} 1_{E_j}$ and $\sum_{j=1}^{n} \lambda_j = 1$. Furthermore, it is a routine computation that

$$\|\psi_n - f\|_1 \leqslant \frac{1}{2}n + 1m(\operatorname{supp}(f))$$

and hence for large enough n, say $\frac{2}{n+1}\operatorname{supp}(f) < \frac{\varepsilon'}{2}$, we are done.

(II) We let ψ satisfy Equation (7), with $\varepsilon' = \frac{\varepsilon\delta}{m(K)}$, provided $m(K) > 0$ (otherwise we let $N = K$ and we are done). Note that if $E, F \subseteq G$ with $E \cap F = \varnothing$ and $x \in G$ then

$$xE \triangle E) \cap (xF \triangle F) = \varnothing$$

so

$$(xE \triangle E) \cup (xF \triangle F) = (x(E \cup F)) \triangle (E \cup F)$$

Write

$$\psi = \sum_{j=1}^{n} \frac{\lambda_j}{m(E_j)} \sum_{i=1}^{j} 1_{E_i \setminus E_{i+1}}$$

(with $E_{n+1} = \varnothing$). We thus have that

$$\begin{aligned}
|x * \psi - \psi| &= \left| \sum_{j=1}^{n} \frac{\lambda_j}{m(E_j)} \sum_{i=1}^{j} (1_{x(E_i \setminus E_{i+1})} - 1_{E_i \setminus E_{i+1}}) \right| \\
&= \left| \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\lambda_j}{m(E_j)} (1_{x(E_i \setminus E_{i+1})} - 1_{E_i \setminus E_{i+1}}) \right| \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\lambda_i}{m(E_j)} |1_{x(E_i \setminus E_{i+1})} - 1_{E_i \setminus E_{i+1}}| \text{ (pairwise disjoint supports)} \\
&= \sum_{j=1}^{n} \frac{\lambda_j}{m(E_j)} \sum_{i=1}^{j} 1_{x(E_i \setminus E_{i+1}) \triangle (E_i \setminus E_{i+1})} \\
&= \sum_{j=1}^{n} \frac{\lambda_j}{m(E_j)} 1_{xE_j \triangle E_j}
\end{aligned}$$

Thus

$$\frac{\delta\varepsilon}{m(K)} > \|x * \psi - \psi\|_1 = \sum_{j=1}^{n} \lambda_j \frac{m(xE_j \triangle E_j)}{m(E_j)}$$

Then we have

$$\delta\varepsilon > \int_K \|x * \psi - \psi\|_1 \mathrm{d}x = \sum_{j=1}^{n} \lambda_j \int_K \frac{m(xE_j \triangle E_j)}{m(E_j)} \mathrm{d}x$$

so at least one $\delta\varepsilon > \int_K \frac{m(xE_j \triangle E_j)}{m(E_j)} \mathrm{d}x$; we let $E = E_j$ for this j. Let

$$N = \left\{ x \in K : \frac{m(xE \triangle E)}{m(E)} \geqslant \varepsilon \right\}$$

which is closed, and thus Borel. Then N satisfies $\varepsilon 1_N(x) \leqslant \frac{m(xE \triangle E)}{m(E)}$, so

$$m(N) \leqslant \frac{1}{\varepsilon} \int_K \frac{m(xE \triangle E)}{m(E)} \mathrm{d}x < \delta$$

$\underline{(2) \implies (3)}$ First note that if G is discrete and m is the counting measure, we could just let $\delta < 1$ and be done. The hard part of the proof, then, is when G is not discrete.

Let $K \subseteq G$ be compact; let $A = K \cup K^2$. Hence if $x \in K$ then $m(xA \cap A) \geqslant m(xK) = m(K)$. Let $0 < \delta < \frac{m(K)}{2}$. If $B \subseteq A$ is Borel with $m(A \backslash B) < \delta$ then for $x \in K$ we have

$$xA \cap A \subseteq (xB \cap B) \cup (x(A\backslash B)) \cup (A\backslash B)$$

so

$$2\delta < m(K) \leqslant m(xA \cap A) \leqslant m(xB \cap B) + 2\underbrace{m(A\backslash B)}_{<\delta}$$

Hence $0 < m(xB \cap B)$. So $xB \cap B \neq \varnothing$, and $x \in BB^{-1}$. Thus $K \subseteq BB^{-1}$. Now for $\varepsilon > 0$ the hypothesis gives a compact $F \subseteq G$ such that $\dfrac{m(xF \triangle F)}{m(F)} < \dfrac{\varepsilon}{2}$ for $x \in A\backslash N$ and $m(N) < \delta$. Let $B = A\backslash N$. Notice for $C, D \subseteq G$ we have $C\backslash D \subseteq (C\backslash F) \cup (F\backslash D)$; so $C \triangle D \subseteq (C \triangle F) \cup (F \triangle D)$. Thus if $x, y \in B^{-1}$ we have

$$\begin{aligned} m(x^{-1}yF \triangle F) &= m(xF \triangle yF) \\ &= m(xF \triangle F) + m(F \triangle yF) \\ &= m(F \triangle x^{-1}F) + m(y^{-1}F \triangle F) \\ &< \varepsilon m(F) \end{aligned}$$

by Equation (7). Hence for $z \in K \subseteq BB^{-1}$ we are done.

$\underline{(3) \implies (4)}$ Straightforward. (Just like Reiter's theorem.)

$\underline{(4) \implies (1)}$ If (F_α) is a Følner net, then $(\frac{1}{m(F_\alpha)} 1_{F_\alpha})$ in $P^1(G)$ is a Reiter net. □ Theorem 13.13

Remark 13.16. The construction of a Følner net above does not provide $F_\alpha \subseteq F_{\alpha'}$ for $\alpha \leqslant \alpha'$. This can be arranged, generally, but is technical. However, in practice, most Følner nets one encounters do satisfy this.

Fact 13.17. *If G is separable and amenable, then $L^1(G)$ is separable. If $L^1(G)$ is separable, then we can extract a Reiter sequence from a Reiter net. If this last holds, then Følner sequences can be found.*

13.1 Hulanicki's theorem

Let $\lambda\colon G \to U(L^2(G))$ be the *left regular representation*: $\lambda(x)h(y) = h(x^{-1}y)$ for almost every $y \in G$. Let

$$A^+(G) = \left\{ \langle h \,|\, \lambda^{(\mathbb{N})}(\cdot)h \rangle = \sum_{j=1}^{\infty} \langle h_j \,|\, \lambda(\cdot)h_j \rangle : h = (h_j)_{j=1}^{\infty} \in L^2(G)^{(\mathbb{N})} \right\}$$

Note that

$$L^2(G)^{(\mathbb{N})} = \left\{ h = (h_j)_{j=1}^{\infty} : \text{each } h_j \in L^2(G), \sum_{j=1}^{\infty} \|h_j\|_2^2 < \infty \right\}$$

Fact 13.18. $A^+(G) \subseteq B^+(G) = \{\, u\colon G \to \mathbb{C} \mid$ *u continuous, positive definite*$\}$.

Notice that for$h = (h_j)_{j=1}^{\infty} \subseteq L^2(G)^{(\mathbb{N})}$ we have

$$\left\| \langle h \,|\, \lambda^{(\mathbb{N})}(\cdot)h \rangle - \sum_{j=1}^{n} \langle h_j \,|\, \lambda(\cdot)h_j \rangle \right\|_\infty = \sum_{j=n+1}^{\infty} \|\langle h_j \,|\, \lambda(\cdot)h_j \rangle\|_\infty$$

Remark 13.19. 1. If $|J| > |\mathbb{N}|$ and $h = (h_j)_{j \in J} \in L^2(G)^{(J)}$, so $\sum_{j \in J} \|h_j\|_2^2 < \infty$, then $h_j \neq 0$ for at most countably many $j \in J$. Hence $\langle h \,|\, \lambda^{(J)}(\cdot)h \rangle \in A^+(G)$. (Easy check.)

2. (Eymard, 64) Each $u \in A^+(G)$ can be written in the form $u = \langle h \mid \lambda(\cdot)h \rangle$ for some $h \in L^2(G)$. (This is the *standard form of von Neumann algebras.*)

Theorem 13.20 (Hulanicki's theorem I)**.** *G is amenable if and only if there is a net (u_α) in $A^+(G)$ such that $\lim_\alpha u_\alpha = 1$ uniformly on compact sets.*

Proof.

($\implies$) Let (r_α) in $P^1(G)$ be a Reiter net. Let $h_\alpha = r_\alpha^{\frac{1}{2}}$; so

$$\|h\|_2 = \left(\int_G |h_\alpha|^2 \mathrm{d}m \right)^{\frac{1}{2}} = \left(\int_G r_\alpha \mathrm{d}m \right)^{\frac{1}{2}} = 1$$

Note for $a, b \geqslant 0$ we have $|a-b|^2 \leqslant |a-b|(a+b) = |a^2 - b^2|$; so for $x \in G$ we have

$$\begin{aligned}
\|\lambda(x)h_\alpha - h_\alpha\|_2^2 &= \int_G |h_\alpha(x^{-1}y) - h_\alpha(y)|^2 \mathrm{d}y \\
&\leqslant \int_G |r_\alpha(x^{-1}y) - r_\alpha(y)| \mathrm{d}y \\
&= \|x * r_\alpha - r_\alpha\|_1
\end{aligned}$$

Hence

$$\begin{aligned}
|1 - \langle h_\alpha \mid \lambda(x)h_\alpha \rangle| &= |\langle h_\alpha \mid h_\alpha \rangle - \langle h_\alpha \mid \lambda(x)h_\alpha \rangle| \\
&\leqslant \underbrace{\|h_\alpha\|_2}_{=1} \|h_\alpha - \lambda(x)h_\alpha\|_2 \text{ (by Cauchy-Schwarz)} \\
&= \|x * r_\alpha - r_\alpha\|_1^{\frac{1}{2}}
\end{aligned}$$

and it follows that $u_\alpha = \langle h_\alpha \mid \lambda(\cdot)h_\alpha \rangle$ converges uniformly on compact sets to 1.

□ Theorem 13.20