Course notes for PMATH 930

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Contents

1 Introduction

Some recent work in algebraic geometry can be understood as equipping a field with an operator (with some additional conditions), and attempting to do algebraic geometry in the context of that operator. (So looking at solutions to polynomials that also involve the operator.) Model theory seems to be the correct framework for studying this.

No overall textbook, though individual sections may have references. He may periodically post notes. A few assignments, maybe every other week, and perhaps a short oral final.

All rings are commutative with unity. (So ring morphisms preserve identity, subrings are assumed to be unital, etc.) After Section 1.3 we assume that all fields are of characteristic zero.

Rahim is numbering the theorems etc. in class; I'll put those numbers in parentheses after the automatic theorem numbering LaTeX does.

I use the convention of conflating L with the L-formulas; so "$\varphi \in L$" means φ is an L-formula, and "$\varphi \in L(A)$" means φ is an L-formula with parameters from A.

1.1 Algebraically closed fields

We view fields as structures in the language $L = \{0, 1, +, -, \times\}$ of rings. (Throughout this course L will always denote the language of rings.) Consider the L-theory T of integral domains; note that T is a universal theory. (In fact it is the universal theory of the theory of fields: the integral domains are precisely the subrings of fields.)

We are interested in the *existentially closed (e.c.)* models of T; i.e. the models $M \models T$ satisfying the following:
Suppose $\varphi(x)$ is a conjunction of *literals* (i.e. atomic or negated atomic, so in this case a system of polynomial equations and inequations) with parameters from M, where $x = (x_1, \ldots, x_n)$ is a tuple of variables; suppose further that φ has a solution in some extension $M \subseteq N \models T$. Then φ has a solution in M.

Remark 1.1. If M is an existentially closed integral domain then it is a field. Indeed, if $a \in M$ is non-zero, then $xa = 1$ has a solution in $\mathrm{Frac}(M)$, and thus in M. It is also algebraically closed: if $P \in M[x]$ is non-constant then $P(x) = 0$ has a solution in M^{alg}, and thus in M.

The converse is a form of Hilbert's Nullstellensatz: every algebraically closed field is an existentially closed integral domain. We use model-completeness of the theory ACF of algebraically closed fields:

Proof. Suppose K is an algebraically closed field; suppose $\varphi(x)$ is some conjunction of literals over K with a solution in some integral domain $R \supseteq K$. Then $K \subseteq R \subseteq \mathrm{Frac}(R) \subseteq \underbrace{\mathrm{Frac}(R)^{\mathrm{alg}}}_{L}$ with L an algebraically closed field; so by model completeness we have $K \preceq L$. But $L \models \exists x \varphi(x)$; so $K \models \exists x \varphi(x)$, and φ is solved in K. $\square$

So the existentially closed integral domains are exactly the algebraically closed fields. Note also that the class of algebraically closed fields are axiomatizable (namely by ACF). We say ACF is a *model companion* of T: the theory of the existentially closed models of T. (Note that for general theories the class of existentially closed models may not be axiomatizable; these have no model companion.)

Fact 1.2. *A model of T always has an extension that is existentially closed.*

The above is proven by iteratively adding solutions.

1.2 Differential fields

Our language is $L_\delta = \{0, 1, +, -, \times, \delta\}$ with δ a unary function symbol.

Definition 1.3. A *differential ring* is a ring R equipped with a *derivation*; i.e. $\delta \colon R \to R$ satisfying

- $\delta(a + b) = \delta a + \delta b$.
- $\delta(ab) = a(\delta b) + (\delta a)b$ (the Leibniz rule).

Example 1.4.

1. Any ring with the trivial derivation $\delta(a) = 0$ for all $a \in R$ is a differential ring.
2. If A is any ring we can consider $R = A[t]$ with $\delta = \frac{\mathrm{d}}{\mathrm{d}t}$.
3. The ring of smooth functions on $\mathbb{R}$ with the usual differentiation.
4. The field of germs of meromorphic functions (functions with isolated singularities) on some open $U \subseteq \mathbb{C}$ equipped with the usual complex differentiation.

Exercise 1.5.

1. Suppose R is an integral domain and $\delta \colon R \to R$ is a derivation. Then δ extends uniquely to a derivation $\delta \colon \mathrm{Frac}(R) \to \mathrm{Frac}(R)$ given by
$$\delta\Big(\frac{a}{b}\Big) = \frac{(\delta a)b - b(\delta a)}{b^2}$$

2. Suppose (R, δ) is a differential ring and $I \subseteq R$ is an ideal. We say I is a *differential ideal* if $\delta a \in I$ for all $a \in I$. In this case δ induces a unique derivation $\delta \colon R/I \to R/I$ given by $\delta(r + I) = \delta(r) + I$ for $r \in R$.

3. Determine the differential ideals of $A[t]$ as in the previous example.

Suppose $R \subseteq S$ is a ring extension. A *derivation* $\delta \colon R \to S$ is an additive map satisfying the Leibniz rule $\delta(ab) = a(\delta b) + (\delta a)b$.

Exercise 1.6.

1. Suppose $\delta \colon R \to F$ is a derivation where $R \subseteq F$ is a field. Then δ extends uniquely to a derivation $\mathrm{Frac}(R) \to F$.

2. Suppose $\delta \colon R \to S$ is a derivation; suppose we have ideals $I \subseteq R$ and $J \subseteq S$ such that $J \cap R = I$ (so $R/I \subseteq S/J$). If $\delta(I) \subseteq J$ then δ extends uniquely to a derivation $\delta \colon R/I \to S/J$ via $\delta(a+I) = \delta(a) + J$.

3. Suppose $\delta \colon R \to S$ is a derivation and we are given a ring homomorphism $\varphi \colon S \to S'$ such that $\varphi \restriction R \colon R \to R'$ is an isomorphism. Then $\varphi \circ \delta \circ (\varphi \restriction R)^{-1} \colon R' \to S'$ is a derivation.

Remark 1.7. $\delta(1) = \delta(1 \cdot 1) = 1 \cdot \delta(1) + \delta(1) \cdot 1 = 2\delta(1)$, so $\delta(1)$ is always 0. Hence $\delta(n) = 0$ for $n \in \mathbb{Z}$.

Proposition 1.8 (1)**.** *Suppose $\delta \colon R \to S$ is a derivation (so $R \subseteq S$). Suppose $P \in R[x_1, \ldots, x_n]$ and $a = (a_1, \ldots, a_n) \in R^n$. Then*

$$\delta(P(a)) = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(a)\delta(a_i) + P^{\delta}(a)$$

where $P^{\delta}(x) \in S[x_1, \ldots, x_n]$ is obtained from P by applying δ to the coefficients.

Proof. We let $x = (x_1, \ldots, x_n)$.

Claim 1.9. *If $P(x)$ is a monomial (which for us means no leading coefficient) then*

$$\delta(P(a)) = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(a)\delta(a_i)$$

Proof. We induct on the total degree. If $\deg(P) = 0$ then $P = 1$, and indeed $\delta(P(a)) = 0$.

Suppose $\deg(P) > 0$. Then $P(x) = x_i Q(x)$ for some $1 \le i \le n$ with Q a monomial. Then

$$\begin{aligned}
\delta(P(a)) &= \delta(a_i Q(a)) \\
&= (\delta a_i)Q(a) + a_i \delta(Q(a)) \\
&= (\delta a_i)Q(a) + a_i \sum_{j=1}^{n} \frac{\partial Q}{\partial x_j}(a)\delta(a_j) \\
&= (\delta a_i)Q(a) + a_i \frac{\partial Q}{\partial x_i}(a)\delta a_i + a_i \sum_{j \neq i} \frac{\partial Q}{\partial x_j}(a)\delta a_j \\
&= \frac{\partial(x_i Q)}{\partial x_i}(a)\delta a_i + \sum_{j \neq i} \frac{\partial(x_i Q)}{\partial x_j}(a)\delta a_j
\end{aligned}$$

as desired. $\square$ Claim 1.9

Now, suppose $P(x) = bQ(x)$ where $b \in R$ an d$Q(x)$ a monomial. Then

$$\begin{aligned}
\delta(P(a)) &= \delta(bQ(x)) \\
&= b\delta(Q(a)) + (\delta b)Q(a) \\
&= b\left(\sum_{i=1}^{n} \frac{\partial Q}{\partial x_i}(a)\delta a_i\right) + (\delta b)Q(a) \\
&= \sum_{i=1}^{n} \frac{\partial bQ}{\partial x_i}(a)\delta(a_i) + \delta(b)Q(a) \\
&= \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(a)\delta(a_i) + P^{\delta}(a)
\end{aligned}$$

The general P is a sum of such; the result follows because

- $\delta((P+Q)(a)) = \delta(P(a)) + \delta(Q(a))$
- $\dfrac{\partial(P+Q)}{\partial x_i}(a) = \dfrac{\partial P}{\partial x_i}(a) + \dfrac{\partial Q}{\partial x_i}(a)$
- $(P+Q)^{\delta}(a) = P^{\delta}(a) + Q^{\delta}(a)$. □ Proposition 1.8

This proposition is why differential algebra works out as nicely as it does. Note in particular that $\delta(a_i)$ always shows up as a linear term, regardless of the degree of the polynomial; so we can solve for it.

Corollary 1.10 (2)**.** *A derivation is determined by its values on generators.*

Proof. Suppose we have a derivation $\delta\colon R \to S$ and $\Lambda \subseteq R$ generates R as a ring. Then if $a \in R$ then $a = P(\lambda)$ for some $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^n$ and $P \in \mathbb{Z}[x_1, \ldots, x_n]$. Then

$$\delta a = \delta(P(\lambda)) = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(\lambda)\delta\lambda_i$$

(Note that P^{δ} is 0 since the coefficients of P are integers.) So $\delta(a)$ is determined by $\delta(\lambda_1), \ldots, \delta(\lambda_n)$. □ Corollary 1.10

Lemma 1.11 (3)**.** *Suppose $\delta\colon R \to S$ is a derivation. Consider the extension of rings $R[t] \subseteq S[t]$. Given $f \in S[t]$ there is a unique extension of δ to a derivation $R[t] \to S[t]$ such that $\delta(t) = f$.*

Proof. Uniqueness is by corollary (2), since $R[t]$ is generated by $R \cup \{t\}$.

For existence, we define $\delta\colon R[t] \to S[t]$ by $\delta(P(t)) = P'(t)f + P^{\delta}(t)$. Taking $P \in R$ a constant polynomial shows that this extends the original $\delta\colon R \to S$. Taking $P = t$ shows that $\delta(t) = f$. One checks that this defines a derivation $R[t] \to S[t]$. □ Lemma 1.11

Corollary 1.12 (4)**.** *Suppose $F \subseteq K$ is an extension of fields of characteristic 0. Suppose $\delta\colon F \to K$ is a derivation; suppose $a \in K$.*

1. *If $a \in F^{\mathrm{alg}}$ then there is a unique extension of δ to a derivation $\delta\colon F(a) \to K$.*
2. *If $a \notin F^{\mathrm{alg}}$ then for any $b \in K$ there is a unique extension of δ to a derivation $\delta\colon F(a) \to K$ such that $\delta a = b$.*

In particular, δ extends to a differential field (K, δ) and then uniquely to a differential field $(K^{\mathrm{alg}}, \delta)$.

Proof.

1. Let $P(t) \in F[t]$ be the minimal polynomial of a over F. Note that $P'(a) \neq 0$ since $\deg(P)$ is minimal and $\mathrm{char}(F) = 0$. Extend $\delta\colon F \to K$ to $\delta\colon F[t] \to K[t]$ by

$$t \mapsto \frac{-P^{\delta}(a)}{P'(a)}$$

by lemma (3). Then

$$\begin{aligned}\delta(P(t)) &= P'(t)\delta t + P^{\delta}(t) \text{ (by proposition 1)}\\ &= P'(t)\left(\frac{-P^{\delta}(a)}{P'(a)}\right) + P^{\delta}(t)\end{aligned}$$

So $\delta(P(t))(a) = 0$, and $\delta(P(t)) \in (t-a)K[t]$; i.e. $\delta\colon F[t] \to K[t]$ takes the ideal $PF[t]$ to $(t-a)K[t]$. Also $(t-a)K[t] \cap F[t] = PF[t]$ (since P is the minimal polynomial for a over K). By Exercise 1.6 we get an induced derivation $\delta\colon F[t]/(P) \to K[t]/(t-a)$ extending δ.

So we have the following picture:

$$\begin{tikzcd} F[t]/(P) \arrow[r, "\delta"] \arrow[d, "\cong"] & K[t]/(t-a) \arrow[d, "\varphi"] \\ F(a) & K \end{tikzcd}$$

where φ is evalation at a, an isomorphism. So we get a derivation $\varphi\delta\varphi^{-1}\colon F(a) \to K$.

For uniqueness: suppose $\delta\colon F(a) \to K$ extends $\delta\colon F \to K$. Then since $0 = P(a)$ we get $0 = \delta(P(a)) = P'(a)\delta a + P^{\delta}(a)$; so

$$\delta a = \frac{-P^{\delta}(a)}{P'(a)}$$

So δ is determined on a, and hence on $F[a]$ (by corollary (2)), and hence on $F(a)$ by the quotient rule.

2. Suppose $a \in K \setminus F^{\mathrm{alg}}$; suppose $b \in K$. We wish to show that there is a unique extension of δ to $F[a] \to K$ such that $\delta(a) = b$.

Use lemma (3) to extend $\delta\colon F \to K$ to $\delta\colon F[t] \to K[t]$ with $\delta(t) = b$. Then we have the following picture:

$$\begin{tikzcd} F[t] \arrow[r, "\delta"] \arrow[d, "\varphi\restriction F[t]"] & K[t] \arrow[d, "\varphi"] \\ F[a] & K \end{tikzcd}$$

where φ is the K-algebra homomorphism given by evaluation at a. In particular $\varphi \restriction F[t]$ is an isomorphism; so by Exercise 1.6 we get a derivation $\varphi\delta(\varphi \restriction F[t])^{-1}\colon F[a] \to K$; so again by Exercise 1.6 we can extend to a derivation $F(a) \to K$. But $(\varphi\delta(\varphi \restriction F[t])^{-1})(a) = b$; so this is our desired derivation. Uniqueness is by corollary (2).

For the "in particular", suppose $\delta\colon F \to K$ is a derivation. Let B be a transcendence basis for K over F; so $K \subseteq F(B)^{\mathrm{alg}}$. Extend to $\delta\colon F(B) \to K$ by repeated use of part 2; then extend to $\delta\colon K \to K$ be repeated application of part 1. By a similar argument using repeated applications of part 1, we can further extend this *uniquely* to $(K^{\mathrm{alg}}, \delta)$. □ Corollary 1.12

1.3 Some motivation: algebraic vector fields

We are actually interested in differential fields of characteristic zero. (From now on we assume that **all** fields have characteristic zero.) We are thus interested in differential integral domains.

Example 1.13. Fix a field k and a finitely generated integral k-algebra R. We are interested in derivations $\delta\colon R \to R$ which are *k-linear*: i.e. $k \subseteq R^\delta$. (Here $R^\delta = \{ r \in R : \delta r = 0 \}$ is the *subring of constants*.) In this case we have $\delta(ar) = a\delta(r)$ for $a \in k$ and $r \in R$.

From algebraic geometry we have a bijective correspondence between finitely generated integral k-algebras R and irreducible affine algebraic varieties V over k. Indeed, given R we can write $R \cong k[x_1, \ldots, x_n]/I$ where $I \subseteq k[x_1, \ldots, x_n]$ is a prime ideal; we then consider $V(I)$ the zero set of I in K^n where $K \supseteq k$ is algebraically closed. Conversely, given such $V \subseteq K^n$ we consider $k[V] = k[x_1, \ldots, x_n]/I(V)$ where $I(V)$ is the polynomials in $k[x_1, \ldots, x_n]$ that vanish on V.

What is the geometric content of a derivation δ on R? What structure is this on V?

Let $V \subseteq \mathbb{A}^n$ be an irreducible affine variety over k. We let $k[V]$ be the coordinate ring of V as above; so $k[V]$ is a finitely generated integral k-algebra and $V = \operatorname{Spec}(k[V])$. In coordinates, take $x_1, \ldots, x_n$ as coordinates for $\mathbb{A}^n$; then $k[V] = k[x_1, \ldots, x_n]/I(V)$ for some prime ideal $I(V) \subseteq k[x_1, \ldots, x_n]$ and

$$x_1 + I(V), \ldots, x_n + I(V) \in k[V]$$

are generators.

TODO 1. *Is this an elaboration of the above?*

If you prefer to avoid spectra, it works to fix a large algebraically closed field $K \supseteq k$ and identify V with $K^n \supseteq V(K) = \{ (a_1, \ldots, a_n) \in K^n : P(a_1, \ldots, a_n) \text{ for all } P \in I(V) \}$. So work with the irreducible Zariski-closed subsets of K^n. (All our varieties will be affine.)

We define the *tangent bundle* $TV \subseteq \mathbb{A}^{2n}$ of V to be the affine algebraic variety defined as the solution set to the follow: for each $P(x_1, \ldots, x_n) \in I(V) \subseteq k[x_1, \ldots, x_n]$

- $P(x_1, \ldots, x_n) = 0$
- $\displaystyle\sum_{i=1}^n \frac{\partial P}{\partial x_i}(x_1, \ldots, x_n) y_i = 0.$

So the elements of TV consist of points in V together with vectors orthogonal to the gradients of all elements of $I(V)$.

Exercise 1.14. It suffices to consider generators of $I(V)$.

We have a projection map $\pi\colon TV \to V$. If $a \in V$, we use T_aV to denote the fibre of π over a: this is $\{ v \in \mathbb{A}^n : (a, v) \in TV \}$. We call T_aV the *tangent space of V at a*.

Lemma 1.15 (5). *Suppose (K, δ) is a differential field with $k \subseteq K^\delta$. Suppose $a \in V(K)$ (here $V(K) \subseteq K^n$ is the set of tuple from K^n satisfying the equations of V). Then $\nabla_a = (a, \delta a) = (a_1, \ldots, a_n, \delta a_1, \ldots, \delta a_n) \in K^{2n}$ lies in $TV(K)$. So $\nabla\colon V(K) \to TV(K)$ defines a section of $\pi\colon TV(K) \to V(K)$.*

Proof. Suppose $a \in V(K)$ and $P \in I(V)$; then $P(a) = 0$. Then

$$0 = \delta(P(a)) = \sum_{i=1}^n \frac{\partial P}{\partial x_i}(a)\delta a_i + \underbrace{P^\delta(a)}_{=0}$$

(since $P \in k[x_1, \ldots, x_n]$ and $k \subseteq K^\delta$). So $(a_1, \ldots, a_n, \delta a_1, \ldots, \delta a_n) \in TV(K)$, as desired. □ Lemma 1.15

So in particular $\delta a \in T_aV$.

Note this ∇ is not an algebraic section (i.e. is not a morphism of varieties): it involves δ, not just polynomial maps.

Definition 1.16. An *affine algebraic vector field over k* is a variety $V \subseteq \mathbb{A}^n$ over k equipped with an algebraic section $s\colon V \to TV$ over k of π (so $\pi s = \mathrm{id}_V$).

In particular, since $V \subseteq \mathbb{A}^n$ and $TV \subseteq \mathbb{A}^n$, we are demanding that s take the form $s(a) = (a, s_1(a), \ldots, s_n(a))$ for some polynomials $s_1, \ldots, s_n \in k[x_1, \ldots, x_n]$; so $(s_1(a), \ldots, s_n(a)) \in T_aV$. (We're being a bit vague about where a lives here; the reader should interpret it as lying in $V(L)$ for some large algebraically closed L if they prefer naive algebraic geometry, or as lying in $V(L')$ with L' any extension of k if they prefer the Grothendieck approach.)

Proposition 1.17 (6)**.** *There is a bijective correspondence between*

- *finitely generated integral k-algebras R equipped with a k-linear derivation δ, and*
- *affine algebraic vector fields (V, s) over k.*

Proof. Given (R, δ) we choose $x_1, \dots, x_n \in R$ generating R over k. Then

$$R = k[x_1, \dots, x_n] = k[X_1, \dots, X_n]/I(V) = k[V]$$

where the X_i are variables, $x_i = X_i + I(V)$, and $V \subseteq \mathbb{A}^n$.

TODO 2. *Presumably we're taking $V = V(I)$?*

For each $i \in \{1, \dots, n\}$ we have $\delta x_i = s_i(x_1, \dots, x_n)$ for some $s_i \in k[X_1, \dots, X_n]$. Define $s \colon \mathbb{A}^n \to \mathbb{A}^{2n}$ by $a \mapsto (a, s_1(a), \dots, s_n(a))$.

Claim 1.18. $s \restriction V \colon V \to TV$.

Proof. In $R = k[V]$, for any $P \in I(V) \subseteq k[X_1, \dots, X_n]$ we have

$$\sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(x_1, \dots, x_n) s_i(x_1, \dots, x_n) = \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(x_1, \dots, x_n)\delta x_i = \delta(P(x_1, \dots, x_n)) = 0$$

since $P^\delta = 0$ and $P(x_1, \dots, x_n) = P(X_1, \dots, X_n) + I(V) = I(V) = 0_R$ since $P \in I(V)$. So in $R = k[X_1, \dots, X_n]/I(V) = k[x_1, \dots, x_n]$ we have

$$0 = \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(x_1, \dots, x_n) s_i(x_1, \dots, x_n) = \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(X) s_i(X) + I(V)$$

where $X = (X_1, \dots, X_n)$. So

$$\sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(X) s_i(X) \in I(V)$$

So for all $a \in V$ we have

$$\sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(a) s_i(a) = 0$$

and thus $(a, s_1(a), \dots, s_n(a)) \in TV$. □ Claim 1.18

Suppose for the other direction we are given (V, s). Let $R = k[V] = k[X_1, \dots, X_n]/I(V)$. Define δ on $k[X_1, \dots, X_n]$ as follows: write $s \colon V \to TV$ as $a \mapsto (a, s_1(a), \dots, s_n(a)$ with $s_1, \dots, s_n \in k[X_1, \dots, X_n]$. We define δ on $k[X_1, \dots, X_n]$ by $\delta \restriction k = 0$ and $\delta(X_i) = s_i$; this is by iterative application of lemma (3).

In order for δ to induce a derivation on R we need that $I(V) \subseteq k[X]$ is a differential ideal. Suppose then that $P \in I(V)$. Then

$$\delta P = \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(X)\delta X_i + \underbrace{P^\delta(x)}_{=0} = \sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(X) s_i(X)$$

since $\delta \restriction K = 0$; it then suffices to check that this lies in $I(V)$. But for $a \in V$ we have

$$\sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(a) s_i(a) = 0$$

since $(a, s_1(a), \dots, s_n(a)) = (a, s(a)) \in TV$. So $\delta P \in I(V)$. We thus induce a k-linear derivation on $R = k[X]/I(V)$.

One checks that this is a bijective correspondence (up to isomorphism?). □ Proposition 1.17

A morphism of affine algebraic vector fields $(V,s) \to (W,t)$ is a morphism of varieties $\varphi\colon V \to W$ such that the following diagram commutes:

$$\begin{array}{ccc} TV & \xrightarrow{\mathrm{d}\varphi} & TW \\ {\scriptstyle s}\uparrow & & {\scriptstyle t}\uparrow \\ V & \xrightarrow{\varphi} & W \end{array}$$

Remark 1.19. If we fix R and V then the above gives a bijection between derivations on R and affine algebraic vector fields on V.

We can extend this correspondence to the case when the base field k has a non-trivial derivation (k,δ).

Definition 1.20. Fix a derivation δ on k. Given an affine variety $V \subseteq \mathbb{A}^n$ over k we define the *prolongation* of V is the affine algebraic variety $\tau V \subseteq \mathbb{A}^{2n}$ as the solution set of following polynomial equations: for each $P \in I(V) \subseteq k[X]$ with $X = (X_1,\ldots,X_n)$ we demand

- $P(X) = 0$
- $\displaystyle\sum_{i=1}^{n} \frac{\partial P}{\partial X_i}(X)Y_i + P^\delta(X) = 0$

in coordinates $(X,Y) = (X_1,\ldots,X_n,Y_1,\ldots,Y_n)$.

We have the projection $\tau V \to V$ onto the first n coordinates.

Remark 1.21. If $\delta = 0$ on k then $\tau V = TV$.

Fix $a \in V$, and consider $\tau_a V = \{\, b \in \mathbb{A}^n : (a,b) \in \tau V \,\}$ the prolongation space of V at a. Then $\tau_a V$ is defined by the equations

$$\sum_i \frac{\partial P}{\partial X_i}(a)Y_i + P^\delta(a) = 0$$

for $P \in I(V)$. This is an inhomogeneous linear equation, which then defines an affine subspace.

Remark 1.22. The linear space $T_a V$ acts on the affine space $\tau_a V$; this is a *torsor*.

Again, given $(K,\delta) \supseteq k,\delta)$ we get a projection $\tau V(K) \xrightarrow{\pi} V(K)$ and a section $\nabla\colon V(K) \to \tau V(K)$ given by $\nabla_a = (a, \delta a) \in \tau V(K)$ for $a = (a_1,\ldots,a_n) \in V(K)$. As before we have a bijective correspondence between derivations on $k[V]$ extending (k,δ) and algebraic sections $s\colon V \to \tau V$ to π over k (i.e. $s = (\mathrm{id}_V, s_1,\ldots,s_n)$).

Definition 1.23. Given (k,δ) an *(affine) D-variety over* k is an (affine) algebraic variety V over k equipped with an algebraic section $s\colon V \to \tau V$ (so $\pi \circ s = \mathrm{id}_V$).

Indeed, given $R = k[V]$ and a derivation δ on R we write $k[V] = k[X]/I(V)$. Write $\delta x_i = s_i(x)$ for some $s_i \in k[X]$ (here $x_i = X_i + I$); set $s = (\mathrm{id}_V, s_1,\ldots,s_n)$. One checks that this works.

Conversely, given $s\colon V \to \tau V$ we define δ on $k[X]$ by $\delta X_i = s_i(X)$ and extending (k,δ) where $s = (\mathrm{id}, s_1,\ldots,s_n)$. One checks that $I(V)$ is a δ-ideal. So this induces δ on $k[X]/I(V) = R$ extending (k,δ).

(This is a literal bijective correspondence, not "up to isomorphism".)

Unfortunately being finitely generated isn't first-order; so to do model theory we instead look at all differential integral k-algebras and hope to recover the finitely generated ones later.

2 Logic of differential rings

We study differential integral domains of characteristic zero. This is first-order axiomatizable in the language $L_\delta = L \cup \{\,\delta\,\}$ where L is (as always) the language of rings and δ is a unary function symbol; furthermore the axiomatization is universal. Call this theory T. We wish to study the existentially closed models of T: for example, are they axiomatizable?

We'll need to study *differential polynomials*: the "free object" in differential algebra. Given $(R,\delta) \models T$ we construct an extension $R\{X\}$ where X is a single differential indeterminate. As a ring we define $R\{X\} = R[X^{(0)}, X^{(1)}, \ldots]$ the polynomial ring over R in algebraic indeterminates $X^{(0)}, X^{(1)}, \ldots$. This is still an integral domain of characteristic 0; it then suffices to equip it with a derivation. We use the derivation δ extending (R, δ) such that $\delta X^{(i)} = X^{(i+1)}$. Then $(R\{X\}, \delta) \models T$ and extends (R, δ).

We call this the *ring of differential polynomials over* R. Given $f \in R\{X\}$ there is a minimal $n \geq 0$ such that $f \in R[X^{(0)}, \ldots, X^{(n)}]$; this is the *order* of f, denoted $\operatorname{ord}(f)$. Notationally we identify $X = X^{(0)}$; so $X^{(i)} = \delta^i X$ is just δ applied i times to X. So any $f \in R\{X\}$ of order n can be expressed uniquely as

$$\sum_{\alpha=(\alpha_0,\ldots,\alpha_n)\in\omega^{n+1}} a_\alpha X^{\alpha_0}(\delta X)^{\alpha_1}\cdots(\delta^n X)^{\alpha_n}$$

where each $a_\alpha \in R$ and cofinitely many are zero.

Each $f \in R\{X\}$ and $(S, \delta) \supseteq (R, \delta)$ we get a map $S \to S$ denoted $b \mapsto f(b)$ given by

$$f(b) = \sum_{\alpha\in\omega^{n+1}} a_\alpha b^{\alpha_0}(\delta b)^{\alpha_1}\cdots(\delta^n b)^{\alpha_n}$$

Iterating we get $R\{X_1, \ldots, X_n\}$ in differential indeterminates $X_1, \ldots, X_n$.

Proposition 2.1 (7)**.** *Every atomic L_δ-formula is T-equivalent to one of the form $f(x_1, \ldots, x_n) = 0$ where $f \in \mathbb{Z}\{X_1, \ldots, X_n\}$.*

(We distinguish between the algebraic variables $X_1, \ldots, X_n$ and the logical variables $x_1, \ldots, x_n$.)

What are the existentially closed models of T?

Lemma 2.2 (8)**.** *If (R, δ) is an existentially closed model of T, then R is an algebraically closed field.*

Proof. Let $F = \operatorname{Frac}(R)$, and extend δ (uniquely) to $(F, \delta) \supseteq (R, \delta)$; then $(F, \delta) \models T$. Fix $a \in R \setminus \{0\}$, and consider the L_δ-atomic formula over R given by $ax = 1$. This is solved in (F, δ) by a^{-1}, and hence in (R, δ) by existential closedness. So a is invertible in R; so $R = F$ is a field.

Fix non-constant $P(t) \in F[t]$, and consider the atomic formula over F given by $P(x) = 0$. We have seen that (F, δ) extends to $(F^{\mathrm{alg}}, \delta) \models T$, and there is a root of P in F^{alg}; so there is one in F by existential closedness. So F is algebraically closed. □ Lemma 2.2

Being algebraically closed is certainly not sufficient.

Example 2.3. Consider $\mathbb{C}$ with the derivation $\delta = 0$. This is an algebraically closed field, but is not existentially closed: the equation $\delta x = 1$ is solved in $(\mathbb{C}[t], \frac{\mathrm{d}}{\mathrm{d}t}) \models T$ by t, but has no solution in $\mathbb{C}$.

An algebraic characterization of which systems of δ-polynomial equations and inequations need to be solved to be existentially closed was given by Lenore Blum in the 70s. She showed that $(K, \delta) \models T$ is existentially closed if and only if for all $f, g \in K\{X\}$ with $\operatorname{ord}(f) > \operatorname{ord}(g)$ we have that $(f(x) = 0) \wedge (g(x) \neq 0$ has a solution in K. (Here X is a singleton variable.)

TODO 3. *Really? What about*

$$f = \delta^2 x - 2\delta x + x, g = \delta x - x$$

The proof is somewhat technical; we instead present the geometric axioms of Pierce-Pillay (90s).

Theorem 2.4 (9)**.** $(K, \delta) \models T$ *is existentially closed if and only if K is an algebraically closed field and*

(*) *Suppose V is an irreducible affine algebraic variety over K and $W \subseteq \tau V$ an irreducible subvariety over K such that $\pi(W)$ is Zariski-dense in V. Then there is $a \in V(K)$ such that $\nabla_a \in W$. (In fact we will have $\nabla_a \in W(K)$.)*

Remark 2.5.

1. Write

$$\begin{aligned} K[V] &= K[X]/I(V) \\ I(V) &= (P_1, \dots, P_\ell) \\ K[W] &= K[X,Y]/I(W) \\ I(W) &= (Q_1, \dots, Q_r) \text{ (for } Q_i \in K[X,Y]) \end{aligned}$$

So $I(\tau V) \subseteq I(W)$. (Here $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ are tuples.) Then there is $a \in V(K)$ such that $\nabla_a \in W$ if and only if there is $a \in K^n$ such that

$$\bigwedge_{i=1}^{\ell} P_i(a) = 0 \wedge \bigwedge_j Q_j(a, \delta a) = 0$$

2. Suppose (V, s) is a D-variety over (K, δ). Then $W = s(V)$ is a subvariety of τV and projects dominantly onto V. So (*) says that there is $a \in V(K)$ such that $\nabla_a \in W$; i.e. $s(a) = \nabla(a)$. ((*) is actually stronger; this is merely a consequence.)

Definition 2.6. If (V, s) is a D-variety over (K, δ) then we define the *D-points* over K to be $(V, s)^\sharp(K) = \{ a \in V(K) : \nabla_a = s(a) \}$.

If $a = (a_1, \dots, a_n)$ and $s = (\mathrm{id}_V, s_1, \dots, s_n)$ for $s_1, \dots, s_n \in K[x_1, \dots, x_n]$ then $\nabla_a = s(a)$ if and only if

$$\bigwedge_{i=1}^{n} \delta(a_i) = s_i(a_1, \dots, a_n)$$

TODO 4. *Label with Rahim's numbering?*

Proof of Theorem 2.4.

($\implies$) Suppose (K, δ) is existentially closed. Then K is an algebraically closed field by (8). Given V and $W \subseteq \tau V$ as in (*). Then $\pi(W)$ is Zariski-dense in V if and only if $K[V] \subseteq K[W]$ (recall $K[W] = K[X,Y]/I(W)$ and $K[V] = K[X]/I(V)$); this is in turn equivalent to $I(W) \cap K[X] = I(V)$. Extend (K, δ) to $\delta\colon K[X] \to K[X,Y]$ by $X_i \mapsto Y_i$ for $i \in \{1, \dots, n\}$. If $P \in I(V)$ then

$$\delta(P(X)) = \sum_i \frac{\partial P}{\partial X_i}(X) Y_i + P^\delta(X) \in I(\tau V) \subseteq I(W)$$

(since $W \subseteq \tau V$). So we get an induced $\delta\colon K[V] \to K[W]$; extend this to $\delta\colon K(V) \to K(W)$, and in turn extend this to $\delta\colon K(W) \to K(W)$. Then $(K, \delta) \subseteq (K(W), \delta) \models T$.

Let $L = K(W)$; we show that (*) holds in L. For each $i \in \{1, \dots, n\}$ let $b_i = X_i + I(V) \in K[V] \subseteq K(W) = L$

TODO 5. *port the preceding convention to earlier?*

Then $b_i = X_i + I(W) \in K[W] \subseteq L$. Then $\delta b_i = \delta(X_i + I(V)) = \delta(X_i) + I(W) = Y_i + I(W)$. Let $b = (b_1, \dots, b_n) \in L^n$; then $b \in V(L)$. But $\nabla_b = (b_1, \dots, b_n, \delta b_1, \dots, \delta b_n) = (X_1, \dots, X_n, Y_1, \dots, Y_n) + I(W) \in W(L)$. So there is $b \in V(L)$ such that $\nabla b \in W(L)$; then by existential closedness of (K, δ) there is $a \in V(K)$ such that $\nabla_a \in W(K)$.

TODO 6. *Check*

($\impliedby$) Suppose $\varphi(x)$ with $x = (x_1, \dots, x_n)$ be a conjunction of L_δ-literals with parameters from K with a realization in some extension $(R, \delta) \models T$ of (K, δ). We wish to find a solution to $\varphi(x)$ in K.

Note that if $t = t(x)$ is a term over K then $t(x) \neq 0$ has a solution in (S, δ) if and only if $t(x)y = 1$ has a solution in $(\mathrm{Frac}(S), \delta)$; applied to $S = K$, we get that $t(x) \neq 0$ has a solution in K if and only if $t(x)y = 1$ does. So we may assume that $\varphi(x)$ is a conjunction of atomic L_δ-formulas over K; say

$$\varphi(x) = \bigwedge_{i=1}^{\ell} f_i(x) = 0$$

where $f_i \in K\{X\}$ where $X = (X_1, \ldots, X_n)$.

Let $L = \mathrm{Frac}(R)$; so $(L, \delta) \models T$ extends (K, δ). Let $b \in L$ realize $\varphi(x)$. Let $r = \max\{\, \mathrm{ord}(f_i) : i \in \{1, \ldots, \ell\}\,\}$. (Here $\mathrm{ord}(f_i)$ is the maximum of its orders with respect to any one variable.) Consider $\overline{b} = (b, \delta b, \ldots, \delta^{r-1} b) \in L^{nr}$. Let $V = \mathrm{loc}(\overline{b}/K)$ be the *locus of* $\overline{b}$ *over* K, that is the smallest affine subvariety of $\mathbb{A}^{nr}$ defined over K and containing $\overline{b}$. Phrased differently: $I(V) = \{\, f \in K[X^{(0)}, \ldots, X^{(r-1)}] : f(\overline{b}) = 0 \,\}$.

TODO 7. *Should this ideal be over a bigger ring of polynomials?*

Then $\nabla_{\overline{b}} \in \tau V(L)$; let $W = \mathrm{loc}(\nabla\overline{b}/K) \subseteq \tau V \subseteq \mathbb{A}^{2nr}$. Let $\pi \colon \tau V \to V$ be the projection. Then $\overline{b} = \pi(\overline{b}, \delta\overline{b}) = \pi\nabla\overline{b}$; so $\overline{b} \in \pi(W) \subseteq \overline{\pi(W)}$ (Zariski closure) and thus $V \subseteq \overline{\pi(W)}$. So $V = \overline{\pi(W)}$. By (*) we then get $\overline{a} \in V(K)$ such that $\nabla\overline{a} \in W(K)$. Write $\overline{a} = (a_0, \ldots, a_{r-1})$.

TODO 8. *Intuitively, $\overline{a}$ is an element of K that "looks like" $\overline{b}$ over K*

We will show that a_0 realizes $\varphi(x)$.

Claim 2.7. *$\delta^i a_0 = a_i$ for $i < r$.*

Proof. Well

$$\begin{aligned} \nabla\overline{a} &= (a_0, a_1, \ldots, a_{r-1}, \delta a_0, \delta a_1, \ldots, \delta a_{r-1}) \\ \nabla\overline{b} &= (b, \delta b, \ldots, \delta^{r-1} b, \delta b, \delta^2 b, \ldots, \delta^r b) \end{aligned}$$

We use the $2nr$ coordinates $(X^{(0)}, \ldots, X^{(r-1)}, Y^{(0)}, \ldots, Y^{(r-1)})$. Then $\nabla\overline{b}$ satisfies

$$\begin{aligned} X^{(1)} &= Y^{(0)} \\ X^{(2)} &= Y^{(1)} \\ &\vdots \\ X^{(r-1)} &= Y^{(r-2)} \end{aligned}$$

But these are polynomial equations, and hence are entailed by W. So they hold of $\nabla\overline{a} \in W$. So

$$\begin{aligned} a_1 &= \delta a_0 \\ a_2 &= \delta a_1 \\ &= \delta^2 a_0 \\ &\vdots \\ a_{r-1} &= \delta a_{r-2} \\ &= \delta^{r-1} a_0 \end{aligned}$$

as desired. $\square$ Claim 2.7

So $\overline{a} = (a_0, \delta a_0, \ldots, \delta^{r-1} a_0)$.

Write $f_i(x) = P_i(x, \delta x, \ldots, \delta^{r-1} x)$ in $(r+1)n$ variables where $P_i \in K[X^{(0)}, \ldots, X^{(r-1)}, Y^{(r-1)}]$. For each $i \in \{1, \ldots, \ell\}$ write $0 = f_i(b) = P_i(b, \delta b, \ldots, \delta^r b)$. So $\nabla\overline{b}$ is a root of $P_i(X^{(0)}, \ldots X^{(r-1)}, Y^{(r-1)}$ (using the coordinates in the claim). So this is entailed by W, and is thus true of

$$\nabla\overline{a} = (\underbrace{a_0}_{X^{(0)}}, \underbrace{\delta a_0}_{X^{(1)}}, \ldots, \underbrace{\delta^{r-1} a_0}_{X^{(r-1)}}, \delta a_0, \ldots, \underbrace{\delta^r a_0}_{Y^{(r-1)}}$$

So $f_i(a_0) = P_1(a_0, \delta a_0, \ldots, \delta^{r-1} a_0, \delta^r a_0) = 0$. $\square$ Theorem 2.4

Morally speaking for the right-to-left direction we converted the order-n equation to an order-1 equation, at which point we can apply (*) (since per Remark 2.5 we can view (*) as guaranteeing the existence of solutions to certain order-1 equations).

2.1 Axiomatizability

If (K, δ) is a differential field with K algebraically closed, we can rephrase property (*) as:

(*) For every irreducible Zariski-closed $V \subseteq K^n$ and irreducible Zariski-closed $W \subseteq \tau V \subseteq K^{2n}$ such that $\pi(W) \subseteq V$ is Zariski-dense (where $\pi\colon K^{2n} \to K^n$ is the projection onto the first n coordinates) there is $a \in V$ such that $\nabla_a = (a, \delta a) \in W$.

Note that any such $V \subseteq K^n$ takes the form $V = \mathcal{V}_c = \{\, d \in K^n : (c, d) \in \mathcal{V} \,\}$ where $\mathcal{V} \subseteq K^m \times K^n$ is Zariski-closed over $\mathbb{Z}$ and $c \in K^m$.

Fact 2.8. *Fix $\mathcal{V} \subseteq K^m \times K^n$. Since K is algebraically closed, we get that $\{\, c \in K^m : \mathcal{V}_c$ is irreducible$\,\}$ is L-definable.*

Roughly speaking, one proves this by showing that if there is a bound on the degree of the generators of some ideal I then there is a bound on the smallest degree pair (f, g) with $fg \in I$ but $f, g \notin I$.

Exercise 2.9. If $V = \mathcal{V}_c$ then $\tau(\mathcal{V}_c) = \tau(\mathcal{V}_{\nabla_c})$. (Note that $\tau\mathcal{V} \subseteq \tau(K^m \times K^n) = \tau K^m \times \tau K^n$ (check). So for $c \in K^m$ we get that $\nabla_c = (c, \delta c) \in \tau K^m$; so the assertion is well-typed.)

Similarly $W = \mathcal{W}_c$ for some $\mathcal{W} \subseteq K^m \times K^{2n}$ and $c \in K^m$. Also $\pi\colon K^{2n} \to K^n$ satisfies the following commuting diagram:

$$\begin{tikzcd} \mathcal{W} \subseteq K^m \times K^{2n} \arrow[rr, "\mathrm{id} \times \pi"] \arrow[dr] & & K^m \times K^n \supseteq \mathcal{V} \arrow[dl] \\ & K^m & \end{tikzcd}$$

So if $c \in K^m$ then $\pi\colon K^{2n} \to K^n$

TODO 9. *Something about sending $\mathcal{W}_c$ to $\mathcal{V}_c$?*

Fact 2.10. *If K is algebraically closed then $\{\, c \in K^m : \pi(\mathcal{W}_c) \subseteq \mathcal{V}_c$ is Zariski-dense$\,\}$ is L-definable.*

For each m, n and $\mathcal{V} \subseteq K^m \times K^n$ Zariski-closed and each $\mathcal{W} \subseteq K^m \times K^{2n}$ Zariski-closed, consider the condition

$(*)_{n,m,\mathcal{V},\mathcal{W}}$ For all $c \in K^m$ such that $\mathcal{V}_c$ is irreducible and $\mathcal{W}_c$ is irreducible and $\mathcal{W}_c \subseteq \tau(\mathcal{V}_c)$ (i.e. $\mathcal{W}_c \subseteq (\tau\mathcal{V})_{\nabla_c}$) and $\pi(\mathcal{W}_c) \subseteq \mathcal{V}_c$ is Zariski-dense, there is $a \in \mathcal{V}_c$ such that $\nabla_a \in \mathcal{W}_c$.

Then (*) is equivalent to $\{\, (*)_{n,m,\mathcal{V},\mathcal{W}} \mid n, m, \mathcal{V}, \mathcal{W} \,\}$.

Remark 2.11. As ACF admits quantifier elimination we get that the above axioms are universal-existential.

Corollary 2.12. *The existentially closed models of T form an elementary class. We let* DCF_0 *be the L_δ-theory of existentially closed models of T; we call them* differentially closed fields *of characteristic zero.*

Remark 2.13. $(\mathrm{DCF}_0)_\forall = T$ (the universal theory entailed by DCF_0).

3 Chapter 2: basic model theory of DCF_0

Remark 3.1. From general nonsense it follows that DCF_0 is model complete. (In general if the class of existentially closed models is axiomatizable then the corresponding theory is model complete.)

Some characterizations of model-completeness:

1. Every L_δ is DCF_0-equivalent to an existential L_δ-formula (and hence also a universal L_δ-formula by looking at negations).

2. If $(K,\delta) \subseteq (L,\delta)$ are models of DCF_0 then $(K,\delta) \preceq (L,\delta)$.

3. Every model of DCF_0 is an existentially closed model of DCF_0.

The above hold generally, though they are above stated for DCF_0; see Hodges 8.3.1. The remark follows from the third condition.

We will prove that DCF_0 admits quantifier-elimination.

Proposition 3.2 (11)**.** *DCF_0 has AP (*amalgamation property*) over arbitrary substructures; i.e. if $(K,\delta),(L,\delta) \models \mathrm{DCF}_0$ with a common substructure (R,δ) then there is $(M,\delta) \models \mathrm{DCF}_0$ and L_δ-embeddings $(K,\delta) \hookrightarrow (M,\delta), (L,\delta) \hookrightarrow (M,\delta)$ such that the embeddings agree on (R,δ); i.e. the following diagram commutes:*

$$\begin{tikzcd} & (M,\delta) & \\ (K,\delta) \arrow[ur, hook, dashed] & & (L,\delta) \arrow[ul, hook', dashed] \\ & (R,\delta) \arrow[ul, hook', "\subseteq"] \arrow[ur, hook, "\subseteq"'] & \end{tikzcd}$$

Proof. Since δ on R has a unique extension to $\mathrm{Frac}(R)^{\mathrm{alg}}$ we may assume that (K,δ) and (L,δ) share a common substructure that is an algebraically closed field; i.e. we may assume $(R,\delta) = (F,\delta)$ for $F \models \mathrm{ACF}$.

Since ACF has AP we can embed K and L as fields in some algebraically closed field U.

Claim 3.3. *Taking* $\mathrm{trdeg}(U/F)$ *big enough we may assume that K and L are* algebraically disjoint *over F: i.e. there is a transcendence basis $B \subseteq K$ for K over F then B remains transcendental over L. Equivalently, for all finite $b_1,\dots,b_\ell \in K$ we have* $\mathrm{trdeg}(b_1,\dots,b_\ell/F) = \mathrm{trdeg}(b_1,\dots,b_\ell/L)$.

Proof. Choose $B' \subseteq U$ transcendental over L with $|B'| = |B|$ (here we're using that U is large enough to find such B'). Then $K = F(B)^{\mathrm{alg}} \cong_F \underbrace{F(B')^{\mathrm{alg}}}_{K'}$ is algebraically disjoint from L over F.

We can then pull δ on K to δ' on K'; so $(K,\delta) \cong (K',\delta')$. ☐ Claim 3.3

Fact 3.4. *If K is algebraically disjoint from L over F in U and $F = F^{\mathrm{alg}}$ then K is* linearly disjoint *from L over F: i.e. there is a linear basis for K over F that is linearly independent from L. Equivalently if $b_1,\dots,b_\ell \in K$ then* $\dim_F(\mathrm{span}\{b_1,\dots,b_\ell\}) = \dim_L(\mathrm{span}\{b_1,\dots,b_\ell\})$.

Fact 3.5. *If we have fields*

$$\begin{tikzcd} & U & \\ K \arrow[ur, "\subseteq"] & & L \arrow[ul, "\subseteq"'] \\ & F \arrow[ul, "\subseteq"'] \arrow[ur, "\subseteq"] & \end{tikzcd}$$

with K linearly disjoint from L over F then letting $R = K[L]$ be the subring of U generated by K and L we have that $K \otimes_F L \cong R$ as rings via the map $a \otimes b \mapsto ab$. (In fact the converse holds as well.)

A proof that doesn't use the above facts: fix a transcendence basis B for K over F that is transcendental over L. Say the derivation on K is δ_1 and the derivation on L is δ_2. Define δ on $L(B)$ by $\delta \restriction L = \delta_2$ and $\delta b = \delta_1 b$ for $b \in B$; there is a unique further extension to δ on $L(B)^{\mathrm{alg}}$. Then $(L(B)^{\mathrm{alg}},\delta)$ extends (L,δ_2) by definition. Also $\delta \restriction F = \delta_1 \restriction F$ and $\delta \restriction B = \delta_1 \restriction B$; so $\delta \restriction F[B] = \delta_1 \restriction F[B]$, and thus $\delta \restriction F(B)^{\mathrm{alg}} = \delta_1 \restriction F(B)^{\mathrm{alg}}$. So $(L(B)^{\mathrm{alg}},\delta)$ extends (K,δ_1). We can then further extend to a model of DCF_0.

Alternate proof using the above facts: we can define δ on $K \otimes_F L$ by $\delta(a \otimes b) = a \otimes \delta_2(b) + \delta_1(a) \otimes b$. Then pass to $\mathrm{Frac}(K \otimes_F L)^{\mathrm{alg}}$, then to an existentially closed model. ☐ Proposition 3.2

Corollary 3.6 (12)**.** *If $(K,\delta),(L,\delta) \models \mathrm{DCF}_0$ with a common differential subfield (F,δ) then $(K,\delta) \equiv_F (L,\delta)$. (i.e. $(K,\delta,a)_{a\in F} \equiv (L,\delta,a)_{a\in F}$; i.e. (K,δ) and (L,δ) satisfy the same L_δ-sentences with parameters from F.) In particular, taking $F = (\mathbb{Q},0)$ we get that all models of DCF_0 are elementarily equivalent; i.e. DCF_0 is complete.*

Proof. By AP we can find $(M,\delta) \models \mathrm{DCF}_0$ and embeddings

$$\begin{tikzcd}
& (M,\delta) & \\
(K,\delta) \arrow[ur, hook, dashed, "\subseteq"] & & (L,\delta) \arrow[ul, hook', dashed, "\subseteq"'] \\
& (F,\delta) \arrow[ul, hook', "\subseteq"] \arrow[ur, hook, "\subseteq"'] &
\end{tikzcd}$$

Then by model-completeness we get that $(K,\delta) \preceq (M,\delta)$ and $(L,\delta) \preceq (M,\delta)$. So of σ is an L_δ-sentence over F then

$$(K,\delta) \models \sigma \iff (M,\delta) \models \sigma \iff (L,\delta) \models \sigma$$

as σ is over F and hence over both K and L. □ Corollary 3.6

Corollary 3.7 (13)**.** *DCF_0 admits quantifier elimination; i.e. every L_δ-formula is DCF_0-equivalent to a quantifier-free L_δ-formula.*

Proof. Given $\varphi(x)$ an L_δ-formula with $x = (x_1,\dots,x_n)$, recall that φ is DCF_0-equivalent to a quantifier-free one if and only if whenever $(K,\delta),(L,\delta) \models \mathrm{DCF}_0$ with (F,δ) a common substructure and $a \in F^n$ we must have $(K,\delta) \models \varphi(a) \iff (L,\delta) \models \varphi(a)$.

But $\varphi(a)$ is an L_δ-formula over F; so this follows since $(K,\delta) \equiv_F (L,\delta)$. □ Corollary 3.7

Recall: given a general model of a theory $M \models T$ and $A \subseteq M$ and $b \in M^n$, we define

$$\mathrm{tp}(b/A) = \{\, \varphi(x_1,\dots,x_n) : \varphi \text{ an } L\text{-formula over } A, M \models \varphi(b) \,\}$$

Note that if $M \preceq N$ and $A \subseteq M, b \in M^n$ then $\mathrm{tp}^M(b/A) = \mathrm{tp}^N(b/A)$.

In DCF_0 we will consider the following situation: we have $(K,\delta) \models \mathrm{DCF}_0$, $(F,\delta) \subseteq (K,\delta)$, and $a \in K^n$. We have $\mathrm{tp}(a/F)$ as above *and* $\mathrm{tp}_L(\underbrace{a,\delta a,\delta^2 a,\dots}_{\nabla_\infty a}/F)$; these will be equal by quantifier elimination. (Recall L is the language of rings.) In general

$$\mathrm{tp}(a_0,a_1,\dots/F) = \bigcup_n \mathrm{tp}(a_0,\dots,a_n/F) = \{\, \varphi(x_0,\dots,x_n) \in L : M \models \varphi(a_0,\dots,a_n) \,\}$$

Proposition 3.8 (14)**.** *Suppose $(K,\delta) \models \mathrm{DCF}_0$ and $(F,\delta) \subseteq (K,\delta)$ is a substructure that is a field. Suppose $a,b \in K^n$. Then the following are equivalent:*

1. *$\mathrm{tp}(a/F) = \mathrm{tp}(b/F)$.*

2. *$I_\delta(a/F) = I_\delta(b/F)$ where $I_\delta(a/F) = \{\, P \in F\{X\} : P(a) = 0 \,\}$ (where $X = (X_1,\dots,X_n)$). This is a differential ideal.*

   Exercise 3.9. This is prime. Is it maximal?

3. *$\mathrm{qftp}_L(a,\delta a,\dots/F) = \mathrm{qftp}_L(b,\delta b,\dots/F)$. (Here $\mathrm{qftp}_L(c_0,c_1,\dots/F) = \{\, \varphi(x_0,\dots,x_\ell) : \ell < \omega, \varphi \in L(F)$ quantifier-free$, K \models \varphi(c_0,\dots,c_\ell) \,\}$ and L is as usual the language of rings.)*

4. *There is an isomorphism of differential fields $f\colon F\langle a\rangle \to F\langle b\rangle$ such that $f(a) = b$ and $f \restriction F = \mathrm{id}_F$. (Here $F\{a\}$ is the δ-subring of (K,δ) generated by F and a; this coincides with $\{\, P(a) : P \in F\{X\} \,\} = F[a,\delta a,\dots]$. We then define $F\langle a\rangle = \mathrm{Frac}(F\{a\}) = F(a,\delta a,\dots)$.)*

Proof.

(1) $\implies$ (2) Immediate.

(2) $\implies$ (3) It suffices to show that if $P \in F[x_0, \dots, x_\ell]$ then $P(a, \delta a, \dots, \delta^\ell a) = 0$ if and only if $P(b, \delta b, \dots, \delta^\ell b) = 0$. But we can write $P(a, \delta a, \dots \delta^\ell a) = Q(a)$ for some $Q \in F\{X\}$; we then wish to show that $Q(a) = 0$ if and only if $Q(b) = 0$ for all $Q \in F\{X\}$. But this is just the hypothesis.

(3) $\implies$ (4) As F-algebras we have

$$F[a, \delta a, \dots, \delta^\ell a] \cong F[x_0, \dots, x_\ell]/I(a, \delta a, \dots, \delta^\ell a/F) = F[x_0, \dots, x_\ell]/I(b, \delta b, \dots, \delta^\ell b/F) \cong F[b, \delta b, \dots, \delta^\ell b]$$

(by the hypothesis). So we get an isomorphism of F-algebras $f \colon F[a, \delta a, \dots, \delta^\ell a] \to F[b, \delta b, \dots, \delta^\ell b]$ that fixes F and sends $\delta^i a \mapsto \delta^i b$.

Exercise 3.10. This implies f is in fact an isomorphism of differential rings.

Then extend f to the fraction fields.

(4) $\implies$ (1) Consider the diagram

$$\begin{tikzcd}
(K, \delta) & & (K, \delta) \\
& (F\langle a \rangle, \delta) \arrow[ul, hook', "\subseteq"'] \arrow[ur, hook, "\subseteq"] \arrow[r, "f"] & (F\langle b \rangle, \delta) \arrow[u, hook, "\subseteq"]
\end{tikzcd}$$

Then corollary (12) implies that $(K, \delta) \models \Theta(a) \iff (K, \delta) \models \Theta(\underbrace{fa}_{=b})$ for all $\Theta \in L_\delta(F)$. □ Proposition 3.8

Corollary 3.11 (15)**.** *Suppose $(K, \delta) \models \mathrm{DCF}_0$ and $A \subseteq K$. Then $\mathrm{acl}(A) = \mathbb{Q}\langle A \rangle^{\mathrm{alg}}$, the field-theoretic algebraic closure of the differential field generated by A. (Recall $\mathrm{acl}(A)$ consists of the b such that $(K, \delta) \models \varphi(b)$ for some $\varphi(x) \in L_\delta(A)$ such that $\varphi(x)$ has finitely many realizations in (K, δ).)*

Note by quantifier elimination for ACF_0 that $\mathrm{acl}_L(A) = \mathbb{Q}(A)^{\mathrm{alg}}$.

Example 3.12. $\delta X = 0$ defines $K^\delta = \{\, a \in K : \delta a = 0 \,\}$; in for example $(\mathbb{C}(t), \frac{\mathrm{d}}{\mathrm{d}t})$ this defines $\mathbb{C}$, which is unfortunately infinite. So the argument from ACF_0 won't work here.

Proof of Corollary 3.11.

($\supseteq$) Suppose $b \in \mathbb{Q}\langle A \rangle^{\mathrm{alg}}$; say $P(b) = 0$ for some $P \in \mathbb{Q}\langle A \rangle[x]$. Then $P(x) = 0$ has finitely many solutions over A.

Exercise 3.13. Convert this to an L_δ-formula over A.

($\subseteq$) Let $F = \mathbb{Q}\langle A \rangle^{\mathrm{alg}}$; we wish to show that $\mathrm{acl}(A) \subseteq F$. Suppose for contradiction that $b \in \mathrm{acl}(A) \setminus F$; suppose $\varphi(x)$ is an L_δ-formula over A with finitely many realizations in (K, δ) with $(K, \delta) \models \varphi(b)$. As usual write $\varphi(K) = \{\, b' \in K : (K, \delta) \models \varphi(b') \,\}$. Write $|\varphi(K)| = n < \omega$. In some algebraically closed field $F \subseteq U \supseteq K$ we can find $K_2 \subseteq U$ algebraically disjoint from K over F and an isomorphism $f \colon K_1 := K \to K_2$ of fields preserving F.

TODO 10. *Reference earlier proof?*

We can use f to define δ_2 on K_2 so we have the following picture:

$$\begin{tikzcd}
& U & \\
K \arrow[ur, "\subseteq"] \arrow[rr, "f", "\cong"'] & & K_2 \arrow[ul, "\subseteq"'] \\
& (F, \delta) \arrow[ul, "\subseteq"] \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

(since F is algebraically closed). By the earlier proof

TODO 11. *ref. Of AP?*

we get

$$\begin{tikzcd}
& (M,\delta) & \\
K \arrow[ur, "\preceq"] & & K_2 \arrow[ul, "\preceq"'] \\
& (F,\delta) \arrow[ul, "\subseteq"] \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

But in fact the proof showed $K \cdot K_2 \subseteq M \subseteq U$. But since $(K,\delta) \models \varphi(b)$ we get $(M,\delta) \models \varphi(b)$; following the isomorphism we also separately get $(K_2,\delta_2) \models \varphi(f(b))$, and thus $(M,\delta) \models \varphi(f(b))$. Also $b \neq f(b)$ since $f(b) \in K_2$ is transcendental over F in U and hence transcendental over K in U by algebraic disjointness, whereas $b \in K$.

So in $(M,\delta) \models \mathrm{DCF}_0$ we get that $\varphi(x)$ has at least 2 solutions, namely b and $f(b)$. Iterating we get $(N,\delta) \succeq (K,\delta)$ in which $\varphi(x)$ has $n+1$ distinct solutions; hence $(N,\delta) \models \exists_{\geq n+1} x \varphi(x)$, and so (K,δ) does as well, a contradiction. $\square$ Corollary 3.11

Note that once we do types and saturation the $\subseteq$ direction will follow more easily.

Proposition 3.14 (16)**.** $\mathrm{dcl}(A) = \mathbb{Q}\langle A\rangle$. *(Recall* $\mathrm{dcl}(A)$ *is the set of* $b \in K$ *such that* $\{ b \}$ *is A-definable.)*

Proof.

($\supseteq$) Suppose $b \in \mathbb{Q}\langle A\rangle$. Then $b = \frac{P(a)}{Q(a)}$ for some $P, Q \in \mathbb{Q}\{X_1, \ldots, X_\ell\}$, $\ell < \omega$, and $a \in A^\ell$. Then $Q(a)x = P(a)$ is our desired formula.

($\subseteq$) Suppose $b \in \mathrm{dcl}(A) \setminus \mathbb{Q}\langle A\rangle$. If $b \notin \mathbb{Q}\langle A\rangle$ then by corollary (15) we get $b \notin \mathrm{acl}(A)$, a contradiction. We may thus assume $b \in \mathbb{Q}\langle A\rangle^{\mathrm{alg}} \setminus \mathbb{Q}\langle A\rangle$. So there is $b' \neq b$ in $\mathbb{Q}\langle A\rangle^{\mathrm{alg}}$ a conjugate of b over $\mathbb{Q}\langle A\rangle$ (i.e. b, b' share a minimal polynomial over $\mathbb{Q}\langle A\rangle$).

TODO 12. *I guess this is because irreducible polynomials are separable in characteristic zero?*

Exercise 3.15. Suppose (L,δ) is a differential field with a differential subfield $(F,\delta) \subseteq (L,\delta)$. Suppose we have an intermediate field $F \subseteq K \subseteq L$ with K algebraic over F. Then K is a differential subfield of L; i.e. $\delta(K) \subseteq K$.

Then we have the following picture:

$$\begin{tikzcd}
& (K,\delta) & \\
(\mathbb{Q}\langle A\rangle(b),\delta) \arrow[ur, "\subseteq"] \arrow[rr, "f", "\cong"'] & & (\mathbb{Q}\langle A\rangle(b'),\delta) \\
& (\mathbb{Q}\langle A\rangle,\delta) \arrow[ul, "\subseteq"] \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

where f is the field-theoretic isomorphism given by conjugacy of b, b'. By the exercise the intermediate fields are differential subfields of (K,δ); then by uniqueness of extensions of δ to algebraic extensions of $\mathbb{Q}\langle A\rangle$ we get that f is an isomorphism of differential fields.

Let $\varphi(x)$ be an L_δ-formula over A such that $\varphi(K) = \{ b \}$. By quantifier elimination we may assume $\varphi(x)$ is quantifier-free. Then

$$\begin{aligned}
&(K,\delta) \models \varphi(b) \\
\implies &(\mathbb{Q}\langle A\rangle(b),\delta) \models \varphi(b) \text{ (since } \varphi \text{ is quantifier-free)} \\
\implies &(\mathbb{Q}\langle A\rangle(b'),\delta) \models \varphi(b') \\
\implies &(K,\delta) \models \varphi(b')
\end{aligned}$$

a contradiction since $b \neq b'$. $\square$ Proposition 3.14

4 Interlude on types and saturation

Chapter 7 of the set theory and model theory notes. For this section L will denote an arbitrary first-order language and T an arbitrary L-theory. We use the numbering from the aforementioned notes.

Proposition 4.1 (7.2)**.** *Suppose $\Phi(x)$ is a set of L-formulas in free variables $x = (x_1, \ldots, x_n)$ (without parameters). Then the following are equivalent:*

1. *$\Phi(x)$ is realized in some model of T.*
2. *Every finite subset of $\Phi(x)$ is realized in a model of T.*
3. *There is $M \models T$ in which all finite subsets of $\Phi(x)$ are realized.*

Proof.

(1) $\Longrightarrow$ (2) Immediate.

(1) $\Longrightarrow$ (3) Immediate.

(3) $\Longrightarrow$ (2) Immediate

(2) $\Longrightarrow$ (1) Consider $L' = L \cup \{ c_1, \ldots, c_n \}$ with $c_1, \ldots, c_n$ new constant symbols; let $c = (c_1, \ldots, c_n)$. Consider the L'-theory $T \cup \Phi(c)$. Then by hypothesis every finite subset of this is consistent, so by compactness there is a model of T in which $\Phi(x)$ is realized. □ Proposition 4.1

Definition 4.2. Any such $\Phi(x)$ is called a *(partial) n-type* in T. We say a partial type $\Phi(x)$ is *complete* if for all $\varphi(x) \in L$ either $\varphi(x) \in \Phi(x)$ or $\neg\varphi(x) \in \Phi(x)$. We write $S_n(T)$ for the set of complete n-types of T.

Remark 4.3. If T is a complete theory and $M \models T$ then $\Phi(x)$ is a type in T if and only if every finite subset of T has a realization in M; this is the third condition plus completeness of T.

Example 4.4. Suppose $p(x)$ is an n-type in T. Then $p(x)$ is complete if and only if $p(x)$ is a maximal n-type in T.

Definition 4.5. Suppose M is an L-structure and $A \subseteq M$. By a *(partial or complete) n-type in M over A* we mean a (complete or partial, respectively) n-type in the $L(A)$-theory $\mathrm{Th}(M_A)$. We write $S_n^M(A)$ for the set of complete n-types in M over A.

Remark 4.6. If $p \in S_n^M(A)$ then every finite subset of p is realized in M.

Exercise 4.7. If $A \subseteq M \preceq N$ then $S_n^M(A) = S_n^N(A)$.

Example 4.8. Suppose $A \subseteq M$ and $b \in M^n$. Then the set $\mathrm{tp}(b/A)$ of $L(A)$-formulas true of b in M has $\mathrm{tp}(b/A) \in S_n^M(A)$. These are precisely the elements of $S_n^M(A)$ realized in M.

Proposition 4.9 (7.12)**.** *Suppose $p \in S_n^M(A)$. Then there is $M \preceq N$ and $b \in N^n$ such that $p = \mathrm{tp}(b/A) \in S_n^N(A) = S_n^M(A)$.*

Proof. Consider $\Sigma(x) = \mathrm{Th}(M_M) \cup p(x)$ (with $x = (x_1, \ldots, x_n)$); this is a set of $L(M)$-formulas. One checks that this is a type in $\mathrm{Th}(M_M)$. So there is $N \models \mathrm{Th}(M_M)$ in which $\Sigma(x)$ is realized; then $N \succeq M$. □ Proposition 4.9

In fact if M is infinite and $|M| \geq |L|$ then we can take $|N| = |M|$; we simply apply Löwenheim-Skolem.

Definition 4.10. Suppose κ is an infinite cardinal and M is an L-structure. We say M is *κ-saturated* if for all $A \subseteq M$ with $|A| < \kappa$ we have that every type in $S_n^M(A)$ is realized in M.

Remark 4.11. If M is κ-saturated then $|M| \geq \kappa$; one sees this by considering $\{ x \neq m : m \in M \}$ and extending to an element of $S_1^M(M)$.

Exercise 4.12 ((Done in the notes)). Let $L = \{ 0, 1, +, -, \times \}$ and $M \models \mathrm{ACF}$. Then M is κ-saturated if and only if $\mathrm{trdeg}(M/\mathbb{F}) \geq \kappa$ where

$$\mathbb{F} = \begin{cases} \mathbb{Q} & \text{if } \mathrm{char}(M) = 0 \\ \mathbb{F}_p & \text{if } \mathrm{char}(M) = p \end{cases}$$

is the prime subfield of $\mathbb{F}$.

Proposition 4.13 (7.26)**.** *M is κ-saturated if and only if every $p(x) \in S_1^M(A)$ is realized for all $A \subseteq M$ with $|A| < \kappa$.*

Proof. One proves by induction on n that $p(x) \in S_n^M(A)$ with $x = (x_1, \ldots, x_n)$ is realized in M. Indeed, $q = p \restriction (x_1, \ldots, x_n)$ is realized by $b \in M^{n-1}$ by the induction hypothesis; we then set $r(x_n) = \{ \varphi(b, x_n) : \varphi(x_1, \ldots, x_n) \in p(x_1, \ldots, x_n) \}$. One then shows that $r(x_n) \in S_1^M(A \cup \{ b_1, \ldots, b_{n-1} \})$, at which point it is realized by hypothesis, say by $b_n \in M$. Then $(b_1, \ldots, b_n)$ is the desired tuple. □ Proposition 4.13

Proposition 4.14 (7.28)**.** *Every infinite L-structure has a κ-saturated elementary extension.*

Proof. Given M we consider an elementary extension $M = M_0 \preceq M_1$ such that every type over M is realized in M_1. Indeed, enumerate $S_1^M(M) = \{ p_\alpha(x) : \alpha < \lambda \}$; then find elementary extensions $M \preceq M^{(1)} \preceq M^{(2)} \preceq \cdots$ such that $M^{(\alpha)}$ realizes p_α by transfinite recursion, and set

$$M_1 = \bigcup_{\alpha < \lambda} M^{(\alpha)}$$

Note that we can take $|M^{(\alpha+1)}| = |M^{(\alpha)}|$. If $\lambda = |S_1^M(M)| \leq |M|$ then $|M_1| = |M|$.

Iterate this κ^+-many times to get $(M_\alpha : \alpha < \kappa^+)$ (taking unions at limit ordinals). So in $M_{\alpha+1}$ every 1-type over M_α is realized. Let

$$N = \bigcup_{\alpha < \kappa^+} M_\alpha \succeq M$$

Then this is κ^+-saturated: if $A \subseteq N$ with $|A| < \kappa^+$ then since κ^+ is a regular cardinal we get that $A \subseteq M_\alpha$ for some $\alpha < \kappa^+$; so types over A are realized in $M_{\alpha+1} \preceq M$. □ Proposition 4.14

Erratum for question 3: we may assume that the P_i generate $I(V)$.

Definition 4.15. We say T is *ω-stable* if whenever $M \models T$ and $A \subseteq M$ we have $|S_1(A)| \leq |A| + \aleph_0$.

TODO 13. *Does this require a countable language to be equivalent to the usual definition?*

Note that in a countable language we always have $|S_1(A)| \leq 2^{|A|+\aleph_0}$.

If T is ω-stable and $M \models T$ is infinite then there is $N \succeq M$ that is *saturated* (i.e. $|N|$-saturated); this follows from the proof of 7.28 and the remarks made therein. Indeed, if $\kappa = |M|$ then we built a κ^+-sized κ^+-saturated model using chains of models $M = M_0 \preceq M_0^{(1)} \preceq \cdots$ of length $|S_1(M)|$. Following the cardinalities through we get our desired saturated model.

Proposition 4.16 (7.37)**.** *Saturation implies* strong homogeneity. *(Definition to follow.)*

Definition 4.17. If M, N are L-structures with $A \subseteq M$ and $f \colon A \to N$ we say f is *elementary* (or a *partial elementary map*) if for all $\varphi(x_1, \ldots, x_n) \in L$ and $a_1, \ldots, a_n \in A$ we have $M \models \varphi(a_1, \ldots, a_n) \iff N \models \varphi(f(a_1), \ldots, f(a_n))$.

Remark 4.18. If $f \colon A \to N$ is elementary and $b \in M, b' \in N$, then f extends to a partial elementary map $A \cup \{ b \} \to N$ with $b \mapsto b'$ if and only if b' realizes $f(\mathrm{tp}(b/A)) = \{ \varphi(x, f(a_1), \ldots, f(a_n)) : \varphi(x, a_1, \ldots, a_n) \in \mathrm{tp}(b/A) \}$.

Definition 4.19. We say M is *strongly homogeneous* if whenever $A \subseteq M$ with $|A| < |M|$ and $f \colon A \to M$ is a partial elementary map, then f extends to an L-automorphism $\widetilde{f} \colon M \to M$.

Proof of Proposition 4.16. Suppose we are given a partial elementary map $f_0 \colon A \to M$; let $\kappa = |M|$. Enumerate $M = \{ b_\alpha : \alpha < \kappa \}$. Define a chain of partial elementary maps with domains having size $< \kappa$ as follows:

- Given f_α define f'_α as follows: if $b_\alpha \in \operatorname{dom}(f_\alpha)$ we set $f'_\alpha = f_\alpha$. Else consider $f_\alpha(\operatorname{tp}(b_\alpha / \operatorname{dom}(f_\alpha)))$; since $|\operatorname{dom}(f_\alpha)| < \kappa$ then by κ-saturation this has a realization $c \in M$. We then extend f_α to f'_α by $b_\alpha \mapsto c$.
- Given f'_α we define $f_{\alpha+1}$ as follows: if $b_\alpha \in \operatorname{Ran}(f'_\alpha)$ then set $f_{\alpha+1} = f'_\alpha$. Else consider $(f'_\alpha)^{-1}(\operatorname{tp}(b_\alpha / \operatorname{Ran}(f'_\alpha)))$; since $|\operatorname{Ran}(f'_\alpha)| < \kappa$ then by κ-saturation this has a realization $c \in M$. We then extend f'_α to $f_{\alpha+1}$ by $c \mapsto b_\alpha$.
- At limit ordinals β we let
$$f_\beta = \bigcup_{\alpha<\beta} f_\alpha$$
This is a partial elementary map, and its domain will have cardinality $\leq \kappa$.

So
$$\widetilde{f} = \bigcup_{\alpha<\kappa} f_\alpha \colon M \to M$$
is an automorphism. □ Proposition 4.16

Corollary 4.20. *Suppose M is saturated, $a, b \in M^n$, and $A \subseteq M$ with $|A| < |M|$. Then $\operatorname{tp}(a/A) = \operatorname{tp}(b/A)$ if and only if there is an automorphism $f \in \operatorname{Aut}_A(M)$ (i.e. $f \restriction A = \operatorname{id}_A$) such that $f(a) = b$.*

Proof. The right-to-left direction is generally true. For the left-to-right, if $\operatorname{tp}(a/A) = \operatorname{tp}(b/A)$ then $g \colon A \cup \{ a \} \to M$ given by $g \restriction A = \operatorname{id}_A$ and $g(a) = b$ is a partial elementary map; then by strong homogeneity this extends to an automorphism. □ Corollary 4.20

Corollary 4.21. *Suppose M is saturated and $D \subseteq M^n$ is a definable set (by which we generally mean with parameters). Suppose $A \subseteq M$ with $|A| < |M|$. Then D is A-definable if and only if $f(D) = D$ for all $f \in \operatorname{Aut}_A(M)$.*

Proof. The left-to-right direction is easy. For the right-to-left suppose $f(D) = D$ for all $f \in \operatorname{Aut}_A(M)$; let D be defined by $\varphi(x, b)$ with $x = (x_1, \ldots, x_n)$, $b \in M^m$, and $\varphi(x, y) \in L$. Let
$$\Phi(x, z) = \{ \psi(x) \leftrightarrow \psi(z) : \psi \in L(A) \} \cup \{ \varphi(x, b), \neg\varphi(z, b) \}$$
(for $z = (z_1, \ldots, z_n)$); then $\Phi \subseteq L(A \cup \{ b_1, \ldots, b_n \})$.

Claim 4.22. *Φ is not a $2n$-type.*

Proof. Otherwise by saturation of M (and since $|A \cup \{ b_1, \ldots, b_n \}| < |M|$) there would be $(a, a') \in M^{2n}$ realizing Φ; i.e. $\operatorname{tp}(a/A) = \operatorname{tp}(a'/A)$ and $a \in D, a' \notin D$. But by previous corollary we get $f \in \operatorname{Aut}_A(M)$ such that $f(a) = a$; so $f(D) \neq D$, contradicting the hypothesis. □ Claim 4.22

So some finite subset of $\Phi(x, z)$ is not realized in M; i.e. there are $\psi_1, \ldots, \psi_\ell \in L(A)$ such that
$$M \models \forall x \forall z \left(\bigwedge_{i=1}^{\ell} (\psi_i(x) \leftrightarrow \psi_i(z)) \to (\varphi(x, b) \leftrightarrow \varphi(z, b)) \right)$$
Now for each $\tau \colon \{ 1, \ldots, \ell \} \to \{ 0, 1 \}$ let
$$\theta_\tau(x) = \bigwedge_{\tau(i)=1} \psi_i(x) \wedge \bigwedge_{\tau(i)=0} \neg\psi_i(x)$$
Then these are finitely many $L(A)$-formulas, and if $\tau \neq \sigma$ then $\theta_\tau(M) \cap \theta_\sigma(M) = \emptyset$. Furthermore by our earlier work we get $\theta_\tau(M) \subseteq D$ or $\theta_\tau(M) \cap D = \emptyset$. Also $\bigcup_\tau \theta_\tau(M) = M^n$. So D is defined by
$$\bigvee_{\substack{\tau \\ \theta_\tau(M) \subseteq D}} \theta_\tau(x)$$
and is thus definable over A. □ Corollary 4.21

Corollary 4.23. *Suppose M is saturated and $A \subseteq M$ with $|A| < |M|$; suppose $b \in M$.*

1. *The following are equivalent:*

   (a) $b \in \mathrm{dcl}(A)$

   (b) $f(b) = b$ *for all* $f \in \mathrm{Aut}_A(M)$

   (c) b *is the only realization of* $\mathrm{tp}(b/A)$ *in* M.

2. *The following are equivalent:*

   (a) $b \in \mathrm{acl}(A)$.

   (b) The orbit of b *under* $\mathrm{Aut}_A(M)$ *is finite.*

   (c) There are finitely many realizations of $\mathrm{tp}(b/A)$ *in* M.

Proof.

1. **(b) $\iff$ (c)** We have seen since M is saturated and $|A| < |M|$ that the set of realizations of $\mathrm{tp}(b/A)$ is the orbit of b under $\mathrm{Aut}_A(M)$.

   (a) $\implies$ (b) Clear.

   (b) $\implies$ (a) $D = \{ b \}$ is definable by $x = b$; by (b) we get $f(D) = D$ for all $f \in \mathrm{Aut}_A(M)$. So by previous corollary we get that D is A-definable; i.e. $b \in \mathrm{dcl}(A)$.

2. **(b) $\iff$ (c)** As above.

   (a) $\implies$ (b) If $b \in \mathrm{acl}(A)$ then $b \in D$ for some finite A-definable D. So if $f \in \mathrm{Aut}_A(M)$ then $f(b) \in D$; so the orbit of b is contained in D, and is thus finite.

   (b) $\implies$ (a) Let D be the orbit of b under $\mathrm{Aut}_A(M)$; then by hypothesis D is finite, and thus definable. But $f(D) = D$ for $f \in \mathrm{Aut}_A(M)$; so by previous corollary we get that D is A-definable, and $b \in \mathrm{acl}(A)$. □ Corollary 4.23

Proposition 4.24. *Saturation implies universality. i.e. If M is saturated and $N \equiv M$ with $|N| \leq |M|$ then there is an elementary embedding $N \hookrightarrow M$.*

Proof. Let $f_0 \colon \emptyset \to M$ be the empty function; since $N \equiv M$ we get that f_0 is a partial elementary map $N \to M$. Enumerate $N = \{ a_\alpha : \alpha < \lambda \}$ with $\lambda = |N| \leq |M|$. Recursively build a chain of partial elementary maps f_α such that

- $|\mathrm{dom}(f_\alpha)| < \lambda$
- $a_\alpha \in \mathrm{dom}(f_{\alpha+1})$.

Given f_α if $a_\alpha \in \mathrm{dom}(f_\alpha)$ we set $f_{\alpha+1} = f_\alpha$; else use saturation of M to find b realizing $f_\alpha(\mathrm{tp}(a_\alpha/\mathrm{dom}(f_\alpha)))$, and set $f_{\alpha+1} = f_\alpha \cup \{ (a_\alpha, b) \}$. At limits we take unions.

So we get a partial elementary map $N \to M$ whose domain is all of N; this is then an elementary embedding. □ Proposition 4.24

Definition 4.25. Suppose κ is an infinite cardinal. We say T is *κ-stable* if for all $M \models T$ and all $A \subseteq M$ with $|A| \leq \kappa$ we have $|S_n(A)| \leq \kappa$ for all $n < \omega$.

Fact 4.26. *It suffices to consider $n = 1$.*

Fact 4.27. *If T is countable and T is ω-stable then T is κ-stable for all κ.*

(See Tent and Ziegler for proofs of these.)

5 Back to model theory of DCF_0

Theorem 5.1 (17)**.** DCF_0 *is ω-stable.*

Proof. Fix a model $(K, \delta) \models \mathrm{DCF}_0$ and countable $A \subseteq K$; we wish to show that $S_1(A)$ is countable. Let $F = \mathrm{dcl}(A) = \mathbb{Q}\langle A\rangle$; then every type in $S_1(A)$ extends uniquely to a type in $S_1(F)$. Also since A is countable so too is F, and F is a differential subfield of (K, δ). By replacing (K, δ) with an $\aleph_1$-saturated elementary extension we may assume all types in $S_1(A)$ are realized in (K, δ).

Consider $b \in K$ *differentially transcendental over* F: that is, $(b, \delta b, \delta^2 b, \ldots)$ is algebraically independent over F. Then $\mathrm{tp}_L(b, \delta b, \ldots/F)$ in variables $x_0, x_1, \ldots$ is completely determined: it is entailed by $\{ P(x_0, \ldots, x_\ell) \neq 0 : 0 \neq P \in F[x_0, \ldots, x_\ell], \ell < \omega \}$. Furthermore by quantifier elimination for DCF_0 this determines $\mathrm{tp}(b/F)$. So there is a unique type of a δ-transcendental element over F.

It remains to count $\mathrm{tp}(b/F)$ where b is differentially algebraic over F (i.e. not differentially transcendental over F). Fix such $b \in K$. Let $\ell < \omega$ be least such that $\delta^\ell \in F(b, \delta b, \ldots, \delta^{\ell-1} b)^{\mathrm{alg}}$; let $P(t) \in F(b, \delta b, \ldots, \delta^{\ell-1})[t]$ be the minimal polynomial of $\delta^\ell b$. Then $0 = P(\delta^\ell b)$; differentiating we find

$$0 = \delta(P(\delta^\ell b)) = P'(\delta^\ell b)\delta^{\ell+1} b + P^\delta(\delta^\ell b)$$

So

$$\delta^{\ell+1} b = \frac{-P^\delta(\delta^\ell b)}{P'(\delta^\ell b)} \in F(b, \delta b, \ldots, \delta^\ell b)$$

Iterating we get that $\delta^{\ell+r} b \in F(b, \delta b, \ldots, \delta^\ell b)$ for all $r > 0$.

We have thus shown:

Claim 5.2. $F\langle b\rangle = F(b, \delta b, \ldots, \delta^\ell b)$.

Claim 5.3. *Write $P(t) = Q(b, \delta b, \ldots, \delta^{\ell-1} b, t)$ with $Q \in F(X_0, \ldots, X_{\ell-1})[t]$. Suppose $c \in K$ is such that $(c, \delta c, \ldots, \delta^{\ell-1} c)$ is algebraically independent over F but $Q(c, \delta c, \ldots, \delta^{\ell-1} c, \delta^\ell c) = 0$. Then $\mathrm{tp}(c/F) = \mathrm{tp}(b/F)$.*

Proof. The map $\alpha \colon F(b, \delta b, \ldots, \delta^{\ell-1} b) \to F(c, \delta c, \ldots, \delta^{\ell-1} c)$ given by fixing F and sending $\delta^i b \mapsto \delta^i c$ is an isomorphism of fields by algebraic independence over F. We then get a map $F(b, \ldots, \delta^{\ell-1} b)[t] \to F(c, \delta c, \ldots, \delta^{\ell-1} c)$ which we denote $f \mapsto f^\alpha$. Then

$$P^\alpha(\delta^\ell c) = Q(\alpha(b), \alpha(\delta b), \ldots, \alpha(\delta^{\ell-1} b), \delta^\ell c) = Q(c, \delta c, \ldots, \delta^{\ell-1}, \delta^\ell c) = 0$$

But P^α is monic and irreducible; so it is the minimal polynomial of $\delta^\ell c$ over $F(c, \delta c, \ldots, \delta^{\ell-1} c)$. Hence we can extend α to an isomorphism $F(b, \delta b, \ldots, \delta^\ell b) \to F(c, \delta c, \ldots, \delta^\ell c)$ sending $\delta^\ell b \mapsto \delta^\ell c$. This is then by previous claim an isomorphism $F\langle b\rangle \to F\langle c\rangle$.

But these are isomorphisms of fields; what about δ? Well $\alpha\delta \upharpoonright F = \delta = \delta\alpha \upharpoonright F$, and for $i \leq \ell - 1$ we have

$$\alpha\delta(\delta^i b) = \alpha\delta^{i+1} b = \delta^{i+1} c = \delta(\delta^i c) = \delta(\alpha(\delta^i b))$$

Transport δ on $F\langle b\rangle$ to δ' on $F\langle c\rangle$ using α; then δ and δ' agree on $F, b, \delta b, \ldots, \delta^{\ell-1}$, and hence as maps $F(b, \delta b, \ldots, \delta^{\ell-1} b)$. By corollary (4) since $\delta^\ell b$ is algebraic over $F(b, \ldots, \delta^{\ell-1} b)$, we get that $\delta = \delta'$. So α is an isomorphism of differential fields $(F\langle b\rangle, \delta) \to (F\langle c\rangle, \delta)$ that sends $b \mapsto c$; so $\mathrm{tp}(c/F) = \mathrm{tp}(b/F)$. $\square$ Claim 5.3

So if b is differentially algebraic then $\mathrm{tp}(b/F)$ is determined by (ℓ, Q) with $\ell < \omega$ and $Q \in F(X_0, \ldots, X_{\ell-1})[t]$. So $|S_1(F)| \leq \aleph_0$. $\square$ Theorem 5.1

Corollary 5.4. DCF_0 *has saturated models. If $(K, \delta) \models \mathrm{DCF}_0$ and $\kappa \geq |K|$ is an infinite regular cardinal then (K, δ) has a saturated elementary extension of size κ.*

Proof. Do the construction producing a κ-saturated elementary extension of (K, δ), and use κ-stability to make sure the extension is of size κ. (Here one uses that ω-stability implies κ-stable for all κ.) $\square$ Corollary 5.4

Our convention: we fix a sufficiently large κ for our purposes, and fix $(\mathcal{U}, \delta) \models \mathrm{DCF}_0$ saturated with $|\mathcal{U}| = \kappa$. We say $(\mathcal{U}, \delta)$ is *sufficiently saturated.* More conventions:

- A *model* is an elementary substructure of $\mathcal{U}$.
- A *parameter set* is some $A \subseteq \mathcal{U}$ with $|A| < |\mathcal{U}|$ unless stated otherwise.
- A *type* is a type in $(\mathcal{U}, \delta)$ over some $A \subseteq \mathcal{U}$ with $|A| < |\mathcal{U}|$.
- A *global type* is an element of $S_n(\mathcal{U})$.

Definition 5.5. Suppose A, B, C are subsets of $\mathcal{U}$. We say A is *independent of B over C*, denoted $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if $\operatorname{acl}(C \cup A) = \mathbb{Q}\langle C, A\rangle^{\mathrm{alg}}$ is algebraically disjoint from $\operatorname{acl}(C \cup B) = \mathbb{Q}\langle C, B\rangle^{\mathrm{alg}}$ over $\operatorname{acl}(C) = \mathbb{Q}\langle C\rangle^{\mathrm{alg}}$. (Recall this means that if a is a finite tuple from $\operatorname{acl}(CA)$ then $\operatorname{trdeg}(a/\operatorname{acl}(C)) = \operatorname{trdeg}(a/\operatorname{acl}(CB))$. Equivalently for some (equivalently any) transcendence basis A_0 for $\operatorname{acl}(CA)$ over $\operatorname{acl}(C)$ we have that A_0 remains algebraically independent over $\operatorname{acl}(CB)$.)

Notation 5.6. If a is a tuple with $a = (a_1, \ldots, a_n)$ then we say $a \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if $\{ a_1, \ldots, a_n \} \mathop{\smile\!\!\!\!\!\!\mid}_C B$.

Definition 5.7. If $C \subseteq B$, $p(x) \in S_n(C)$, and $p(x) \subseteq q(x) \in S_n(B)$, we say q is a *free* extension of p if for some (equivalently any) $a \models q$ we have $a \mathop{\smile\!\!\!\!\!\!\mid}_C B$.

Proposition 5.8 (18)**.**

(Symmetry) *$A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ implies $B \mathop{\smile\!\!\!\!\!\!\mid}_C A$.*

(Transitivity) *Suppose $A \subseteq B \subseteq C$ and a is a tuple. Then $a \mathop{\smile\!\!\!\!\!\!\mid}_A C$ if and only if $a \mathop{\smile\!\!\!\!\!\!\mid}_B C$ and $a \mathop{\smile\!\!\!\!\!\!\mid}_A B$.*

(Invariance) *If $\alpha \in \operatorname{Aut}(\mathcal{U})$ then $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ implies $\alpha(A) \mathop{\smile\!\!\!\!\!\!\mid}_{\alpha(C)} \alpha(B)$.*

(Finite character) *$A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ if and only if $A_0 \mathop{\smile\!\!\!\!\!\!\mid}_C B_0$ for all finite $A_0 \subseteq A$ and $B_0 \subseteq B$.*

(Non-triviality) *$A \mathop{\smile\!\!\!\!\!\!\mid}_C A$ if and only if $A \subseteq \operatorname{acl}(C)$.*

(Superstability) *If a is a finite tuple and B a set then $A \mathop{\smile\!\!\!\!\!\!\mid}_{B_0} B$ for some finite $B_0 \subseteq B$.*

(Extension) *Suppose $C \subseteq B$. Then every $p(x) \in S_n(C)$ has a free extension to $S_n(B)$.*

(Stationarity) *Suppose $C \subseteq B$ and $C = \operatorname{acl}(C) = F$. Then every $p \in S_n(C)$ has a unique free extension to $S_n(C)$.*

Proof.

(Symmetry) Let $F = \mathbb{Q}\langle C\rangle^{\mathrm{alg}} = \operatorname{acl}(C)$; let A_0 be a transcendence basis for $\operatorname{acl}(AC)/F$ and B_0 a transcendence basis for $\operatorname{acl}(BC)/F$. Then since $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ we get that A_0 is algebraically independent over $\operatorname{acl}(BC) \supseteq F(B_0)$; then since B_0 is algebraically independent over F we get that $A_0 \cup B_0$ is algebraically independent over F. So B_0 is algebraically independent over $F(A_0)$, and thus over $F(A_0)^{\mathrm{alg}} = \operatorname{acl}(AC)$; so $B \mathop{\smile\!\!\!\!\!\!\mid}_C A$.

(Transitivity) ($\implies$) Suppose $a \mathop{\smile\!\!\!\!\!\!\mid}_A C$, and let A_0 be a transcendence basis for $\operatorname{acl}(Aa) = \operatorname{acl}(A \cup \{ a_1, \ldots, a_n \})$ (where $a = (a_1, \ldots, a_n)$). Then since $a \mathop{\smile\!\!\!\!\!\!\mid}_A C$ we get that A_0 is algebraically independent over $\mathbb{Q}\langle C\rangle^{\mathrm{alg}}$, and hence $a \mathop{\smile\!\!\!\!\!\!\mid}_A B$. We defer the proof that $a \mathop{\smile\!\!\!\!\!\!\mid}_B C$

TODO 14. *And I guess the converse?*

(Invariance) α preserves acl; so $\alpha(\operatorname{acl}(X)) = \operatorname{acl}(\alpha(X))$. Also α is a field automorphism of $\mathcal{U} \models \mathrm{ACF}_0$; so K algebraically disjoint from L over F implies $\alpha(K)$ is algebraically disjoint from $\alpha(L)$ over $\alpha(F)$.

(Finite character) ($\implies$) By symmetry and transitivity we get

$$A \mathop{\smile\!\!\!\!\!\!\mid}_C B \implies A_0 \mathop{\smile\!\!\!\!\!\!\mid}_C B \implies B \mathop{\smile\!\!\!\!\!\!\mid}_C A_0 \implies B_0 \mathop{\smile\!\!\!\!\!\!\mid}_C A_0 \implies A_0 \mathop{\smile\!\!\!\!\!\!\mid}_C B_0$$

($\impliedby$) If $A \not\mathop{\smile\!\!\!\!\!\!\mid}_C B$ then it is witnessed by some algebraic dependence, which can only involve finite subsets of A and B; so $A_0 \not\mathop{\smile\!\!\!\!\!\!\mid}_C B_0$ for some finite $A_0 \subseteq A$ and $B_0 \subseteq B$.

(Non-triviality) We have that $\operatorname{acl}(AC)$ is algebraically disjoint from $\operatorname{acl}(AC)$ over $\operatorname{acl}(C)$ if and only if $\operatorname{acl}(AC) = \operatorname{acl}(C)$, which occurs if and only if $A \subseteq \operatorname{acl}(C)$.

(Superstability)

(Step 1) Reduce to a a singleton. If we assume the case where a is a singleton, and want to do say the case $a = (a_1, a_2)$, then there is finite $B_1 \subseteq B$ such that $a_1 \mathop{\smile\!\!\!\!\!\!|}_{B_1} B$ for some finite $B_1 \subseteq B$, and there is $B_2 \subseteq B$ finite such that $a_2 \mathop{\smile\!\!\!\!\!\!|}_{B_2 a_1} Ba_1$. One then shows that $(a_1, a_2) \mathop{\smile\!\!\!\!\!\!|}_{B_1 B_2} B$; one uses transitivity and symmetry, and instead shows that $B \mathop{\smile\!\!\!\!\!\!|}_{B_1 B_2 a_1} a_2$ and $B \mathop{\smile\!\!\!\!\!\!|}_{B_1 B_2} a_1$.

(Step 2) Let $F = \operatorname{acl}(B) = \mathbb{Q}\langle B\rangle^{\mathrm{alg}}$. If a is δ-transcendental over F then $A \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} B$. Indeed, in this case $\{\, \delta^i a : i < \omega \,\}$ is algebraically independent over F (and hence also over $\mathbb{Q}^{\mathrm{alg}}$), and $\operatorname{acl}(a) = \mathbb{Q}\langle a\rangle^{\mathrm{alg}}$; so $\{\, \delta^i a : i < \omega \,\}$ is a transcendence basis for $\operatorname{acl}(a)$ over $\operatorname{acl}(\emptyset)$, and it is independent from $\operatorname{acl}(B) = F$. So $\operatorname{acl}(a)$ is algebraically disjoint from $\operatorname{acl}(B)$ over $\operatorname{acl}(\emptyset)$, and $a \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} B$.

Suppose then that a is δ-algebraic over F; let $\ell < \omega$ be least such that $\delta^\ell a \in F(a, \dots, \delta^{\ell-1} a)^{\mathrm{alg}}$; let $P(t) = Q(a, \dots, \delta^{\ell-1} a, t)$ be the minimal polynomial of $\delta^\ell a$ over $F(a, \dots, \delta^{\ell-1} a)$ (so $Q \in F(X^{(0)}, \dots, X^{(\ell-1)})[t]$). Then Q involves finitely many parameters from $F = \mathbb{Q}\langle B\rangle^{\mathrm{alg}}$; let $B_0 \subseteq B$ be finite such that if $F_0 = \mathbb{Q}\langle B_0\rangle^{\mathrm{alg}}$ then $Q \in F_0(X^{(0)}, \dots, X^{(\ell-1)})[t]$. Then since $0 = P(\delta^\ell a) = Q(a, \dots, \delta^{\ell-1} a, \delta^\ell a)$ we get that $\delta^\ell a \in F_0(a, \dots, \delta^{\ell-1} a)^{\mathrm{alg}}$, and hence $F_0\langle a\rangle^{\mathrm{alg}} = F_0(a, \dots, \delta^{\ell-1} a)^{\mathrm{alg}}$. (Actually $F_0\langle a\rangle = F_0(a, \dots, \delta^\ell a) \subseteq F_0(a, \dots, \delta^{\ell-1} a)^{\mathrm{alg}}$.) So $\{\, \delta^i a : i < \ell \,\}$ is a transcendence basis for $F_0\langle a\rangle^{\mathrm{alg}} = \operatorname{acl}(B_0 a)$ over $F_0 = \operatorname{acl}(B_0)$ which remains independent over $F = \operatorname{acl}(B)$; so $a \mathop{\smile\!\!\!\!\!\!|}_{B_0} B$.

(Extension) Write

$$
\begin{aligned}
p(x) &= \operatorname{tp}(a/C) \\
F &= \operatorname{acl}(C) \\
L &= \operatorname{acl}(B) \\
K &= \operatorname{acl}(Ca) = F\langle a\rangle^{\mathrm{alg}}
\end{aligned}
$$

So $F \subseteq L$. Let K' be a field-isomorphic (over F) copy of K that is algebraically disjoint from L over F.

TODO 15. *Ref to earlier instance of this?*

Let δ' be a derivation on K' such that $(K', \delta') \cong (K, \delta)$ over F. Then we have the diagram

$$
\begin{tikzcd}
 & (K'(L)^{\mathrm{alg}}, \delta') & \\
(K', \delta') \arrow[ur, "\subseteq"] & & (L, \delta) \arrow[ul, "\subseteq"'] \\
 & (F, \delta) \arrow[ul, "\subseteq"] \arrow[ur, "\subseteq"'] &
\end{tikzcd}
$$

We can then extend $(K'(L)^{\mathrm{alg}}, \delta')$ to $(M, \delta') \models \mathrm{DCF}_0$. By universality there is an elementary embedding $\upharpoonright \colon (M, \delta) \hookrightarrow \mathcal{U}$

TODO 16. *Fixing L?*

Then $\widehat{K} := \rho(K')$ is a differential subfield of $\mathcal{U}$. Then $\widehat{K} = F\langle b\rangle^{\mathrm{alg}}$ for some b, and the map $F\langle a\rangle \to F\langle b\rangle$ given by $a \mapsto b$ is an isomorphism of differential fields. So by (14) we get $\operatorname{tp}(a/F) = \operatorname{tp}(b/F)$, and hence that $p(x) = \operatorname{tp}(a/C) = \operatorname{tp}(b/C)$. Since K' is algebraically disjoint from L over F we get $\operatorname{acl}(Cb) = \widehat{K} = \rho(K')$ is algebraically disjoint from $\operatorname{acl}(B) = L = \rho(L)$ over $\operatorname{acl}(C) = F = \rho(F)$. So $b \mathop{\smile\!\!\!\!\!\!|}_{C} B$, and $\operatorname{tp}(b/B) \in S_n(B)$ is a free extension of $\operatorname{tp}(b/C) = p \in S_n(C)$.

(Stationarity) We again reduce to the case $n = 1$; we do this reduction in the case $n = 2$ to illustrate. Suppose we have $(a_1, a_2) \mathop{\smile\!\!\!\!\!\!\mid}_F B$ and $(b_1, b_2) \mathop{\smile\!\!\!\!\!\!\mid}_F B$, and that $\mathrm{tp}(a_1 a_2/F) = \mathrm{tp}(b_1 b_2/F)$; we wish to show that $\mathrm{tp}(a_1 a_2/B) = \mathrm{tp}(b_1 b_2/B)$. By the case of 1-types we get that $\mathrm{tp}(a_1/B) = \mathrm{tp}(b_1/B)$; so by strong homogeneity there is $\sigma \in \mathrm{Aut}_B(\mathcal{U})$ such that $\sigma(a_1) = b_1$. Also by strong homogeneity there is $\tau \in \mathrm{Aut}_F(\mathcal{U})$ such that $\tau(a_i) = b_i$. Then $\sigma \circ \tau^{-1} \in \mathrm{Aut}_{Fb_1}(\mathcal{U})$, so $\mathrm{tp}(b_2/\mathrm{acl}(Fb_1)) = \mathrm{tp}(\sigma(a_2)/\mathrm{acl}(Fb_1))$. By symmetry and transitivity we get $a_2 \mathop{\smile\!\!\!\!\!\!\mid}_{Fa_1} B$; so applying σ we get that $\sigma(a_2) \mathop{\smile\!\!\!\!\!\!\mid}_{Fb_1} B$, and hence $\sigma(a_2) \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{acl}(Fb_1)} B$. Also $b_2 \mathop{\smile\!\!\!\!\!\!\mid}_{Fb_1} B$, so $b_2 \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{acl}(Fb_1)} B$. So by the case of 1-types we get $\mathrm{tp}(\sigma(a_2)/Bb_1) = \mathrm{tp}(b_2/Bb_1)$; let $\rho \in \mathrm{Aut}(Bb_1)$ such that $\rho(\sigma(a_2)) = b_2$. Then $\rho \circ \sigma$ sends a_i to b_i, and ρ, σ both fix B. So $\mathrm{tp}(a_1 a_2/B) = \mathrm{tp}(b_1 b_2/B)$.

We now do the case of 1-types. Suppose $a, b \in \mathcal{U}$ with $a \mathop{\smile\!\!\!\!\!\!\mid}_F B$ and $b \mathop{\smile\!\!\!\!\!\!\mid}_F B$ and $\mathrm{tp}(a/F) = \mathrm{tp}(b/F)$ (and $F = \mathrm{acl}(F)$). We want $\mathrm{tp}(a/L) = \mathrm{tp}(b/L)$ where $L = \mathrm{acl}(B) \supseteq F$. Then if a is δ-transcendental over F then a is δ-transcendental over L (since $a \mathop{\smile\!\!\!\!\!\!\mid}_F L$), and also b is δ-transcendental over L; so $\mathrm{tp}(a/L) = \mathrm{tp}(b/L)$. Assume then that a is δ-algebraic over F; let (ℓ_a, Q_a) determine $\mathrm{tp}(a/F)$ (so $\delta^\ell a \in F(a, \ldots, \delta^{\ell-1}a)^{\mathrm{alg}}$ and $Q_a(a, \ldots, \delta^{\ell-1}a, t)$ is its minimal polynomial, with $Q_a \in F(X^{(0)}, \ldots, X^{(\ell_a-1)})[t]$).

Claim 5.9. *Since $a \mathop{\smile\!\!\!\!\!\!\mid}_F L$, we get that (ℓ_a, Q_a) determine $\mathrm{tp}(a/L)$ in the same way.*

We need that $P(t) \in F(a, \ldots, \delta^{\ell_a-1}a)[t]$ is still irreducible over $L(a, \ldots, \delta^{\ell_a-1}a)[t]$. Since $F \subseteq L$ is an algebraically closed field, we get that $\{ a_1, \ldots, \delta^{\ell_a-1}a \}$ is algebraically independent over L.

TODO 17. *Transition word?*

$F(x)^{\mathrm{alg}} \cap L(x) = F(x)$, so irreducible over $F(x)$ implies irreducible over $L(x)$.

But now the same is true of b: if (ℓ_b, Q_b) determine $\mathrm{tp}(b/F)$ then they also determine $\mathrm{tp}(b/L)$. But $\mathrm{tp}(a/F) = \mathrm{tp}(b/F)$; so $\ell_a = \ell_b$ and $Q_a = Q_b$. So $\mathrm{tp}(b/L) = \mathrm{tp}(a/L)$, as desired. $\square$ Proposition 5.8

Definition 5.10. Suppose $p(x) = \mathrm{tp}(a/B) \in S_n(B)$. We define $\dim(p) = \dim(a/B)$ to be

$$(\mathrm{trdeg}(F(\nabla_\ell a)/F) : \ell < \omega)$$

where $F = \mathrm{acl}(B)$ and $\nabla_\ell x = (x, \delta x, \ldots, \delta^\ell x)$; we put the lexicographic ordering on these.

Remark 5.11.

1. This doesn't depend on $a \models p(x)$: by homogeneity if $a' \models p(x)$ there is $\sigma \in \mathrm{Aut}_B(\mathcal{U})$ such that $\sigma(a) = a'$; so $\mathrm{trdeg}(F(\nabla_\ell a')/F) = \mathrm{trdeg}(F(\nabla_\ell a)/F)$ for all ℓ since $\varphi(F) = F$ and $\sigma(\nabla_\ell a) = \nabla_\ell a'$.

2. This is in some sense the wrong definition: it isn't invariant under definable bijections. Consider for example $B = \emptyset$, so $F = \mathbb{Q}^{\mathrm{alg}}$, and consider $a \in \mathcal{U}$ differentially transcendental over $\mathbb{Q}$. Then $\mathrm{trdeg}(F(\nabla_\ell a)/F) = \ell + 1$; so $\dim(a) = (1, 2, 3, \ldots)$. But $\nabla \colon \mathcal{U} \to \mathcal{U} \times \mathcal{U}$ given by $x \mapsto (x, \delta x)$ is a definable injection, and a and ∇_a are interdefinable (i.e. $a \in \mathrm{dcl}(\nabla a)$ and $\nabla a \in \mathrm{dcl}(a)$). But $\nabla_\ell(\nabla a) = ((a, \delta a), (\delta a, \delta^a), \ldots)$ so $\dim(\nabla_a) = (2, 3, 4, \ldots)$.

3. Even more wrongness: using the Blum axioms and saturation we can find an algebraically independent pair of elements over $\mathbb{Q}$ such that $\delta b = \delta c = 0$; then $\nabla_\ell(b, c) = ((b, c,), (0, 0), \ldots)$, and so $\dim(b, c) = (2, 2, 2, \ldots)$. Then if a is as above we have $\dim(a) < \dim(b, c)$, which is weird: note for example that $\mathrm{trdeg}(F\langle a\rangle/F)$ is infinite, whereas $\mathrm{trdeg}(F\langle b, c\rangle/F) = 2$, so in this sense the first extension is much larger.

We are interested in how dimension behaves with extensions of types, rather than comparing arbitrary pairs of types.

Proposition 5.12 (19)**.** *Suppose $C \subseteq B$ and a is a tuple.*

1. $\dim(a/C) \geq \dim(a/B)$.

2. $\dim(a/C) = \dim(a/B)$ *if and only if* $a \mathop{\smile\!\!\!\!\!\!\mid}_C B$.

Proof.

1. If $F = \operatorname{acl}(C)$ and $K = \operatorname{acl}(B)$ then $F \subseteq K$; so $\operatorname{trdeg}(F(\nabla_\ell a)/F) \geq \operatorname{trdeg}(K(\nabla_\ell a)/K)$.

2. We have $a \downarrow_C B$ if and only if $F\langle a\rangle^{\mathrm{alg}}$ is algebraically disjoint from K over F; this occurs if and only if $\operatorname{trdeg}(F(\nabla_\ell a)/F) = \operatorname{trdeg}(K(\nabla_\ell a)/K)$ for all $\ell < \omega$, i.e. $\dim(a/C) = \dim(a/B)$. □ Proposition 5.12

Definition 5.13. Suppose $p(x) = \operatorname{tp}(a/B) \in S_n(B)$. We say p is *finite-dimensional* if $\dim(p)$ is eventually constant. In this case we write $\dim(p) = d$.

Remark 5.14. In this case $d = \operatorname{trdeg}(F\langle a\rangle/F)$.

5.1 Back to D-varieties

Suppose k is a differential field. Then an affine algebraic variety over k will be some Zariski-closed k-irreducible $V \subseteq \mathcal{U}^n$, and a regular section on V will be some $s\colon \underbrace{V}_{\subseteq \mathcal{U}^n} \to \underbrace{\tau V}_{\subseteq \mathcal{U}^{2n}}$ given by $s(x) = (x, s_1(x), \ldots, s_n(x))$ with $s_i \in k[x]$.

Definition 5.15. A *D-subvariety* of (V, s) over k is a subvariety $W \subseteq V$ such that $s(W) \subseteq \tau W$.

(Note that since $W \subseteq V$ we have $\tau W \subseteq \tau V$; so $s \restriction W \colon W \to \tau V$ always.)

Recall that if $\delta = 0$ on k then $\tau V = TV$ is the tangent bundle. So we get a definition of sub-vector-field as well.

Let

$$\Sigma(x) = \{\, x \in V, \nabla(x) = s(x) \,\} \cup \{\, x \notin W : W \subsetneq V \text{ a proper } D\text{-subvariety over } k \,\}$$

(The second formula is equivalent to $\delta(x_i) = s_i(x)$.)

Claim 5.16. *$\Sigma(x)$ is a type; i.e. every finite subset has a realization.*

Proof. Take proper D-subvarieties $W_1, \ldots, W_\ell$, and let

$$W = \bigcup_{i=1}^{\ell} W_i$$

Then $S(V) \subseteq \tau V$ is a subvariety that projects onto V; so by the geometric axiom there is $a \in V$ such that $\nabla(a) \in s(V)$, and thus $\nabla(a) = s(a)$. We want $a \notin W$. Note that $s(W) \subsetneq s(V)$ is a proper Zariski-closed subset; so $U = s(V) \setminus s(W)$ is non-empty Zariski-open (and proper by k-irreducibility of V) and by remark following the geometric axiom we can take $a \notin W$. □ Claim 5.16

Claim 5.17. *$\Sigma(x)$ determines a unique complete n-type over k.*

Proof. Suppose $a \models \Sigma(x)$; we show that $\operatorname{tp}(a/k)$ is determined. By quantifier elimination it suffices to show that $\operatorname{tp}_L(a, \delta a, \delta^a, \ldots/k)$ is determined. Since $s(a) = \nabla(a)$, if we write $a = (a_1, \ldots, a_n)$ and $s = (\mathrm{id}, s_1, \ldots, s_n)$ then $s_i(a) = \delta(a_i)$. Differentiating we get that $\delta^\ell(a_i)$ is a polynomial for all $\ell < \omega$. So $\operatorname{tp}_L(a, \delta a, \delta^a, \ldots/k)$ is determined by $\operatorname{tp}_L(a/k)$.

By quantifier elimination in ACF we get that $\operatorname{tp}_L(a/k)$ is determined by $\operatorname{loc}(a/k)$ (the smallest Zariski-closed subset of $\mathcal{U}^n$ containing a). (See Aside 5.18.) It then suffices to show that $\operatorname{loc}(a/k) = V$. Let $W = \operatorname{loc}(a/k) \subseteq V$; so W is Zariski closed over k. Is it a D-subvariety? Well $s(a) = \nabla(a) \in \tau_a W$, and "$s(x) \in \tau_x W$" is a Zariski-closed condition over k; i.e. $\{\, b \in W : s(b) \in \tau_b W \,\} \subseteq W$ is Zariski-closed over k. So by definition of W we get $W = \{\, b \in W : s(b) \in \tau_b W \,\}$. So W is a D-subvariety containing a, and thus since $a \models \Sigma(x)$ we get that W is not a proper D-subvariety, and $W = V$, as desired. □ Claim 5.17

Aside 5.18 (Types in ACF are determined by their loci). Let $W = \operatorname{loc}(a/k)$.

1. $k(W) = k(a)$.

Proof. Consider the k-algebra homomorphism $\varphi\colon k[x_1,\dots,x_n] \to k[a]$ given by $x_i \mapsto a_i$. Then $\ker(\varphi) \supseteq I(W)$ since $a \in W$; so $W \supseteq V(\ker(\varphi)) \ni a$, and $V(\ker(\varphi))$ is a subvariety over k, and by definition of locus we get that $W = V(\ker(\varphi))$, and $\ker(\varphi) = I(W)$. So $k[W] = k[x]/I(W) = k[a]$, and $k(W) = k(a)$. ☐

2. Suppose $\mathrm{loc}(b/k) = W$. Then we get k-algebra isomorphisms

$$\begin{aligned} k[a] &\xrightarrow{\cong} k[x]/I(W) \xrightarrow{\cong} k[b] \\ a &\mapsto\ \ x + I(W)\ \ \mapsto b \end{aligned}$$

So $k(a) \cong k(b)$ over k via a map sending $a \mapsto b$; so by quantifier elimination in ACF we get $\mathrm{tp}_L(a/k) = \mathrm{tp}_L(b/k)$.

Definition 5.19. The complete type over k determined by $\Sigma(x)$ is called the *generic type* of (V, s) over k. If $a \models \Sigma(x)$ we say that a is a *generic D-point of* (V, s) *over* k.

Remark 5.20.

1. A generic D-point of (V, s) over k is algebro-gemoetrically a generic point of V over k. i.e. if a is a generic D-point then $\mathrm{loc}(a/k) = V$ (by the proof).

   TODO 18. *ref*

   i.e. the generic D-points coincide with the D-points that are algebraically generic.

2. The Zariski-closure of $(V, s)^\sharp$ is all of V. (The Zariski closure is taken over all varieties, not just those over k.) So "generically" on V our points are D-points.

   Proof. Let W be the Zariski closure of $(V, s)^\sharp$; so $W \subseteq V$. In principle W may not be over k. But if $\sigma \in \mathrm{Aut}_k(\mathcal{U})$ then $\sigma((V, s)^\sharp) = (V, s)^\sharp$ since $\sigma \restriction k = \mathrm{id}$ and σ is an L_δ-automorphism; so $\sigma(W) = W$. So W is L_δ-definable over k.

   Fact 5.21. *Being L-definable with arbitrary parameters and L_δ-definable over k implies being L-definable over k.*

   So W is a subvariety of V over k. But $a \in (V, s)^\sharp \subseteq W$; so $W = V$ by (a). ☐

3. If $b \in (V, s)^\sharp$ then $\mathrm{loc}(b/k)$ is a D-subvariety of V over k.

Proposition 5.22 (20)**.** *The generic type of a D-variety (V, s) over a differential field k is finite-dimensional. In fact it is of dimension* $\dim(V)$ *(say the Krull dimension of the algebraic variety V).*

Proof. Let a be a generic D-point of (V, s) over k. Then

$$\mathrm{trdeg}(k(\nabla_\ell a)/k) = \mathrm{trdeg}(k(a, \delta a, \dots, \delta^\ell a)/k) = \mathrm{trdeg}(k(a)/k) = \dim(V)$$

(since $\nabla(a) = s(a)$ and since $\mathrm{loc}(a/k) = V$ (or alternatively by Aside 5.18)). (One notes that $\dim(V) = \mathrm{trdeg}(k(V)/k)$ in general.) ☐ Proposition 5.22

We now prove part of

TODO 19. *ref*

the previous remark.

Proposition 5.23 (21)**.** *Suppose (V, s) is a D-variety over a differential field k. Then $(V, s)^\sharp$ is Zariski-dense in V.*

Proof. Let W be the Zariski-closure of $(V,s)^\sharp$; so $W \subseteq V \subseteq \mathcal{U}^n$ is Zariski-closed (with arbitrary parameters, not necessarily from k).

Fact 5.24 (Weil)**.** *Every Zariski-closed set has a* minimal field of definition *F; i.e.*

1. *F is a* field of definition *of W; i.e. $I(W) \subseteq \mathcal{U}[x]$ is generated by some $P_1, \dots, P_\ell \in F[x]$.*
2. *For any L-automorphism $\sigma \in \mathrm{Aut}(\mathcal{U})$ if $\sigma(W) = W$ then $\sigma \restriction_F = \mathrm{id}_F$.*

Note that this implies that F is the intersection of all fields of definition of W.

Suppose $\sigma \in \mathrm{Aut}_k(\mathcal{U})$ is an L_δ-automorphism. Then $\sigma(V,s)^\sharp = (V,s)^\sharp$; so $\sigma(W) = W$) and by minimality we get $\sigma \restriction_F = \mathrm{id}$. So by homogeneity we get that $F \subseteq \mathrm{dcl}(k) = k$; so W is over k.

Let a be a generic D-point of $(V,s)^\sharp$. By earlier remark

TODO 20. *ref*

we have $\mathrm{loc}(a/k) = V$ and $a \in (V,s)^\sharp \subseteq W$. So $V \subseteq W$, and $V = W$. □ Proposition 5.23

Theorem 5.25 (22)**.** *Suppose k is a differential field and $p = \mathrm{tp}(a/k) \in S_n(k)$ is finite-dimensional. Then there is a D-variety (V,s) over k and a generic D-point b of (V,s) over k such that $\mathrm{dcl}(ka) = \mathrm{dcl}(kb)$.*

Remark 5.26. The following are equivalent:

1. $\mathrm{dcl}(ka) = \mathrm{dcl}(kb)$.
2. $k\langle a\rangle = k\langle b\rangle$.
3. There are k-definable sets $a \in X \subseteq \mathcal{U}^n$ and $b \in Y \subseteq \mathcal{U}^m$ (where $n = |x|$ and $m = |b|$) and a bijective k-definable map $f\colon X \to Y$ such that $a \mapsto b$.

Remark 5.27. $q = \mathrm{tp}(b/k)$ is the generic type of (V,s) over k.

Proof of Theorem 5.25. Write $a = (a_1, \dots, a_n)$. By finite-dimensionality there is $\ell > 0$ such that $\delta^\ell(a) \in k(a, \delta a, \dots, \delta^{\ell-1}a)^{\mathrm{alg}}$ (since $\mathrm{trdeg}(k\langle a\rangle/k)$ is finite). Now $\delta^{\ell+1}a \in k(a, \delta a, \dots, \delta^\ell a)$, and more generally $k\langle a\rangle = k(a, \dots, \delta^\ell a)$. Let $b = (a, \delta a, \dots, \delta^\ell a) \in \mathcal{U}^{(\ell+1)n}$; let $V = \mathrm{loc}(b/k) \subseteq \mathcal{U}^{(\ell+1)n}$. For $i \in \{1, \dots, n\}$ write

$$\delta^{\ell+1}a_i = \frac{P_i(b)}{Q_i(b)}$$

for $P_i, Q_i \in k[X^{(0)}, \dots, X^{(\ell)}]$ where $X^{(i)}$ is an n-tuple of variables.

A simplifying assumption: assume $Q_i = 1$ for all $i \in \{1, \dots, n\}$. So $\delta^{\ell+1}a_i = P_i(b)$ for all $i \in \{1, \dots, n\}$.

Now define $s\colon V \to V \times \mathcal{U}^{(\ell+1)n}$: if $x = (x^{(0)}, \dots, x^{(\ell)})$ then we send $x \mapsto (x, x^{(1)}, \dots, x^{(\ell)}, P_1(x), \dots, P_n(x))$. So s is a regular polynomial map. Also

$$s(b) = (a, \delta a, \dots, \delta^\ell a, \delta^a, \delta^a, \dots, \delta^\ell a, \underbrace{\delta^{\ell+1}a_1, \dots, \delta^{\ell+1}a_n}_{\delta^{\ell+1}a}) = \nabla(a, \delta a, \dots, \delta^\ell a) = \nabla(b)$$

In particular $s(b) \in \tau_b V$; but this is a Zariski-closed condition and $V = \mathrm{loc}(b/k)$, so $s(x) \in \tau_x V$ for all $x \in V$. So $s\colon V \to \tau V$ is a regular section and (V,s) is a D-variety over k. So $b \in (V,s)^\sharp$ and $\mathrm{loc}(b/k) = V$; so b is a generic D-point of (V,s). Also $k\langle b\rangle = k\langle a\rangle$; so a, b are interdefinable over k.

It remains to do the case where some $Q_i \neq 1$. Write

$$Q = \prod_{i=1}^n Q_i \in k[X^{(0)}, \dots, X^{(\ell)}]$$

(Note we assume $Q_i(b) \neq 0$.)

Claim 5.28.

$$\delta(Q(b) = \frac{P(b)}{Q(b)}$$

for some $P \in k[X^{(0)}, \dots, X^{(\ell)}]$.

Proof. Indeed, we know $\delta(Q(b))$ is a polynomial in $\nabla(b) = (b, \delta b)$, and

$$\delta b = (\delta a, \dots, \delta^\ell a, \underbrace{\frac{P_1(b)}{Q_1(b)}, \dots, \frac{P_n(b)}{Q_n(b)}}_{\delta^{\ell+1}a})$$

with δb only appearing linearly. □ Claim 5.28

Then

$$\delta\left(\frac{1}{Q(b)}\right) = \frac{-P(b)}{Q(b)^3}$$

Let

$$\widetilde{b} = \left(b, \frac{1}{Q(b)}\right)$$

and

$$\widetilde{V} = \mathrm{loc}(\widetilde{b}/k) \subseteq \mathcal{U}^{(\ell+1)n+1}$$

In coordinates (x, y) we have that $\widetilde{V}$ says $x \in V$ and $Q(x)y = 1$. Also $k\langle a \rangle = k\langle \widetilde{b} \rangle$. Then

$$\begin{aligned} \widetilde{s} \colon \widetilde{V} &\to \widetilde{V} \times \mathcal{U}^{(\ell+1)n+1} \\ (x, y) &\mapsto \left(x, y, x^{(1)}, \dots, x^{(\ell)}, y\frac{QP_1}{Q_1}(x), \dots, y\frac{QP_n}{Q_n}(x), -y^3P(x)\right) \end{aligned}$$

(where $x = (x^{(0)}, \dots, x^{(\ell)})$). Then $\widetilde{s}$ is a regular section with $\widetilde{s}(\widetilde{b}) = \nabla(\widetilde{b})$, and as before we're done. □ Theorem 5.25

Essentially what we're doing with $\widetilde{V}$ is writing $V \setminus V(Q)$ as a closed subvariety by adding an extra variable.

5.2 Kolchin topology

Fix a differential field k.

Definition 5.29. Suppose $\Lambda \subseteq k\{X\}$ with $X = (X_1, \dots, X_n)$. We define $V(\Lambda) = \{\, a \in \mathcal{U}^n : P(a) = 0 \text{ for all } P \in \Lambda \,\}$. We call these the *Kolchin-closed* subsets of $\mathcal{U}^n$ over k; these form the closed sets of the *Kolchin topology* on $\mathcal{U}^n$ over k.

Remark 5.30. This refines the Zariski topology on $\mathcal{U}^n$ over k since $k[X] \subseteq k\{X\}$; i.e. Zariski-closed sets are Kolchin closed.

Example 5.31. Consider the case $n = 1$. There are many Kolchin closed sets that are proper and infinite; we have for example that $\delta X = 0$ defines the constants $\mathcal{C}$ in $\mathcal{U}$, and $\mathcal{C} = (\mathcal{U}, 0)^\sharp$ is Zariski-dense in $\mathcal{U}$. Alternatively note by universality that $\left(\mathbb{C}(t), \frac{\mathrm{d}}{\mathrm{d}t}\right) \subseteq (\mathcal{U}, \delta)$, and consider $\delta x = p(x)$ for $p \in k[X]$.

Aside 5.32. In fact $\mathcal{C}$ is an algebraically closed subfield of $\mathcal{U}$.

Definition 5.33. Given $V \subseteq \mathcal{U}^n$ we define $I(V/k) = \{\, f \in k\{X\} : f(a) = 0 \text{ for all } a \in V \,\}$.

Remark 5.34. This is then a radical differential ideal of $k\{X\}$. In for example the case $n = 1$ if $f(X) = P(X, \delta X, \dots, \delta^\ell X) \in I(V/k)$ then

$$\delta f = \sum_{i=0}^{\ell} \frac{\partial P}{\partial X^{(i)}}(X, \dots, \delta^\ell X)\delta^{i+1}X + P^\delta(X, \dots, \delta^\ell X)$$

So $(\delta f)(a) = \delta(f(a))$, and thus $\delta f \in I(V/k)$.

Fact 5.35 (Differential Nullstellensatz)**.** *Suppose $\Lambda \subseteq k\{X\}$. Then $I(V(\Lambda)) = \sqrt{[\Lambda]}$ where $[\Lambda]$ is the differential ideal of $k\{X\}$ generated by Λ.*

Fact 5.36. *Note that this coincides with the smallest radical differential ideal containing Λ; this is simply because if I is a differential ideal then so is $\sqrt{I}$. This last seems hard to see though.*

Fact 5.37 (Ritt-Raudenbush basis theorem)**.** *Every radical differential ideal is finitely generated as a radical differential ideal. (i.e. if I is a radical differential then $I = \sqrt{[\Lambda]}$ for some finite $\Lambda \subseteq k\{X\}$.)*

Note though that not every differential ideal (even if we assume it's prime) is finitely generated as a differential ideal.

Corollary 5.38. *The Kolchin topology on $\mathcal{U}^n$ over k is Noetherian.*

Definition 5.39. V is an *irreducible* Kolchin-closed set over k if whenever $V = W_1 \cup W_2$ with W_i Kolchin-closed over k we must have $V = W_1$ or $V = W_2$.

Corollary 5.40. *Every Kolchin closed set over k can be written uniquely as an irredundant finite union of irreducible Kolchin closed sets over k, called its irreducible components.*

The proof is just topology. Here "irredundant" means that no component is contained in the union of the others, and "unique" means up to reordering.

Theorem 5.41 (23)**.** *There are natural bijective correspondences*

$$\delta\text{-spec}(k\{X\}) \longleftrightarrow S_n(k) \longleftrightarrow \left\{ {\textit{irreducible Kolchin-closed} \atop \textit{subsets of } \mathcal{U}^n \textit{ over } k} \right\}$$

$$I_p \xleftarrow[\Phi]{\quad} p \xmapsto{\quad\Psi\quad} V_p$$

where $I_p = \{ f \in k\{X\} : p(x) \vdash f(x) = 0 \}$ (so Φ is injective by quantifier elimination) and $V_p = V(I_p)$.

Proof.

(Correct codomains) Fix $a \models p(x)$ (so $p(x) = \mathrm{tp}(a/k)$). Then $I_p = I(\{ a \}/k) = I(a/k)$ is a radical differential ideal and is prime.

Claim 5.42. *V_p is the Kolchin-locus of a over k, the smallest Kolchin-closed subset of $\mathcal{U}^n$ over k containing a.*

Proof. Suppose $V(\Lambda) \ni a$ with $\Lambda \subseteq k\{X\}$. Then $\Lambda \subseteq I_p$, and thus $V(\Lambda) \supseteq V(I_p) = V_p$. □ Claim 5.42

So V_p is irreducible. So the claimed codomains of Φ, Ψ are correct.

(Ψ injective) If $V_p = V_q$ then $V(I_p) = V(I_q)$, and thus $I(V(I_p)) = I(V(I_q))$. So by the differential Nullstellensatz we get that $I_p = I_q$, and thus $p = q$ by injectivity of Φ.

(Φ surjective) Suppose $I \subseteq k\{X\}$ is a prime differential ideal. Then $k \subseteq k\{X\}/I \subseteq \mathrm{Frac}(k\{X\}/I) \subseteq K \models \mathrm{DCF}_0$, so by universality we can embed K into $\mathcal{U}$ over k. Let a be the image of $(X_1+I, \ldots, X_n+I) \in K^n$ in $\mathcal{U}$ under this embedding; let $p = \mathrm{tp}(a/k)$. Then for $f \in k\{X\}$ we have

$$f \in I \iff f(x) + I = 0 \text{ in } k\{X\}/I \iff f(X_1 + I, \ldots, X_n + I) = 0 \text{ in } k\{X\}/I \iff f(a) = 0$$

by the embedding. So $I = I_p$.

(Ψ surjective) Suppose $V \subseteq \mathcal{U}^n$ is irreducible Kolchin-closed over k. Let $p(x)$ be the complete type over k determined by $x \in V$ and $x \notin W$ for any proper Kolchin-closed $W \subsetneq V$ over k. (This p is the *generic type* of V over k.) Then irreducibility of V implies p is a type, which by quantifier elimination determines a complete type. One then shows that $V = V(I_p) = V_p$. □ Theorem 5.41

The map from irreducible Kolchin-closed subsets of $\mathcal{U}^n$ over k to $S_n(k)$ sends X to the Kolchin-generic type of V over k: the type that says $x \in X$ and $x \notin Y$ for any proper Kolchin-closed $Y \subsetneq X$ over k.

Remark 5.43. Quantifier elimination implies that every definable set is Kolchin-constructible (i.e. a Boolean combination of Kolchin-closed sets).

Suppose (V, s) is a D-variety over k with $s = (\mathrm{id}, s_1, \ldots, s_n)$. Then $(V, s)^\sharp$ is Kolchin-closed: if $\Lambda = \{ p \in k[X] : p(a) = 0 \text{ for all } a \in V \}$ then $(V, s)^\sharp = V_\delta(\Lambda)$. (So Λ is the algebraic ideal of V; I think from now on we use $I(X)$ and $V(I)$ to denote the algebraic operations and I_δ, V_δ to denote the differential ones.)

Proposition 5.44 (24)**.** *Suppose (V, s) is a D-variety over k and $W \subseteq V$ is a subvariety over k. Then W is a D-subvariety if and only if $W \cap (V, s)^\sharp$ is Zariski dense in W.*

Proof.

($\Longrightarrow$) We have that $(W, s \restriction W)^\sharp = W \cap (V, s)^\sharp$ and we have seen that the former is Zariski-dense in W.

($\Longleftarrow$) We have a Zariski-dense set of points in W such that $s(a) = \nabla(a) \in (\tau W)_a$, and this is a Zariski-closed condition; so $s(x) \in (\tau W)_x$ for all $x \in W$, and W is a D-subvariety. ☐ Proposition 5.44

Corollary 5.45 (25)**.** *$(V, s)^\sharp$ is irreducible over k.*

Proof. Suppose $X \subseteq (V, s)^\sharp \subseteq \mathcal{U}^n$ is Kolchin-closed over k; say $X = V_\delta(\Lambda)$ for $\Lambda vmk\{X\}$. For $P \in \Lambda$ write $P = Q(X, \delta X, \ldots, \delta^\ell X)$; consider $\widetilde{P} \in k[X]$ obtained from P by replacing δX_i by $s_i(X)$ everywhere. Let $\widetilde{\Lambda} = \{ \widetilde{P} : P \in \Lambda \}$. Then $X = V(\widetilde{\Lambda}) \cap (V, s)^\sharp$. Now $\widetilde{X} = V(\widetilde{\Lambda}) \cap V$ is a Zariski-closed subset of V, and $X = \widetilde{X} \cap (V, s)^\sharp$.

What we have shown is that the Kolchin topology on $(V, s)^\sharp$ is the topology induced by the Zariski topology. Suppose now that $(V, s)^\sharp = X \cup Y$ with X, Y Kolchin-closed over k; let $a \in (V, s)^\sharp$ be a generic D-point. Say $a \in X$; then $X = \widetilde{X} \cap (V, s)^\sharp$, so $a \in \widetilde{X}$, and thus $\widetilde{X} = V$ (since $\mathrm{loc}(a/k) = V$). So $X = (V, s)^\sharp$. ☐ Corollary 5.45

Remark 5.46. We remarked in the above proof that the Kolchin topology on $(V, s)^\sharp$ is induced by the Zariski topology; it follows from this that $a \in (V, s^\sharp)$ is D-generic in (V, s) over k if and only if a is Kolchin-generic in $(V, s)^\sharp$ over k.

Indeed, suppose a is a D-generic point of $(V, s)^\sharp$ over k. If $a \in X \subsetneq (V, s)^\sharp$ is Kolchin-closed over k then $X = \widetilde{X} \cap (V, s)^\sharp$ for some Zariski-closed $\widetilde{X}$. But $\mathrm{loc}(a/k) = V$, so $\widetilde{X} = V$, and $X = (V, s)^\sharp$. Conversely suppose $a \in (V, s)^\sharp$ is Kolchin-generic over k. Let $W = \mathrm{loc}(a/k) \subseteq V$; then $a \in W \cap (V, s)^\sharp$, and the latter is a Kolchin-closed subset of $(V, s)^\sharp$ over k. So $V = \overline{(V, s)^\sharp} \subseteq \overline{W}$, and W is Zariski-dense in V, and is thus all of V.

Fact 5.47 (Kolchin irreducibility theorem)**.** *In a D-variety (V, s) we have that V is irreducible as a Kolchin-closed set.*

Remark 5.48. $(V, s)^\sharp \subseteq V \subseteq \mathcal{U}^n$ can be quite far apart. Indeed if $a \in (V, s)^\sharp$ then $\mathrm{trdeg}(k\langle a \rangle / k) \leq \dim(V)$ (since $k\langle a \rangle = k(a)$) in particular is finite, whereas the geometric axioms for DCF show that there is $b \in V$ such that $\nabla(b) \in \tau V$ is generic (taking $W = \tau V$ in the geometric axiom). But $\dim(\tau V) = 2\dim(V)$; so $\mathrm{trdeg}(k\langle b \rangle / k) \geq 2\dim(V)$. In fact for each ℓ we can iterate to find $b \in V$ such that $\nabla_\ell(b)$ has transcendence degree $\ell \dim(V)$ over k. Iterating, we get $b \in V$ such that $\mathrm{trdeg}(k\langle b \rangle / k)$ is infinite.

5.3 Constants

Remark 5.49.

1. We have $\mathcal{C} = \{ a \in \mathcal{U} : \delta a = 0 \} = (\mathcal{U}, 0)^\sharp$. So $\mathcal{C}$ is an irreducible Kolchin closed set over $\mathbb{Q}$ with generic type of dimension 1.

2. $\mathcal{C} \subseteq \mathcal{U}$ is a subfield.

3. $\mathcal{C}$ is algebraically closed. Indeed, if $a \in \mathcal{C}^{\mathrm{alg}}$, say with minimal polynomial $P \in \mathcal{C}[x]$, then $0 = \delta(P(a)) = P'(a)\delta a$; so $\delta a = 0$ and $a \in \mathcal{C}$.

4. The full induced structure on $\mathcal{C}$ is that of a pure field; i.e. if $D \subseteq \mathcal{U}^n$ is definable in $\mathcal{U}$ (with parameters) then $D \cap \mathcal{C}^n$ is a definable set in $(\mathcal{C}, 0, 1, +, -, \times)$.

   Proof. By quantifier elimination it suffices to do this when D is Kolchin-closed over $k \subseteq \mathcal{U}$ a subfield. Note that $\mathcal{C}^n = (\mathcal{U}^n, 0)^\sharp$. Then $X = D \cap \mathcal{C}^n$ is Kolchin-closed in $(\mathcal{U}^n, 0)^\sharp$ implies that $X = \widetilde{X} \cap \mathcal{C}^n$ with $\widetilde{X} \subseteq \mathcal{U}^n$ Zariski-closed over k.

   TODO 21. *By earlier stuff?*

   Say $\widetilde{X} = V(P_1, \ldots, P_\ell)$. So $X = \{\, a \in \mathcal{C}^n : P_1(a) = \cdots = P_\ell(a) = 0 \,\}$. So we're almost done except that $P_1, \ldots, P_\ell \in k[X]$ and $k \not\subseteq \mathcal{C}$.

   How do we get that $k \subseteq \mathcal{C}$?

   Fact 5.50 (Definability of types in stable theories). *Suppose we have a type $\mathfrak{p}(x) \in S_n(\mathcal{C})$ (note that $\mathcal{C}$ is not small; indeed $|\mathcal{C}| = |\mathcal{U}|$). Suppose $\varphi(x, y)$ is a formula over $\emptyset$. Then there is a formula $\psi(y) = d_{\mathfrak{p}} x \varphi(x, y)$ such that for all $b \in \mathcal{C}^n$ we have $\varphi(x, b) \in \mathfrak{p}(x)$ if and only if $\models d_{\mathfrak{p}} x \varphi(x, y)$; we call this the* φ-definition *of* $\mathfrak{p}$.

   Question 5.51. Can this be proven differential-algebraically?

   We now have $D \subseteq \mathcal{U}^n$ definable over k, say defined by $\varphi(a, y)$ with $a \in \mathcal{U}^m$. Let $\mathfrak{p} = \operatorname{tp}(a/\mathcal{C})$. Then by definability of types we get $\varphi(a, y) \wedge (\delta y = 0) \equiv d_{\mathfrak{p}} x \varphi(x, y) \wedge (\delta y = 0))$; but the former defines $D \cap \mathcal{C}^n$, and the latter has parameters in $\mathcal{C}$. So we can take $D \subseteq \mathcal{U}^n$ defined over $\mathcal{C}$ and show that $D \cap \mathcal{C}^n$ is definable in $\mathcal{C}, 0, 1, +, -, \times)$; but we have already done this. ☐

   So algebraic geometry lives on $\mathcal{C}$ in $\mathcal{U}$. Given $V \subseteq \mathcal{U}^n$ Zariski-closed, V should be viewed as an infinite-dimensional Kolchin-closed set; this is *not* algebraic geometry in $\mathcal{C}$. To do algebraic geometry in $\mathcal{U}$: if $V \subseteq \mathcal{U}^n$ is a variety over $k \subseteq \mathcal{C}$ we consider $V(\mathcal{C}) = (V, 0)^\sharp$; this is finite dimensional (of dimension $\dim(V)$).

Example 5.52. Consider the set X defined by $\delta x = 1$; then $\tau\mathcal{U} = TV = \mathcal{U} \times \mathcal{U}$, so $X = (\mathcal{U}, 1)^\sharp$. Then $X \cap \mathcal{C} = \emptyset$, and moreover X is *weakly orthogonal* to $\mathcal{C}$: for any $a \in X$ and $b \in \mathcal{C}^m$ we have $a \mathop{\smile\!\!\!\!\!\mid}\, b$. Indeed if $F = \operatorname{acl}(b) = \mathbb{Q}(b)^{\mathrm{alg}} \subseteq \mathcal{C}$ then $a \notin F$. Then

$$\dim(a/F) = (1, 1, 1, \ldots)$$
$$\dim(a/\mathbb{Q}) = (1, 1, 1, \ldots)$$

So by

TODO 22. *ref*

$a \mathop{\smile\!\!\!\!\!\mid}_{\mathbb{Q}} F$ and $a \mathop{\smile\!\!\!\!\!\mid}\, b$.

But if we fix $a \in X$ we get a map $X \to \mathcal{C}$ given by $b \mapsto b - a$; this is a definable bijection. (Since $\delta(b - a) = \delta b - \delta a = 1 - 1 = 0$.) The point is this bijection is definable over $\{\, a \,\}$.

Definition 5.53. Suppose X is a definable set. We say X is *internal* to $\mathcal{C}$ if there is a definable bijection $f \colon X \to D \subseteq \mathcal{C}^n$. (Note that f is allowed to use parameters not used by X.)

We say X is *almost internal* to $\mathcal{C}$ if there is $D \subseteq \mathcal{C}^n$ definable and a definable finite-to-one surjection $f \colon X \to D$ (over some parameters). (Here finite-to-one means each fibre is finite.)

Question 5.54.

1. What are the dimension 1 almost $\mathcal{C}$-internal sets? We know $\delta x = t$ is, and $\mathcal{C}$ is as well.

2. What do the dimension 1 sets that are not almost $\mathcal{C}$-internal look like?

Up to interdefinability the dimension 1 sets are $(V, s)^\sharp$ with $\dim(V) = 1$; that is, the D-points of D-curves. We restrict our attention to $V = \mathcal{U}$; so equations of the form $\delta(x) = f(x)$ where $f \in k[X]$. Let's allow $f \in k(X)$. We restrict to $k \subseteq \mathcal{C}$.

Fact 5.55 (Kolchin)**.** *Suppose $X \subseteq \mathcal{U}$ is defined by $\delta x = f(x)$ where $f \in k(X)$ and $k \subseteq \mathcal{C}$. Then X is almost $\mathcal{C}$-internal if and only if one of the following holds:*

1. $f = 0$
2. $\frac{1}{f} = g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(g)$ *for some* $g \in k(X)$
3. $\frac{1}{f} = \frac{cg'(x)}{g(x)}$ *for some* $g \in k(X)$ *and* $c \in k$.

Example 5.56. Consider $\delta x = x$; so $f(x) = x$ and

$$\frac{1}{f} = \frac{1}{x} = \frac{\frac{\mathrm{d}}{\mathrm{d}x}x}{x}$$

So it defines an almost $\mathcal{C}$-internal set. We can also see this directly: fix $0 \neq a \in X$ (where X is defined by $\delta x = x$). Consider the map $X \to \mathcal{C}$ given by $x \mapsto \frac{x}{a}$. Then $\delta\left(\frac{x}{a}\right) = \frac{-x\delta(a)}{a^2} + \frac{\delta x}{a} = -\frac{x}{a} + \frac{x}{a} = 0$.

Example 5.57. Consider $f(x) = x^3 - x^2$. Then the fractional decomposition of f is

$$\frac{1}{f} = \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2}$$

So

$$\frac{1}{f} = \frac{\frac{\mathrm{d}\frac{x-1}{x}}{\mathrm{d}x}}{\frac{x-1}{x}} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)$$

One then checks that this doesn't fit into the cases of Kolchin's theorem; so X defined by $\delta x = x^3 - x^2$ is not almost $\mathcal{C}$-internal.

What do the dimension 1 sets that are not almost $\mathcal{C}$-internal look like?

Theorem 5.58 (Hrushovski)**.** *In this case X is ω-categorical; i.e. the theory of the full induced structure on X is. In this case this means that if A is a finite set then $\mathrm{acl}(A) \cap X$ is finite; this is equivalent to saying that in X^2 there are only finitely many Kolchin-closed subsets over k that project cofinitely onto X in both coordinates.*

Ideas: suppose $k \subseteq \mathcal{C}$ and X over k has dimension 1 is not almost $\mathcal{C}$-internal; write $X = (V, s)^\sharp$. Suppose a is a finite tuple. Then $\mathrm{acl}(ka) = k\langle a\rangle^{\mathrm{alg}}$.

TODO 23. *Transition word?*

There are no non-constant differential-rational maps $X \to \mathcal{C}$ over $k\langle a\rangle^{\mathrm{alg}}$. So there are no dominant D-rational morphisms $(V, s) \to (\mathcal{U}, 0)$. For general D-varieties (V, s), if there were such $F\colon (V, s) \to (\mathcal{U}, 0)$ and $c \in k$ then we would have $F^{-1}(c)$ a D-subvariety of (V, s) of codimension 1.

The converse (for vector fields) is a theorem of Joanolou. Hrushovski generalized this to our setting of D-varieties over the algebraic closure of a finitely generated differential field over k.

So there are only finitely many D-points in (V, s) over $k\langle a\rangle^{\mathrm{alg}}$; so $\mathrm{acl}(ka) \cap X$ is finite.

6 Zilber dichotomy

We have discussed a strong dichotomy for 1-dimensional Kolchin-closed sets (in fact in one variable): namely X is almost $\mathcal{C}$-internal ("has much to do with algebraic geometry in $\mathcal{C}$") or ω-categorical ("very little induced structure on X^2").

The Zilber dichotomy generalizes this to n-dimensional Kolchin-closed sets at the cost of weakening "ω-categorical". We work towards a proof of this.

Definition 6.1. We say an infinite definable set X is *strongly minimal* if every definable subset of X is either finite or cofinite. (Here "definable" means with parameters.)

This is intended to generalize curves in ACF.

At this point we give up on numbering theorems.

Proposition 6.2. *Suppose (V, s) is a D-variety over an algebraically closed differential field k. Let $X = (V, s)^\sharp$. Then X is strongly minimal if and only if (V, s) has no proper infinite D-subvarieties over any differential field $L \supseteq k$.*

Proof. First note that (V, s) is a D-variety over L; i.e. V is still irreducible as a Zariski-closed set over L (since k is algebraically closed).

By quantifier elimination X is strongly minimal if and only if for any $L \supseteq k$ and $Y \subseteq X$ Kolchin closed over L we have that Y is finite or all of X. Write $Y = W \cap X$ with W the Zariski closure of Y in V; so W is a D-subvariety of V over L.

TODO 24. *ref*

Then Y is finite if and only if W is, and $Y = X$ if and only if $W = X$. So X is strongly minimal if and only if any such W is all of V or finite. □ Proposition 6.2

Example 6.3. $(V, s)^\sharp$ with $\dim(V) = 1$ satisfies the hypothesis of the proposition since V has no proper infinite subvarieties at all. There are also many strongly minimal $(V, s)^\sharp$ with $\dim(V) > 1$.

(Marker) Consider $(\delta^2 x = \frac{\delta x}{x}) \wedge (\delta x \neq 0)$. Why does this take the desired form? Roughly speaking, we introduce a variable y for δx, and to express that $\delta x \neq 0$ we introduce a variable z for $\frac{1}{xy}$. Formally speaking our variety is $V \subseteq \mathcal{U}^3$ given by $xyz = 1$, and our section is $s(x, y, z) = (x, y, z, y, y^2 z, -(xy^2 z + y^2)z^2)$. One then checks that $s\colon V \to TV$ (our variety is over $\mathbb{Q}^{\mathrm{alg}}$ so this coincides with the prolongation) and that $(V, s)^\sharp = \{ (x, \delta x, \frac{1}{x\delta x}) : x \in X \}$ (which in particular is in 0-definable bijection with X). Also $\dim(V) = 2$ and $(V, s)^\sharp$ is strongly minimal. (This is also ω-categorical.)

(Painlevé I) Consider the equation $\delta^2 x = 6x^2 + t$ over $k = \mathbb{C}(t)^{\mathrm{alg}}$ with $\delta = \frac{\mathrm{d}}{\mathrm{d}t}$. This is (in definable bijection with) $(\mathcal{U}^2, s)^\sharp$ where $s(x, y) = (x, y, y, 6x^2 + t)$. It is strongly minimal: this is due to Kolchin, Nishioka, and Umemura. (This is also ω-categorical.)

(j-function) Satisfies an important order-3 differential equation of the form $\delta^3 x = f(x, \delta x, \delta^2 x)$ where f is rational. This gives rise to some $(V, s)^\sharp$ with $\dim(V) = 3$; this was recently (2000s, 2010s) shown to be strongly minimal by Freitag-Scanlon using work of Pila. (This is not ω-categorical but is *locally modular*, which we'll define soon.)

(Manin Kernels) Use the theory of abelian varieties. (These aren't constructed by explicit equations, though.) These were studied by Buium and Hrushovski in the 90s, 80s. (This is not ω-categorical.)

The Zilber dichotomy in DCF_0: if $X = (V, s)^\sharp$ is strongly minimal then X is almost $\mathcal{C}$-internal or X is locally modular.

6.1 Elimination of imaginaries

Definition 6.4. Suppose X is a definable set. A *canonical parameter* or *code* for X is a tuple c such that for all $\sigma \in \mathrm{Aut}(\mathcal{U})$ we have $\sigma(X) = X$ if and only if $\sigma(c) = c$.

Example 6.5. In ACF_0 with $V \subseteq \mathcal{U}^n$ Zariski-closed, we have that c is a code for V in $(\mathcal{U}, 0, 1, +, -, \times)$ if and only if $\mathbb{Q}(c)$ is the minimal field of definition for V. (Note $\sigma(c) = c$ if and only if $\sigma \restriction \mathbb{Q}(c) = \mathrm{id}$.)

Example 6.6. Every finite set has a code (in any expansion of the theory of fields). Indeed, if $X = \{ a_1, \ldots, a_\ell \}$ then $(x - a_1) \cdots (x - a_\ell) = x^\ell + c_{\ell-1} x^{\ell-1} + \cdots + c_1 x + c_0$ for some $c_0, \ldots, c_{\ell-1}$; then $c = (c_0, \ldots, c_{\ell-1})$ is a code for X.

Lemma 6.7. *c is a code for X if and only if there is $\varphi(x, y) \in L_\delta$ such that $\varphi(x, c)$ defines X and if $d \neq c$ then $\varphi(x, d)$ does not define X.*

Proof.

($\Longleftarrow$) Suppose $\sigma \in \mathrm{Aut}(\mathcal{U})$. Then since $\sigma(X) = X$ we have $\varphi(x, \sigma(c)) \equiv \varphi(x, c)$, and thus $\sigma(c) = c$ by hypothesis.

($\Longrightarrow$) By saturation X is defined over c; say by $\psi(x, c)$ with $\psi \in L_\delta$.

Suppose $d \models p(y)$ where $p(y) = \mathrm{tp}(c)$, and that $\psi(x, d)$ defines X; then there is $\sigma \in \mathrm{Aut}(\mathcal{U})$ such that $\sigma(c) = d$. But then $\sigma(X)$ is defined by $\psi(x, \sigma(c)) = \psi(x, d)$, and is thus X; so by hypothesis $c = \sigma(c) = d$.

Consider now the type $p(y) \cup \{ \forall x(\psi(x, y) \leftrightarrow \psi(x, c)) \}$ over $\{ c \}$; we have just shown that this type entails $y = c$ (by saturation). So by compactness there is $\theta \in p$ such that

$$\mathcal{U} \models \forall y(\theta(y) \wedge (\forall x(\psi(x, y) \leftrightarrow \psi(x, c))) \to y = c)$$

Let $\varphi(x, y) = \psi(x, y) \wedge \theta(y)$. Then this is our desired formula: $\varphi(x, c)$ defines X, and if $\varphi(x, d)$ defines X then $\theta(d)$ holds and $\psi(x, d) \equiv \psi(x, c)$, so by above observation we have $c = d$. $\square$ Lemma 6.7

We write $c = \ulcorner X \urcorner$ to mean that c is a canonical parameter for X.

TODO 25. *Corners?*

Remark 6.8. It follows that for $\sigma \in \mathrm{Aut}(\mathcal{U})$ we have $\sigma(\ulcorner X \urcorner) = \ulcorner \sigma(X) \urcorner$.

Proposition 6.9. *Kolchin-closed sets have codes.*

Proof. We use the Ritt-Raudenbush basis theorem. Suppose $Z \subseteq \mathcal{U}^n$ is Kolchin-closed over k. Recall by Ritt-Raudenbush basis theorem that $I_\delta(Z) = \{ f \in k\{X\} : f(a) = 0 \text{ for all } a \in Z \}$ (with $X = (X_1, \ldots, X_n)$) is radically differentially generated by some $f_1, \ldots, f_\ell \in k\{X\}$. (Since $I_k(Z)$ is a radical differential ideal.) Let $N = \max(\mathrm{ord}(f_1), \ldots, \mathrm{ord}(f_\ell))$; let $J = I_\delta(Z) \cap k[X^{(0)}, X^{(1)}, \ldots, X^{(N)}]$. Then J is a radical ideal in the polynomial ring over k in $(N+1)n$ variables. Then $V = V(J)$ is Zariski closed in $\mathcal{U}^{(N+1)n}$; let F be the minimal field of definition of V. Then for $\sigma \in \mathrm{Aut}(\mathcal{U})$ we have $\sigma \upharpoonright F = \mathrm{id}$ if and only if $\sigma(V) = V$ if and only if $J^\sigma = J$ (where $J^\sigma = \{ f^\sigma : f \in J \}$ and f^σ is f with σ applied to the coefficients). But note that $J^\sigma = J$ if and only if $I_\delta(Z)^\sigma = I_\delta(Z)$, since $I_\delta(Z)$ is radically differentially generated by J. Since $I_\delta(Z)^\sigma = I_\delta(Z)$ if and only if $\sigma(Z) = Z$, if we write $F = \mathbb{Q}(c)$ then c is a code for Z. $\square$ Proposition 6.9

Corollary 6.10. *Every definable set has a code. (i.e.* DCF_0 *has elimination of imaginaries.)*

Proof. By quantifier elimination any definable set takes the form $X = A_1 \setminus B_1 \cup \cdots \cup A_\ell \setminus B_\ell$ with $A_1, \ldots, A_\ell$ (absolutely) irreducible Kolchin closed sets with $B_i \subsetneq A_i$ proper Kolchin-closed subsets with

$$A_i \setminus B_i \not\subseteq \bigcup_{j \neq i} A_j \setminus B_j$$

and this expression is unique up to reordering. Let $a_i = \ulcorner A_i \urcorner$ and $b_i = \ulcorner B_i \urcorner$. Then for $\sigma \in \mathrm{Aut}(\mathcal{U})$ we have $\sigma(A_i \setminus B_i) = A_i \setminus B_i$ implies $\sigma(A_i) = A_i$ since $A_i \setminus B_i$ is Kolchin-dense in A_i. But also $\sigma(A_i \setminus B_i) = \sigma(A_i) \setminus \sigma(B_i)$; so in this case we have $\sigma(a_i) = a_i$ and $\sigma(b_i) = b_i$. So $(a_i, b_i) = \ulcorner A_i \setminus B_i \urcorner$. Then $\ulcorner \{ (a_1, b_1), \ldots, (a_\ell, b_\ell) \} \urcorner = \ulcorner X \urcorner$. (The former exists as finite sets have codes.) This uses that $\sigma(\ulcorner A_i \setminus B_i \urcorner) = \ulcorner \sigma(A_i) \setminus \sigma(B_i) \urcorner$. $\square$ Corollary 6.10

6.2 Minimal types

(These definitions work in any stable theory, though our attention is of course on DCF_0.)

Definition 6.11. Suppose $p \in S_n(A)$. We say $p \in S_n(A)$ is *minimal* if it is non-algebraic and stationary and for all $B \supseteq A$ every non-algebraic extension of p to B is free.

Recall that $p(x)$ is algebraic if it has finitely many realizations in $\mathcal{U}$; equivalently, if $a \models p$ then $a \in \mathrm{acl}(A)$. Recall further that $p \in S_n(A)$ is stationary if for all $B \supseteq A$ there is a unique free extension of p to B. In particular, all types over $\mathrm{acl}(A)$ are stationary.

So for p non-algebraic and stationary, p is minimal if and only if whenever $a \models p$ with $a \not\mathop{\smile}\limits^{\mid}_A B$ then $a \in \mathrm{acl}(B)$. (Note that algebraic extensions of p are certainly not free.)

General note: this coincides with types of U-rank 1.

The following proposition is again specific to DCF_0:

Proposition 6.12. *If p is minimal then* $\dim(p) < \infty$.

We first prove:

Lemma 6.13. *If $p(x,y) = \mathrm{tp}(ab/A)$ is minimal (where $|a| = n$ and $|b| = m$) then $\mathrm{tp}(a/A)$ is minimal or algebraic.*

Proof. Let $r(x) = \mathrm{tp}(a/A)$.

(Stationarity) Suppose $a_1, a_2 \models r$ with $a_i \mathop{\smile\!\!\!\!\!\mid}_A B$ for $i \in \{1,2\}$ and some $B \supseteq A$. Extend a_i to $(a_i, b_i) \models p$: pick $\sigma_i \in \mathrm{Aut}_A(\mathcal{U})$ such that $\sigma_i(a) = a_i$ (since $a_i, a \models r$), and set $b_i = \sigma_i(b)$. By existence of free extensions there is $b_i' \models \mathrm{tp}(b_i/Aa_i)$ with $b_i' \mathop{\smile\!\!\!\!\!\mid}_{Aa_i} B$. Then (a_i, b_i') still realizes p and $(a_i, b_i') \mathop{\smile\!\!\!\!\!\mid}_A B$ by transitivity. Since p is stationary we get that $\mathrm{tp}((a_1, b_1')/B) = \mathrm{tp}((a_2, b_2')/B)$; so $\mathrm{tp}(a_1/B) = \mathrm{tp}(a_2/B)$.

(Algebraic or minimal) Suppose r is non-algebraic; we show that r is minimal. Suppose $B \supseteq A$, $a' \models r = \mathrm{tp}(a/A)$, and $a' \not\mathop{\smile\!\!\!\!\!\mid}_A B$. Let $\sigma \in \mathrm{Aut}_A(\mathcal{U})$ be such that $\sigma(a) = a'$; let $b' = \sigma(b)$. Then $(a', b') \models p(x,y)$, and $(a', b') \not\mathop{\smile\!\!\!\!\!\mid}_A B$; so by minimality of p we get that $(a', b') \in \mathrm{acl}(B)$, and in particular that $a' \in \mathrm{acl}(B)$. $\square$ Lemma 6.13

Proof of Proposition 6.12. Suppose $p = \mathrm{tp}(a_1, \ldots, a_n/k)$ is not finite-dimensional with k a differential field. Let $m \leq n$ be least such that $\mathrm{tp}(a_1, \ldots, a_m/k)$ is not finite-dimensional. Let $L = k\langle a_1, \ldots, a_{m-1}\rangle$. Then $\mathrm{trdeg}(L/k)$ is finite, by minimality of m. We must then have that a_m is differentially transcendental over L; else we would have $\mathrm{trdeg}(L\langle a_m\rangle/L)$ finite, and thus $\mathrm{trdeg}(k\langle a_1, \ldots, a_m\rangle/k) < \infty$, contradicting our assumption that $\mathrm{tp}(a_1, \ldots, a_m/k)$ is not finite-dimensional.

So $a_m \notin L(\delta a_m, \delta^2 a_m, \ldots)^{\mathrm{alg}} = L\langle \delta a_m\rangle^{\mathrm{alg}} = \mathrm{acl}(L\delta a_m)$. So $\mathrm{tp}(a_1, \ldots, a_m/k\delta a_m)$ is non-algebraic. So $(a_1, \ldots, a_m) \not\mathop{\smile\!\!\!\!\!\mid}_k \delta a_m$, since $\delta a_m \in \mathrm{acl}(ka_1, \ldots, a_m) \setminus \mathrm{acl}(k)$. i.e. $\mathrm{tp}(a_1, \ldots, a_m/k\delta a_m)$ is a non-free non-algebraic extension $\mathrm{tp}(a_1, \ldots, a_m/k)$. So $\mathrm{tp}(a_1, \ldots, a_m/k)$ is not minimal non-algebraic. So by previous lemma we get $\mathrm{tp}(a_1, \ldots, a_n/k)$ is not minimal. $\square$ Proposition 6.12

So we can focus on D-varieties to study minimal types.

Proposition 6.14. *Suppose (V,s) is a D-variety over a differential field $k = k^{\mathrm{alg}}$; suppose $X = (V,s)^\sharp$ is strongly minimal. Then the generic type of X over k is minimal.*

Proof. Let p be the generic type of X over k. (Recall that this says $x \in (V,s)^\sharp$ and the Zariski-locus $\mathrm{loc}(a/k) = V$.) Then since X is infinite we get that p is non-algebraic. Since $k = k^{\mathrm{alg}}$ we get that p is stationary. Let $K \supseteq k$ be a differential field and $q \in S_n(K)$ a non-free extension of p. Let $a \models q$; so $a \not\mathop{\smile\!\!\!\!\!\mid}_k K$. So by assignment we get $X = \mathrm{Kloc}(a/k) \neq \mathrm{Kloc}(a/K)$ (Kolchin loci); let $Y = \mathrm{Kloc}(a/K)$. Then $Y \subseteq X$ is a proper Kolchin-closed subset, and is thus finite since X is strongly minimal; so $q = \mathrm{tp}(a/K)$ is algebraic. $\square$ Proposition 6.14

Definition 6.15. We say (V,s) is *minimal* (or $(V,s)^\sharp$ is minimal) if the generic type of $(V,s)^\sharp$ is minimal.

More generally:

Definition 6.16. If X is an irreducible Kolchin-closed set over k we say X is *minimal* if its generic type over k is minimal.

We say (essentially) that strongly minimal implies minimal.

Recall: if $k = k^{\mathrm{alg}}$ and (V,s) is a D-variety over k, then $(V,s)^\sharp$ is strongly minimal if and only if (V,s) has no proper infinite D-subvarieties (over any parameters).

What of minimality? The characterization should be: for any differential field $K \supseteq k$, there does not exist a proper infinite D-subvariety $W \subseteq V$ over K with some $a \in W$ generic in $(V,s)^\sharp$ over k. (One checks that this equivalence follows from the above.)

Terminology note: this notion of minimality appears to be unrelated to the notion in Tent and Ziegler (and in fact came after the definition of strongly minimal).

An exercise (that maybe should have gone in the elimination of imaginaries section):

Exercise 6.17. Suppose X is a Kolchin-closed set over a differential field k. Let $L \supseteq k$ be a differential field extension. Let X_0 be an irreducible component of X over L. Then X_0 is defined over $L \cap k^{\mathrm{alg}}$.

One uses elimination of imaginaries: we see that $\ulcorner X_0 \urcorner \in k^{\mathrm{alg}}$ as it has a finite orbit under $\mathrm{Aut}_k(\mathcal{U})$ (by the decomposition of X into finitely many irreducible components over L).

Corollary 6.18. *If X is irreducible over k and $k = k^{\mathrm{alg}}$ then X is irreducible over any $L \supseteq k$ (i.e. absolutely irreducible).*

(The proof is also an exercise.)

Exercise 6.19. Show that $p \in S_n(k)$ is stationary if and only if for any $a \models p$ the Kolchin-locus $\mathrm{Kloc}(a/k)$ is absolutely irreducible.

We want to adapt the notion of "almost $\mathcal{C}$-internal" to types, rather than definable sets.

Definition 6.20. Suppose $p \in S_n(A)$ is stationary. We say p is *orthogonal* to $\mathcal{C}$ if for any $B \supseteq A$ and any $a \models p$ with $a \mathop{\smile\!\!\!\!\!\!\mid}_A B$ (i.e. $\mathrm{tp}(a/B)$ is a free extension of p) and any finite tuple c from $\mathcal{C}$, we have $a \mathop{\smile\!\!\!\!\!\!\mid}_B c$. (Equivalently, any free extension of p is weakly orthogonal to the constants.) This is denoted $p \perp \mathcal{C}$.

Lemma 6.21. *Suppose p is minimal. Then $p \not\perp \mathcal{C}$ if and only if there is $B \supseteq A$ and $a \models p$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_A B$ and $a \in \mathrm{acl}(Bc)$ for some finite tuple c from $\mathcal{C}$.*

Proof.

($\Longrightarrow$) By hypothesis there is $B \supseteq A$ and c from $\mathcal{C}$ such that $a \models p$, $a \mathop{\smile\!\!\!\!\!\!\mid}_A B$, and $a \not\mathop{\smile\!\!\!\!\!\!\mid}_B c$. So by transitivity we get $a \not\mathop{\smile\!\!\!\!\!\!\mid}_A Bc$. So $\mathrm{tp}(a/Bc)$ is a non-free extension of $\mathrm{tp}(a/A) = p$. So $a \in \mathrm{acl}(Bc)$ by minimality.

($\Longleftarrow$) Given such a, B, c we see that $a \notin \mathrm{acl}(B)$ since $a \mathop{\smile\!\!\!\!\!\!\mid}_A B$ and $a \notin \mathrm{acl}(A)$ since p is non-algebraic. So $a \not\mathop{\smile\!\!\!\!\!\!\mid}_B c$; so $p \not\perp \mathcal{C}$. $\square$ Lemma 6.21

(In general (without assuming minimality), the right-hand-side condition is the correct generalization of "almost $\mathcal{C}$-internal" to types; "$\mathcal{C}$-internal" should be the same thing but with dcl replacing acl. The lemma then says that for p minimal non-orthogonality coincides with almost internality.)

We restrict our attention to minimal types; since we know these are finite-dimensional, we restrict our attention to sets of the form $(V, s)^\sharp$.

TODO 26. *The next proposition is a better version of this.*

Proposition 6.22. *Consider a D-variety (V, s) over $k = k^{\mathrm{alg}}$; let p be the generic type of $(V, s)^\sharp$. Suppose p is minimal. Then $p \not\perp \mathcal{C}$ if and only if there is $K = K^{\mathrm{alg}} \supseteq k$ an extension of differential fields and $f \in K(V) \setminus K$ such that $\delta f = 0$. (i.e. $(K(V))^\delta \supsetneq K^\delta$).*

(Such an f gives a "D-rational" dominant map $(V, s) \to (\mathcal{U}^1, 0)$.)

Proof.

($\Longleftarrow$) Let $a \in V$ be a D-generic point of (V, s) over K; so $\mathrm{Kloc}(a/K) = (V, s)^\sharp$. Also $\mathrm{Kloc}(a/k) = (V, s)^\sharp$ (since a is then generic over k). So by assignment we have $a \mathop{\smile\!\!\!\!\!\!\mid}_k K$; also $a \models p$. Note that $a \not\mathop{\smile\!\!\!\!\!\!\mid}_K f(a)$ since $f \notin K$ implies $f(a) \notin K = \mathrm{acl}(K)$ but $f(a) \in \mathrm{acl}(Ka)$. Also

$$\delta(f(a)) = (\delta f)(a) = 0$$

(since $\nabla_a = s(a)$; one should check this). So $f(a) \in \mathcal{C}$. So $p \not\perp \mathcal{C}$.

($\Longrightarrow$) Suppose $a \models p$, $L \supseteq k$ a differential field, and c from $\mathcal{C}$ satisfy $a \mathop{\smile\!\!\!\!\!\!\mid}_k L$ and $a \not\mathop{\smile\!\!\!\!\!\!\mid}_L c$. Write $c = (c_1, \ldots, c_n) \in \mathcal{C}^n$. Let $m \leq n$ be least such that $a \not\mathop{\smile\!\!\!\!\!\!\mid}_{Lc_1 \cdots c_{m-1}} c_m$ (applying transitivity to $L \subseteq Lc_1 \subseteq Lc_1c_2 \subseteq \cdots \subseteq Lc$). So $c_m \notin \mathrm{acl}(Lc_1 \cdots c_{m-1})$. So c_m is a generic point of $\mathcal{C}$ over $K = \mathrm{acl}(Lc_1 \cdots c_{m-1}) = L(c_1, \ldots, c_{m-1}^{\mathrm{alg}})$. But $\mathcal{C}$ is minimal and $a \not\mathop{\smile\!\!\!\!\!\!\mid}_K c_m$; so $c_m \in \mathrm{acl}(Ka)$.

TODO 27. *Word?*

Since $\mathrm{tp}(c_m/Ka)$ is a non-free extension of $\mathrm{tp}(c_m/K)$, we get that the orbit D of c_m over $\mathrm{Aut}_{Ka}(\mathcal{U})$ is finite. Let $d = \ulcorner D \urcorner$. Then $d \in \mathrm{dcl}(Ka)$ and $c_m \in \mathrm{acl}(d)$; so $d \notin K$. Write $d = (d_1, \ldots, d_\ell) \in \mathcal{C}^\ell$. Then $d \in \mathcal{C}^\ell$ since $D \subseteq \mathcal{C}$ by stable embeddedness.

TODO 28. *?*

Say $d_1 \notin K$. Then $d_1 \in \mathrm{dcl}(Ka) = K\langle a\rangle = K(a) = K(V)$. But $a \notin \mathrm{acl}(K)$; so by minimality we get $a \mathop{\smile\!\!\!\!\!\!\mid}_k K$. So $\mathrm{Kloc}(a/K) = (V, s)^\sharp$; so a is generic in (V, s) over K. So a is Zariski-generic in V over K. So $K(a) = K(V)$. So $d_1 \in K(V) \setminus K$ and $\delta(d_1) = 0$. So we can take $f = d_1$. □ Proposition 6.22

An improvement of the previous theorem:

Proposition 6.23. *Suppose p is stationary over k; so p is the generic type of some Kolchin-closed absolutely irreducible X over k. Then $p \not\perp \mathcal{C}$ if and only if there is a δ-rational dominant map $f\colon X \to \mathcal{C}$ over some $L \supseteq k$. (Here "dominant" means Kolchin-dominant, which coincides with Zariski-dominant because the induced structure on $\mathcal{C}$ is the pure field structure. But this is a one-dimensional set; so dominant just means infinite. So this is equivalent to $f(a) \notin \mathrm{acl}(L)$. Note that f is not assumed to be total on X.)*

Proof.

($\implies$) By non-orthogonality there is $K \supseteq k$ with $a \models p$ and $c = (c_1, \ldots, c_\ell) \in \mathcal{C}^\ell$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_k K$ and $a \not\mathop{\smile\!\!\!\!\!\!\mid}_K c$. Let m be least such that $a \not\mathop{\smile\!\!\!\!\!\!\mid}_{Kc_1\cdots c_{m-1}} c_m$; let $L = Kc_1 \cdots c_{m-1}$. Then since $\mathcal{C}$ is minimal we get that $c_m \in \mathrm{acl}(La) \setminus \mathrm{acl}(L)$. Let D be the orbit of c_m under $\mathrm{Aut}_{La}(\mathcal{U})$; let $d = \ulcorner D \urcorner$. Then $d \in \mathrm{dcl}(La)$, and

TODO 29. *I guess since it's definable over the set of orbits of c_m, which are all constants.*

$d \in \mathcal{C}^N$. Also $d \notin \mathrm{acl}(L)$. So for some coordinate of d, say d_1, we have $d_1 \in \mathrm{dcl}(La) \setminus \mathrm{acl}(L)$ and $d_1 \in \mathcal{C}$. So $d_1 = f(a)$ for some δ-rational function f over L. But now $a \in \{\, a' \in \mathrm{dom}(f) : f(a') \in \mathcal{C} \,\} = \mathrm{dom}(f) \cap Y$ where Y is Kolchin-closed over L. So $a \in Y$, and $X \subseteq Y$. So $f(X \cap \mathrm{dom}(f)) \subseteq \mathcal{C}$; so we have a δ-rational $f\colon X \to \mathcal{C}$, which is dominant since $f(a) = d_1 \notin \mathrm{acl}(L)$.

($\impliedby$) Suppose we have $L \supseteq k$ and $f\colon X \to \mathcal{C}$ dominant over L. By existence of free extensions there is $a \models p$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_k L$; so $f(a) \in \mathcal{C}$. Then since f is dominant we get that $f(a) \notin \mathrm{acl}(L)$; so $f(a) \not\mathop{\smile\!\!\!\!\!\!\mid}_L a$ since $f(a) \in \mathrm{dcl}(La)$. So $a \not\mathop{\smile\!\!\!\!\!\!\mid}_L f(a)$, and $p \not\perp \mathcal{C}$. □ Proposition 6.23

Remark 6.24.

1. If $(X = (V, s)^\sharp$ then f given on the right-hand side extends to $f \in L(V)$ a (non-differential) rational function on V, and we have $\delta f = 0$ (since $(\delta f)(b) = \delta(f(b)) = 0$ for any $b \in X$). This is

   TODO 30. *ref, proposition before last*

2. If p is minimal then the f in the right-hand side is finite-to-one on a Kolchin-dense definable subset. More generally, we have the following:

   Lemma 6.25. *Say we have Kolchin-closed X over k that is minimal. Then for any $L \supseteq k$ and any δ-rational $f\colon X \to Y$ over L to any Kolchin-closed Y, either f is (constant or f is finite-to-one (onto its image)) on a Kolchin-dense definable set of X.*

   (Note that definable $D \subseteq X$ is Kolchin-dense if and only if there is $U \subseteq D \subseteq X$ such that U is non-empty and Kolchin-open.)

   Proof. Take $a \in X$ generic over L; let Z be the absolute Kolchin-closure of $f^{-1}(f(a))$, which is defined over $L(f(a))^{\mathrm{alg}}$. By minimality

   TODO 31. *ref*

   we get that Z is not proper and infinite. If $Z = X$ we get that f is constant; if Z is finite then f is finite-to-one on a Kolchin-dense definable set.

   TODO 32. *Generically?*

□ Lemma 6.25

TODO 33. *Couldn't we just note that any element of any fibre forks over*
uh
the generic type? I guess if it's not a generic element then it's already algebraic over L.

6.3 1-based

Definition 6.26. Suppose $p \in S_n(k)$ is stationary. We say p is 1-*based* if for any $a_1, \dots, a_n$ realizing p and any $d \in \operatorname{dcl}(ka_1, \dots, a_n)$ and $L = L^{\text{alg}} \supseteq k$ we have

TODO 34. *algebraically closed necessary?*

$\ulcorner \operatorname{Kloc}(d/L) \urcorner \in \operatorname{acl}(kd)$. (Equivalently for any $d \in \operatorname{dcl}(p(\mathcal{U}), k)$.)

(In the general case we would replace "Kolchin locus" with "canonical base".) For some intuition: write $X = \operatorname{Kloc}(d/k)$ and $Y = \operatorname{Kloc}(d/L)$. Let $e = \ulcorner Y \urcorner$; then $\operatorname{Kloc}(d/k\langle e\rangle) = \operatorname{Kloc}(d/L) = Y$, so we may assume $L = k\langle e\rangle$. Then 1-basedness is saying that $e \in \operatorname{acl}(kd)$; i.e. there are only finitely many conjugates of Y over kd. The idea is that there are no "rich" families of Kolchin-closed subsets of X passing through the generic point d.

If p is minimal and $d \models p$ this is vacuous. However we still have to check the case for say $d = (a_1, a_2)$ where $a_1, a_2 \models p$. Let $Z = \operatorname{Kloc}(a_1/k) = \operatorname{Kloc}(a_2/k)$ with $a_1 \mathop{\smile\!\!\!\!\!\mid}_k a_2$. Then $X = \operatorname{Kloc}(d/k) = Z^2$. But $Y = \operatorname{Kloc}(d/\langle a_1\rangle) \subsetneq X$ via $x_1 = a_1$. This is not a counterexample to 1-basedness since the conjugates of Y don't go through d.

Example 6.27. Let p be the generic type of $\mathcal{C}$ over k; so p says $\delta x = 0$ and $x \notin k^{\text{alg}}$. Then p is not 1-based. Let $m, b \in \mathcal{C}$ be algebraically independent. Let $Y \subseteq \mathcal{C}^2$ be defined by $y = mx + b$ (and $\delta x = \delta y = 0$). Then let $d \in Y$ be generic over $L := k(mb)$. (So in terms of the above example our $X = \mathcal{C}^2$.)

First note that a_1, a_2 realize p. Indeed we have $\operatorname{trdeg}(k(m, b, a_1, a_2)/k) = 3$; so if for some $i \in \{1, 2\}$ we had $a_i \in k^{\text{alg}}$ then since $a_2 = ma_1 + b$ we would have $\operatorname{trdeg}(k(m, b, a_1, a_2)/k) = 2$, a contradiction. Note as well that $\ulcorner Y \urcorner = (m, b)$. Finally we remark that $(m, b) \notin \operatorname{acl}(kd) = k(a_1, a_2)^{\text{alg}}$ since $\operatorname{trdeg}(k(a_1, a_2, m, b)/k) = 3$.

In general one can show that for a minimal type one only needs to check the case of two realizations.

Theorem 6.28 (The Zilber dichotomoy for DCF_0)**.** *If p is a minimal type then either p is 1-based or $p \not\perp \mathcal{C}$.*

This theorem was done in an unpublished manuscript by Hrushovski-Sokolovich in 1992; it was used to show the Mordell-Lang conjecture for function fields.

Fact 6.29. *For p minimal 1-basedness is equivalent to a separate notion called* local modularity*.*

We will do the proof of Pillay and Pillay-Ziegler from circa 2004. Our proof goes via the canonical base property.

6.4 Internality (again?)

Definition 6.30. If $p \in S(k)$ is stationary we say p is *almost $\mathcal{C}$-internal* if for some $a \models p$ and $L \supseteq k$ with $a \mathop{\smile\!\!\!\!\!\mid}_k L$ we have $a \in \operatorname{acl}(L\mathcal{C})$.

Note the previous definition of almost internality was for strongly minimal definable sets

TODO 35. *really?*

whereas this is for types.

Exercise 6.31. p is almost $\mathcal{C}$-internal if and only if its Kolchin locus X over k admits a generically finite-to-one δ-rational map $f\colon X \to \mathcal{C}^n$ (over possibly additional parameters).

Further recall that $p \not\perp \mathcal{C}$ if and only if there is a dominant δ-rational function $f\colon X \to \mathcal{C}$.

TODO 36. *ref*

Hence if p is non-algebraic and almost $\mathcal{C}$-internal then $p \not\perp \mathcal{C}$. (Just compose the δ-rational map witnessing almost-internality with a projection.)

Remark 6.32. If p is algebraic then p is almost $\mathcal{C}$-internal, but $p \perp \mathcal{C}$.

We have seen

TODO 37. *ref*

that if p is minimal then p is almost $\mathcal{C}$-internal if and only if $p \not\perp \mathcal{C}$.

Theorem 6.33 (Canonical base property for DCF_0)**.** *Suppose $p \in S(k)$ is stationary and finite-dimensional. Suppose $a \models p$ and $L \supseteq k$; let $e = \ulcorner \mathrm{Kloc}(a/L) \urcorner$. Then $\mathrm{tp}(e/\operatorname{acl}(ka))$ is almost $\mathcal{C}$-internal.*

Idea of statement: let $X = \mathrm{Kloc}(a/k)$ and $Y = \mathrm{Kloc}(a/L)$. As before we may assume $L = k\langle e \rangle$ where $e = \ulcorner Y \urcorner$. Then the theorem says that the set of k-conjugates of Y passing through a, while not necessarily finite as in 1-basedness, does admit a finite-to-one map to $\mathcal{C}^n$. (Here we identify the k-conjugates of Y with their codes (which coincide with the k-conjugates of e, I guess).)

Note that $Y = Z_e$ where

$$Z = \mathrm{Kloc}((e,a)/k)$$
$$E = \mathrm{Kloc}(e/k)$$

and we take fibres of the projection $Z \to E$. Note these are all Kolchin-closed over k. So $Z \to E$ is a family of Kolchin-closed subsets of X parametrized by E, and Y is the generic member of this family. Let $E_a = \mathrm{Kloc}(e/\operatorname{acl}(ka)) \subseteq E$. Essentially, E_a is $\{\, e' \in E : a \in Z_{e'} \,\}$. (The $\subseteq$ containment is clear; this will be good enough for our purposes.) Then the theorem says there is a generically finite-to-one δ-rational map $f\colon E_a \to \mathcal{C}^n$; i.e. up to finite noise we can distinguish between the $Z_{e'}$ passing through a by f.

Idea of proof: suppose we have Z_e and $Z_{e'}$ that are different. One way to show they're distinct is to show they have different "tangent spaces at a". (These will be Kolchin-tangent spaces; to be defined later. For the purposes of this idea, denote this $T(Z_e)_a \neq T(Z_{e'})_a$.) Consider the map Φ sending $e \mapsto T(Z_e)_a$; note $T(Z_e)_a$ is a linear subspace of $T(X)_a$. Because our type is finite-dimensional

TODO 38. *Over k, not $\mathcal{C}$? Is that enough?*

these tangent spaces will be finite-dimensional $\mathcal{C}$-vector spaces. So we can view Φ as a map from E_a to the Grassmannian of $T(X)_a$; roughly speaking, this is the set of all $\mathcal{C}$-linear subspaces. Of course since $T(X)_a \cong \mathcal{C}^m$ we can view this as the Grassmannian of $\mathcal{C}^m$, which is a subvariety of $\mathcal{C}^N$.

So if Z_e and $Z_{e'}$ have different tangents at a then our Φ suffices. What if they share a tangent at a? We use the "higher Kolchin tangent spaces". We can similarly get $\Phi^{(n)}(e) = T^{(n)}(Z_e)_a \subseteq T^{(n)}(X)_a$, which we can view as living in the Grassmanian of $T^{(n)}(X)_a$), which can be viewed as living in $\mathcal{C}^M$ for large M.

Because all the Z_e are from the same family Z, there is some n such that if Z_e and $Z_{e'}$ share an n-tangent space then they are equal; so $\Phi^{(n)}$ is injective. (Much as given a family of polynomials there is a bound on their degree; hence if the n^{th} derivatives of two elements of the family agree for n less than the degree bound, then they are equal.)

6.5 The proof of CBP in detail

What are the "higher-order Kolchin tangent spaces"?

Since our types are finite-dimensional we consider $X = (V,s)^\sharp$ for some D-variety (V,s) over k. If $a \in X$ we let $\mathfrak{m}_{V,a} = \{\, f \in \mathcal{U}[V] : f(a) = 0 \,\} \subseteq \mathcal{U}[V] = \mathcal{U}[x]/I(V)$ be the maximal ideal of a in V; so $\mathfrak{m}_{V,a} = (x_1 - a_1, \ldots, x_n - a_n)$.

Claim 6.34. *$\mathfrak{m}_{V,a}$ is a differential ideal of $\mathcal{U}[V]$.*

Proof. If $\alpha \in \mathfrak{m}_{V,a}$ then $\alpha = P(x) + I(V)$ for some $P \in \mathcal{U}[x]$ such that $P(a) = 0$. Then

$$\delta\alpha = \left(\underbrace{\sum_j \frac{\partial P}{\partial x_j}(x) s_j(x) + P^\delta(x)}_{Q(x)} \right) + I(V)$$

(where $s = (\mathrm{id}, s_1, \ldots, s_n)$). This is by the definition of δ (induced from s) on $\mathcal{U}[V]$. We wish to show $Q(a) = 0$. Since $0 = P(a)$ we get

$$0 = \delta(P(a)) = \sum_j \frac{\partial P}{\partial x_j}(a)\delta a_j + P^\delta(a) = \sum_j \frac{\partial P}{\partial x_j}(a) s_j(a) + P^\delta(a) = Q(a)$$

□ Claim 6.34

Also $\mathfrak{m}_{V,a}^{m+1}$ is a differential ideal of $\mathcal{U}[V]$ for ≥ 0. (Indeed in the case $m = 1$ if $\alpha, \beta \in \mathfrak{m}_{V,a}$ we have $\delta(\alpha\beta) = \delta(\alpha)\beta + \alpha\delta(\beta) \in \mathfrak{m}_{V,a}^2$.) So δ on $\mathcal{U}[V]$ induces a morphism of additive grapes $\delta\colon \mathfrak{m}_{V,a}/\mathfrak{m}_{V,a}^{m+1} \to \mathfrak{m}_{V,a}/\mathfrak{m}_{V,a}^{m+1}$. Note that $\mathfrak{m}_{V,a}/\mathfrak{m}_{V,a}^{m+1}$ is a $\mathcal{U}$-vector space.

Claim 6.35. *For all $\lambda \in \mathcal{U}$ we have $\delta(\lambda\alpha) = \delta(\lambda)\alpha + \lambda\delta(\alpha)$.*

(This is some kind of Leibniz rule for $\mathcal{U}$-scalar multiplication; such structures are called δ-modules.)

Proof. For $\alpha = P(x) + \mathfrak{m}_{V,a}^{m+1}$ we have

$$\delta(\lambda\alpha) = \delta(\lambda P + \mathfrak{m}_{V,a}^{m+1}) = \delta(\lambda P) + \mathfrak{m}_{V,a}^{m+1} = (\delta(\lambda)\cdot P + \lambda\delta(P)) + \mathfrak{m}_{V,a}^{m+1} = \delta(\lambda)(P + \mathfrak{m}_{V,a}^{m+1}) + \lambda\delta(P + \mathfrak{m}_{V,a}^{m+1}) = \delta(\lambda)\alpha + \lambda\delta(\alpha)$$

as desired. □ Claim 6.35

Definition 6.36. The dual to $\mathfrak{m}_{V,a}/\mathfrak{m}_{V,a}^{m+1}$ is called the m^{th} *Jet space of V at a*:

$$\mathrm{Jet}_a^{(m)} V := \hom_{\mathcal{U}}(\mathfrak{m}_{V,a}/\mathfrak{m}_{V,a}^{m+1}, \mathcal{U})$$

(Note this is a $\mathcal{U}$-vector space.)

In particular $\mathrm{Jet}_a^{(1)} V = T_a V$.
Specializing to $X = (V, s)^\sharp$:

Definition 6.37. $\mathrm{Jet}_a^{(m)}(X) = \{ f \in \mathrm{Jet}_a^{(m)} V : \delta(f(\alpha)) = f(\delta a) \text{ for all } \alpha \in \mathfrak{m}_{V,a}/\mathfrak{m}_{V,a}^{m+1} \}$ is the m^{th} *differential Jet space of X at a.*

Claim 6.38. *$\mathrm{Jet}_a^{(m)} X$ is a $\mathcal{C}$-vector space.*

Proof. Suppose $f, g \in \mathrm{Jet}_a^{(m)} X$; so $f, g\colon \mathfrak{m}_{V,a}/\mathfrak{m}_{V,a}^{m+1} \to \mathcal{U}$. Then

$$\delta((f+g)(\alpha)) = \delta(f(\alpha) + g(\alpha)) = \delta f(\alpha) + \delta g(\alpha) = f(\delta\alpha) + g(\delta\alpha) = (f+g)(\delta\alpha)$$

Also if $c \in \mathcal{C}$ then

$$\delta((cf)(\alpha)) = \delta(cf(\alpha)) = c\delta(f(\alpha)) = cf(\delta v\alpha) = (cf)(\delta\alpha)$$

□ Claim 6.38

Claim 6.39. *$\mathrm{Jet}_a^{(m)} X$ is a finite-dimensional vector space over $\mathcal{C}$. In fact* $\dim_{\mathcal{C}}(\mathrm{Jet}_a^{(m)} X) = \dim_{\mathcal{U}}(\mathrm{Jet}_a^{(m)} V)$.

(Note the latter is finite since $\mathfrak{m}_{V,a}/\mathfrak{m}_{V,a}^{m+1}$ is spanned by things of the form

$$\prod_{i=1}^{n}(x_i - a_i)^r$$

where $1 \leq \sum_i r_i \leq m$. So in fact there is a $\mathcal{C}$-basis for $\operatorname{Jet}_a^{(m)} X$ that is a $\mathcal{U}$-basis for $\operatorname{Jet}_a^{(m)} V$.)

So in fact we get the stronger statement:

Proposition 6.40. *There is a finite $\mathcal{C}$-basis for $\operatorname{Jet}_a^{(m)} X$ which is also a $\mathcal{U}$-basis for $\operatorname{Jet}_a^{(m)} V$.*

Proof. Identify $\operatorname{Jet}_a^{(m)} V \cong \mathcal{U}^\ell$ for some ℓ. Then $\operatorname{Jet}_a^{(m)} X \subseteq \mathcal{U}^\ell$ is the set of solutions to a system of linear δ-equations

(*) $\delta y = Ay$ where $y = \begin{pmatrix} y_1 \\ \vdots \\ y_\ell \end{pmatrix}$ and $A \in M_\ell(\mathcal{U})$.

(One checks this.)

Subclaim 6.41. *If $y^{(1)}, \ldots, y^{(r)}$ are $\mathcal{C}$-linearly independent solutions to (*) then they are $\mathcal{U}$-linearly independent.*

Proof. Suppose $y^{(1)} = a_2 y^{(2)} + \cdots + a_r y^{(r)}$ for $a_2, \ldots, a_r \in \mathcal{U}$. Then

$$0 = \delta(y^{(1)}) - Ay^{(1)} = \sum_{i=2}^{r}(\delta a_i y^{(i)} + a_i \delta y^{(i)}) - \sum_{i=2}^{r} a_i A y^{(i)} = \sum_{i=2}^{r}(\delta a_i y^{(i)} + a_i A y^{(i)}) - \sum_{i=2}^{r} a_i A y^{(i)} = \sum_{i=2}^{r} \delta a_i y^{(i)}$$

By induction hypothesis we may assume $y^{(2)}, \ldots, y^{(r)}$ are $\mathcal{U}$-linearly independent; so $\delta a_i = 0$ for $i \in \{2, \ldots, r\}$. So $a_2, \ldots, a_r \in \mathcal{C}$, and $y^{(1)}, \ldots, y^{(r)}$ are $\mathcal{C}$-linearly dependent, a contradiction. □ Subclaim 6.41

So $\dim_{\mathcal{C}}(\operatorname{Jet}_a^{(m)} X) \leq \dim_{\mathcal{U}}(\operatorname{Jet}_a^{(m)} V)$.

But by the Blum axioms in $\mathcal{U}^\ell$ we can find ℓ-many $\mathcal{U}$-linearly independent solutions to (*); let $y^{(1)}, \ldots, y^{(\ell)}$ be such, and so form a basis for $\operatorname{Jet}_a^{(m)} V$ over $\mathcal{U}$. Hence by subclaim they also form a basis for $\operatorname{Jet}_a^{(m)} X$ over $\mathcal{C}$. □ Proposition 6.40

Let W be a D-subvariety of (V, s) over k. Let $Y = (W, s)^\sharp \subseteq X$; suppose $a \in Y$. We get an embedding $\operatorname{Jet}_a^{(m)} W \hookrightarrow \operatorname{Jet}_a^{(m)} V$ via the restriction map $\mathcal{U}[V] \twoheadrightarrow \mathcal{U}[W] = \mathcal{U}[V]/I(W)$, which sends $\mathfrak{m}_{V,a} \twoheadrightarrow \mathfrak{m}_{W,a}$ (and hence $\mathfrak{m}_{V,a}/\mathfrak{m}_{V,a}^{m+1} \twoheadrightarrow \mathfrak{m}_{W,a}/\mathfrak{m}_{W,a}^{m+1}$).

This induces an inclusion $\operatorname{Jet}_a^{(m)} Y \subseteq \operatorname{Jet}_a^{(m)} X$; in fact $\operatorname{Jet}_a^{(m)} Y = \operatorname{Jet}_a^{(m)} X \cap \operatorname{Jet}_a^{(m)} W$. In diagram, the following is a pullback:

$$\begin{tikzcd}
\operatorname{Jet}_a^{(m)} W \arrow[r, hook] & \operatorname{Jet}_a^{(m)} V \\
\operatorname{Jet}_a^{(m)} Y \arrow[r, hook] \arrow[u, hook] & \operatorname{Jet}_a^{(m)} X \arrow[u, hook]
\end{tikzcd}$$

We wish to show that $\ulcorner Y \urcorner$ is $\mathcal{C}$-internal over $\operatorname{acl}(ka)$; i.e. $\ulcorner Y \urcorner \in \operatorname{dcl}(\operatorname{acl}(ka), \mathcal{C}, b)$ and $\ulcorner Y \urcorner \mathop{\smile\!\!\!\!\!\mid}_{ka} b$. Let b_m be a $\mathcal{C}$-basis for $\operatorname{Jet}_a^{(m)} X$ that is also a $\mathcal{U}$-basis for $\operatorname{Jet}_a^{(m)} V$. Then we get a definable over (k, a, b_m) isomorphism $\Phi\colon \operatorname{Jet}_a^{(m)} V \to \mathcal{U}^\ell$; furthermore $\Phi \restriction \operatorname{Jet}_a^{(m)} X$ yields an isomorphism $\operatorname{Jet}_a^{(m)} X \to \mathcal{C}^\ell$. Also $\operatorname{Jet}_a^{(m)} Y \subseteq \operatorname{Jet}_a^{(m)} X$; so Φ restricts to an isomorphism $\operatorname{Jet}_a^{(m)} Y \to L$ where L is the image.

For ease of thinking, we identify

$$\operatorname{Jet}_a^{(m)} V = \mathcal{U}^\ell$$
$$\operatorname{Jet}_a^{(m)} X = \mathcal{C}^\ell$$
$$\operatorname{Jet}_a^{(m)} Y = L$$

Let c_m be a $\mathcal{C}$-basis for $\operatorname{Jet}_a^{(m)} Y$; so we view c_m as a tuple from $\mathcal{C}^\ell$. We will argue that $\ulcorner Y \urcorner \in \operatorname{dcl}(k, a, b_m, c_m)_{m<\omega}$. It suffices to show:

Proposition 6.42. *Suppose $k = k^{\mathrm{alg}}$ and $L = L^{\mathrm{alg}}$. Suppose (V, s) is a D-variety; suppose W_1, W_2 are D-subvarieties of V (all absolutely irreducible) and $Y_i := (W_i, s)^\sharp \subseteq X = (V, s)^\sharp$; suppose $a \in Y_1 \cap Y_2$. If $\mathrm{Jet}_a^{(m)} Y_1 = \mathrm{Jet}_a^{(m)} Y_2$ for all m then $Y_1 = Y_2$.*

This suffices because then any $\sigma \in \mathrm{Aut}(\mathcal{U})$ that fixes k, a, b_m, c_m for all m will fix Y setwise, and hence will fix $\lceil Y \rceil$ pointwise.

We also need that $b_m \mathop{\smile\!\!\!\!\!\;\mid}_{ka} \lceil Y \rceil$; but we can retroactively have chosen b_m to satisfy this. (All we require of the b_m is that it be a $\mathcal{C}$-basis for $\mathrm{Jet}_a^{(m)} V$. So given any choice of b_m we can choose $\widetilde{b_m} \mathop{\smile\!\!\!\!\!\;\mid}_{ka} \lceil Y \rceil$ with $\mathrm{tp}(b_m/ka) = \mathrm{tp}(\widetilde{b_m}/ka)$; then $\widetilde{b_m}$ is also a $\mathcal{C}$-basis for $\mathrm{Jet}_a^{(m)} X$ and a $\mathcal{U}$-basis for $\mathrm{Jet}_a^{(m)} V$. So we can replace b_m by $\widetilde{b_m}$.)

Proof. First observe the following:

Claim 6.43. *If $\mathrm{Jet}_a^{(m)} W_1 = \mathrm{Jet}_a^{(m)} W_2$ for all m then $W_1 = W_2$. (Note W_1, W_2 are absolutely irreducible as varieties over $k = k^{\mathrm{alg}}$.)*

Proof. Work in $\mathcal{U}[V] \supseteq I(W_1), I(W_2)$. It suffices to show that $I(W_1) \subseteq I(W_2)$ (by symmetry). Suppose then that $P \in I(W_1) \subseteq \mathcal{U}[V]$. If $f \in \mathrm{Jet}_a^{(m)} W_1$, so $f \colon \mathfrak{m}_{W_1,a}/\mathfrak{m}_{W_1,a}^{m+1} \to \mathcal{U}$, then $f(P) = 0$ since f is viewed in $\mathrm{Jet}_a^{(m)} V$ by $f \circ \iota^*$ (where $\iota \colon \mathcal{U}[V] \to \mathcal{U}[W_1]$ is induced by the inclusion map $\iota \colon W_1 \hookrightarrow V$).

Suppose $P \notin I(W_2)$; then by Noetherianity we have

$$\bigcap_m \mathfrak{m}_{V,a}^{m+1} = (0)$$

and likewise with W_1, W_2. So

$$\bigcap_m (\mathfrak{m}_{V,a}^{m+1} + I(W_2) = I(W_2)$$

(by the map $\mathcal{U}[V] \to \mathcal{U}[V]/I(W_2) = \mathcal{U}[W_2]$). So $P \notin \mathfrak{m}_{V,a}^{m+1} + I(W_2)$ for some m; so there is $f \in \mathrm{Jet}_a^{(m)} W_2$ such that $f(P) \neq 0$. $\square$ Claim 6.43

Lemma 6.44. *If (V, s) is a D-variety and $a \in (V, s)^\sharp = X$. Then $\mathrm{Jet}_a^{(m)} X$ is Zariski-dense in $\mathrm{Jet}_a^{(m)} V$. (Here Zariski-dense means over any set of parameters.)*

Proof. Let b be a $\mathcal{U}$-basis for $\mathrm{Jet}_a^{(m)} V$ and a $\mathcal{C}$-basis for $\mathrm{Jet}_a^{(m)} X$. Then we get an isomorphism Φ_b as follows:

$$\begin{tikzcd} \mathrm{Jet}_a^{(m)} V \arrow[r, "\Phi_b", "\cong"'] & \mathcal{U}^\ell \\ \mathrm{Jet}_a^{(m)} X \arrow[r, "\cong"] \arrow[u, "\subseteq"] & \mathcal{C}^\ell \arrow[u, "\subseteq"] \end{tikzcd}$$

Then $\mathcal{C}^\ell = (\mathcal{U}^\ell, 0)^\sharp$ and hence is Zariski-dense in $\mathcal{U}^\ell$. $\square$ Lemma 6.44

We now prove our proposition. Taking Zariski closures in $\mathrm{Jet}_a^{(m)} V$, we get $\mathrm{Jet}_a^{(m)} W_1 = \mathrm{Jet}_a^{(m)} W_2$ for all m. So by claim we get $W_1 = W_2$, and hence $Y_1 = Y_2$. $\square$ Proposition 6.42

We are now ready to state and prove the canonical base property.

Theorem 6.45 (CBP)**.** *Suppose $k = k^{\mathrm{alg}}$; suppose $p \in S(k)$ is finite-dimensional and $a \models p$. Then for any $L \supseteq k$ with $L^{\mathrm{alg}} = L$ we have $\mathrm{tp}(\lceil \mathrm{Kloc}(a/L) \rceil / \mathrm{acl}(ka))$ is $\mathcal{C}$-internal.*

Proof. We may assume $\mathrm{Kloc}(a/k) = (V, s)^\sharp =: X$ and $\mathrm{Kloc}(a/L) = (W, s)^\sharp =: Y$.

TODO 39. *Something about finite-dimensionality implying it's the generic type of a Kolchin-closed set? Maybe up to definable bijection? Maybe Proposition (20) or Theorem (22).*

Let $e = \lceil Y \rceil$. For each m let b_m be

(*) a $\mathcal{U}$-basis for $\operatorname{Jet}_a^{(m)} V$ that is also a $\mathcal{C}$-basis for $\operatorname{Jet}_a^{(m)} X$

such that $(b_1, \ldots, b_m) \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.4ex\hbox{$|$}}_{ka} e$. (Possible since (*) is part of $\operatorname{tp}(b_1 \cdots b_m/ka)$.)

Let c_m be a $\mathcal{U}$-basis for $\operatorname{Jet}_a^{(m)} W$ that is also a $\mathcal{C}$-basis for $\operatorname{Jet}_a^{(m)} Y$. We get

$$\begin{tikzcd} \operatorname{Jet}_a^{(m)} W \arrow[r, "\subseteq"] & \operatorname{Jet}_a^{(m)} V \arrow[r, "\Phi_{b_m}", "\cong"'] & \mathcal{U}^{\ell_m} \\ \operatorname{Jet}_a^{(m)} Y \arrow[r, "\subseteq"] \arrow[u, "\subseteq"] & \operatorname{Jet}_a^{(m)} X \arrow[r, "\cong"] \arrow[u, "\subseteq"] & \mathcal{C}^{\ell_m} \arrow[u, "\subseteq"] \end{tikzcd}$$

where $\ell_m = \dim(\operatorname{Jet}_a^{(m)} V)$. Note $\Phi_{b_m}(c_m)$ lies in $\mathcal{C}$.

Claim 6.46. $e \in \operatorname{dcl}(ka, b_m, \Phi_{b_m}(c_m))_{m \geq 1}$.

Proof. Fix $\sigma \in \operatorname{Aut}(\mathcal{U})$ such that $\sigma \upharpoonright (k, a, b_m, \Phi_{b_m}(c_m))_{m\geq 1}$. Then $(V, s)^\sigma = (V^\sigma, s^\sigma) = (V, s)$ since V, s are over k; so $X^\sigma = X$. Also $W^\sigma \subseteq V$ is a D-subvariety (over $\sigma(L)$), and $Y^\sigma = (W^\sigma, s)^\sharp$.

$(\operatorname{Jet}_a^{(m)} X)^\sigma = \operatorname{Jet}_a^{(m)} X$ is also preserved, whereas $(\operatorname{Jet}_a^{(m)} Y)^\sigma = \operatorname{Jet}_{\sigma(a)}^{(m)} Y^\sigma = \operatorname{Jet}_a^{(m)} Y^\sigma$. But $\sigma(\Phi_{b_m}(c_m)) = \Phi_{b_m}(c_m)$ and $\sigma(b_m) = b_m$; so $\sigma(c_m) = c_m$. But c_m is a $\mathcal{C}$-basis for $\operatorname{Jet}_a^{(m)} Y$; so $(\operatorname{Jet}_a^{(m)} Y)^\sigma = \operatorname{Jet}_a^{(m)} Y$.

So $\operatorname{Jet}_a^{(m)} Y^\sigma = \operatorname{Jet}_a^{(m)} Y$ for all m. So by above we get $Y^\sigma = Y$. But $e = \ulcorner Y \urcorner$; so $\sigma(e) = e$. $\square$ Claim 6.46

So there is $m \geq 1$ such that $e \in \operatorname{dcl}(k, a, b_1, \ldots, b_m, \Phi_{b_1}(c_1), \ldots, \Phi_{b_m}(c_m))$. But each $\Phi_{b_i}(c_i) \in \mathcal{C}$ and $e \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.4ex\hbox{$|$}}_{ka} b_1 \cdots b_m$. So $\operatorname{tp}(e/\operatorname{acl}(ka))$ is $\mathcal{C}$-internal.

TODO 40. *See remark after 6.21*

$\square$ Theorem 6.45

6.6 ...and back to the Zilber Dichotomy in DCF_0

Aside 6.47. There is some kind of "analysis" of finite-dimensional types that "decomposes" a finite-dimensional type in some way into a finite set of minimal types. This is the usual motivation for studying minimal types.

Theorem 6.48. *Suppose $k = k^{\mathrm{alg}}$ and $p \in S(k)$ is minimal. Then either p is 1-based or $p \not\perp \mathcal{C}$.*

Proof. Suppose p is not 1-based; so there are

- $a_1, \ldots, a_n \models p$
- $d \in \operatorname{dcl}(ka_1 \cdots a_n)$
- $L = L^{\mathrm{alg}} \supseteq k$

such that $\ulcorner \operatorname{Kloc}(d/L) \urcorner \notin \operatorname{acl}(kd)$.

Note that p minimal implies p is finite-dimensional.

TODO 41. *ref*

Hence $q = \operatorname{tp}(d/k)$ is also finite-dimensional (since $k\langle d\rangle \subseteq k\langle a_1, \ldots, a_d\rangle$ and each $k\langle a_i\rangle$ has finite transcendence degree over k as $a_i \models p$). So we can apply CBP to q: we get that $\operatorname{tp}(e/\operatorname{acl}(kd))$ is $\mathcal{C}$-internal (but non-algebraic).

Claim 6.49. $e \in \operatorname{dcl}(k, b_1, \ldots, b_\ell)$ *for some* $b_1, \ldots, b_\ell \models p$.

Proof. We use the following general proposition:

Proposition 6.50. *Suppose $q \in S(k)$ and $d \models q$; suppose $L = L^{\mathrm{alg}} \supseteq k$. Suppose* $\operatorname{Kloc}(d/k) = (V, s)^\sharp$ *and* $\operatorname{Kloc}(d/L) = (W, s)^\sharp$ *where $W \subseteq V$ is a subvariety. Let $e = \ulcorner (W, s)^\sharp \urcorner$. Then*

1. $k\langle e\rangle = k\langle$*minimal field of definition of* $W\rangle$.

2. $e \in \mathrm{dcl}(kd_1, \ldots, d\ell)$ *where* $d_1, \ldots, d_\ell \models q$.

Proof. 1. Done orally.

2. Consider $W \cap q(\mathcal{U})$ (the generic points of W), which is relatively definable in $q(\mathcal{U})$; then by stable embeddedness (or stable definability, if you want) it is relatively definable using some parameters $d_1, \ldots, d_\ell \in q(\mathcal{U})$. Then $e \in \mathrm{dcl}(kd_1 \cdots d_\ell)$.

☐ Proposition 6.50

☐ Claim 6.49

By internality there is $B \supseteq \mathrm{acl}(kd)$ and $e \mathop{\smile\!\!\!\!\!\mid}_{kd} B$ and $c \in \mathcal{C}$ such that $e \in \mathrm{dcl}(Bc)$. Since $e \notin \mathrm{acl}(kd)$, $e \notin \mathrm{acl}(B)$, we get $e \not\mathop{\smile\!\!\!\!\!\mid}_B c$. So $(b_1, \ldots, b_\ell)_B \not\mathop{\smile\!\!\!\!\!\mid} c$. Let $(b_{i_1}, \ldots, b_{i_s})$ be a subtuple of $(b_1, \ldots, b_\ell)$ that is independent over B (i.e. for all j we have $b_{i_j} \mathop{\smile\!\!\!\!\!\mid}_k Bb_{i_1} \cdots b_{i_{j-1}}$) and $\mathrm{acl}(Bb_{i_1} \cdots b_{i_s}) = \mathrm{acl}(Bb_1 \cdots b_\ell)$. (Note that since p is minimal we get that $b_{i_j} \mathop{\smile\!\!\!\!\!\mid}_k Bb_{i_1} \cdots b_{i_{j-1}}$ if and only if $b_{i_j} \notin \mathrm{acl}(Bb_{i_1} \cdots b_{i_{j-1}})$.)

By transitivity and symmetry, for some j we get that $b_{i_j} \not\mathop{\smile\!\!\!\!\!\mid}_{Bb_{i_1} \cdots b_{i_{j-1}}} c$. But $b_{i_j} \mathop{\smile\!\!\!\!\!\mid}_k Bb_{i_1} \cdots b_{i_{j-1}}$; so $p \not\perp \mathcal{C}$. ☐ Theorem 6.48