Course notes for PMATH 930

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Lectures by Rahim N. Moosa, Fall 2015

Contents

1 Preliminaries

We start with chapter 4 of Tent and Ziegler. (Chapters 1-3 are preliminaries.)

Assignments are roughly biweekly. No midterm, but will be a final.

2 Chapter 4

2.1 Partial types

Definition 1. Fix a first-order language L. For any $n \geq 0$, by a *partial n-type*, we mean a set $\Sigma(x_1, \dots, x_n)$ of L-formulae. **Note:** we don't require consistency.

Definition 2. We say $\Sigma(x_1, \dots, x_n)$ is *realized* in an L-structure $\mathcal{A}$ if there is $a = (a_1, \dots, a_n) \in A^n$ such that $\mathcal{A} \models \sigma(a)$ for all $\sigma \in \Sigma$. We also say *a realizes Σ in $\mathcal{A}$*; this is denoted $\mathcal{A} \models \Sigma(a)$.

Definition 3. $\Sigma(x_1, \dots, x_n)$ is *consistent* if and only if it is realized in some L-structure.

Remark 4. The compactness theorem tells us that Σ is consistent if and only if every finite subset of Σ is consistent.

Proof. Suppose $\Sigma(x_1, \dots, x_n)$ is finitely consistent. Let $L(c_1, \dots, c_n) = L \cup \{ c_1, \dots, c_n \}$ where c_i are new constant symbols. Let

$$\Sigma(c_1, \dots, c_n) = \{ \sigma(c_1, \dots, c_n) : \sigma \in \Sigma \}$$

Then this is an $L(c_1, \dots, c_n)$-theory. Then since every finite subset of $\Sigma(x_1, \dots, x_n)$ is realized in some L-structure, we have that every finite subset of $\Sigma(c_1, \dots, c_n)$ is consistent. Applying compactness, we get a model of $\Sigma(c_1, \ldots, c_n)$: an $L(c_1, \ldots, c_n)$-structure $\mathcal{A}' = (\mathcal{A}, a_1, \ldots, a_n)$ realizing $\Sigma(c_1, \ldots, c_n)$. Then $\mathcal{A} \models \Sigma(a_1, \ldots, a_n)$. $\square$ Remark 4

Definition 5. Suppose T is an L-theory. Then $\Sigma(x_1, \ldots, x_n)$ is *consistent with* T if and only if it is realized in some model of T.

Remark 6. This occurs if and only if $T \cup \Sigma(x_1, \ldots, x_n)$ is consistent.

Remark 7. Σ is consistent with T if and only if every finite subset is.

Question 8. When does T have a model in which Σ is not realized (or is *omitted*)?

Definition 9. A partial n-type $\Sigma(x_1, \ldots, x_n)$ is *isolated* in a theory T if and only if there is an L-formula $\varphi(x_1, \ldots, x_n)$ such that

1. $\varphi(x_1, \ldots, x_n)$ is consistent with T
2. Given $\mathcal{A} \models T$ and $(a_1, \ldots, a_n) \in A^n$ such that $\mathcal{A} \models \varphi(a_1, \ldots, a_n)$, we have $\mathcal{A} \models \Sigma(a_1, \ldots, a_n)$.

We then say φ *isolates* Σ *in* T.

Remark 10. This is equivalent to requiring

$$T \models \forall x_1 \ldots x_n(\varphi(x_1, \ldots, x_n) \to \sigma(x_1, \ldots, x_n))$$

for all $\sigma \in \Sigma$.

Remark 11. When T is a complete theory, if Σ is isolated in T, then it is realized in every model of T.

Proof. Suppose $\mathcal{A} \models T$. Then since $\varphi(x_1, \ldots, x_n)$ is consistent and since T is complete, we have

$$\mathcal{A} \models \exists x_1 \ldots x_n \varphi(x_1, \ldots, x_n)$$

But then we have $a \in A^n$ such that

$$\mathcal{A} \models \varphi(a)$$

Then a realizes Σ. $\square$ Remark 11

Definition 12. A *theory* is countable if and only if the language is countable (i.e. has cardinality $\leq \aleph_0$).

Theorem 13 (Omitting types theorem (4.1.2))**.** *If T is a countable, complete, consistent theory and $\Sigma(x_1, \ldots, x_n)$ is not isolated in T, then T has a model omitting $\Sigma(x_1, \ldots, x_n)$.*

Proof. We'll prove it for $n = 1$. Consider a partial type $\Sigma(x)$ that is. Let C be a countably infinite set of new constant symbols. We wish to construct an L^*-theory $T^* \supseteq T$ that is consistent and such that

1. T^* is a *Henkin theory*; i.e. for any L^*-formula $\psi(x)$ there is $c \in C$ such that
$$T^* \vdash \exists x \psi(x) \to \psi(c)$$
2. For each $c \in C$ there is some $\sigma \in \Sigma$ such that
$$T^* \vdash \neg\sigma(c)$$

Suppose we have such a T^*. Let $\mathcal{A}^* \models T^*$; say $\mathcal{A}^* = (\mathcal{A}, a_c)_{c \in C}$. Then $A \models T$. Let $B = \{ a_c : c \in C \}$. Then Item 1 implies that B is the universe of an elementary substructure $\mathcal{B} \preceq \mathcal{A}$. (It's not hard to see that it's the universe of a substructure; see 2.2.3 in Tent and Ziegler to check that it's elementary. Proof is essentially Tarski-Vaught test.) Thus $\mathcal{B} \models T$. Then Item 2 tells us that $\mathcal{B}$ omits $\Sigma(x)$, since if $a_c \in \mathcal{B}$, then by Item 2, there is $\sigma \in \Sigma$ such that

$$\begin{aligned} T^* &\models \neg\sigma(c) \\ \implies \mathcal{A}^* &\models \neg\sigma(c) \\ \implies \mathcal{A} &\models \neg\sigma(a_c) \\ \implies \mathcal{B} &\models \neg\sigma(a_c) \end{aligned}$$

and thus that a_c does not realize $\Sigma(x)$ in $\mathcal{B}$.

It remains to construct T^*. We will make T^* the union of

$$T = T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$$

of L^*-theories where each T_{i+1} is consistent and a finite extension of T_i (i.e. $T_{i+1} \setminus T_i$ is finite). We will take care of Item 1 in odd steps and Item 2 in even steps. Enumerate $C = \{ c_i : i < \omega \}$ and the L^*-formulae as $\{ \psi_i(x) : i < \omega \}$. Having constructed T_{2i}, in T_{2i+1} we make sure that Item 1 is true of $\psi_i(x)$. Choose $c \in C$ that does not appear in T_{2i} nor in $\psi_i(x)$ and set

$$T_{2i+1} = T_{2i} \cup \{ \exists x(\psi_i(x) \to \psi_i(c)) \}$$

Then T_{2i+1} is consistent since, c being new, we can interpret it in a model of T_{2i} as we wish.

Now construct T_{2i+2} so that Item 2 holds for c_i. Not we can assure T_{2i+1} is of the form $T \cup \{ \delta \}$ where δ is an L^*-sentence, since $T_{2i+1} \setminus T$ is finite. Write $\delta = \varphi(c_i, \overline{c})$ where $\varphi(x, \overline{y})$ is an L-formula and $\overline{c}$ is a tuple of new constants not including c_i. Then $\Sigma(x)$ is not isolated in T by $\exists \overline{y} \varphi(x, \overline{y})$; so there is $\mathcal{A} \models T$ and $a \in A$ such that

$$\mathcal{A} \models \exists \overline{y} \varphi(a, \overline{y})$$

but $\mathcal{A} \models \neg\sigma(a)$ for some $\sigma \in \Sigma$. i.e.

$$\{ \exists \overline{y} \varphi(x, y), \neg\sigma(x) \}$$

is consistent with T. So $T \cup \{ \varphi(x, \overline{y}), \neg\sigma(x) \}$ is consistent. Thus

$$T \cup \{ \varphi(c_i, \overline{c}) \} \cup \{ \neg\sigma(c_i) \}$$

is a consistent L^*-theory, as we can interpret $c_i, \overline{c}$ as we like in a model of T. We can thus let

$$T_{2i+2} = T_{2i+1} \cup \{ \neg\sigma(c_i) \} = T \cup \{ \varphi(c_i, \overline{c}) \} \cup \{ \neg\sigma(c_i) \}$$

□ Theorem 13

Remark 14 (Ed.). I don't think we need T to be complete for the above direction; just for the equivalence.

2.2 Complete types

Fix a theory T. Fix $n \geq 0$.

Definition 15. An *n-type* (or *complete n-type*) is a partial n-type $p(x_1, \dots, x_n)$ that is maximally consistent with T. We use $S_n(T)$ to denote the collection of complete n-types of T.

Remark 16. Let $p(x_1, \dots, x_n)$ be a partial n-type. Then p is an n-type if and only if for all $\varphi(x_1, \dots, x_n)$, we have either $\varphi(x_1, \dots, x_n)$ or $\neg\varphi(x_1, \dots, x_n)$ is in p.

There is a natural topology on $S_n(T)$:

Definition 17. We define the *Stone topology* on $S_n(T)$ to be the topology whose basic open sets are

$$[\varphi] = \{ p \in S_n(T) : \varphi \in p \}$$

for $\varphi(x_1, \dots, x_n)$ an L-formula.

Remark 18. For this to generate a topology, the basic open sets must be closed under finite intersections. In fact, they are closed under all Boolean combinations:

- $[\varphi] \cap [\psi] = [\varphi \wedge \psi]$
- $[\varphi] \cup [\psi] = [\varphi \vee \psi]$
- $S_n(T) \setminus [\varphi] = [\neg\varphi]$
- $\emptyset = [\perp]$

- $S_n(T) = [\top]$

The basic open sets are thus clopen. Thus $S_n(T)$ is totally disconnected; i.e. the only non-empty connected sets are the singletons.

Remark 19. $[\varphi] = [\psi]$ if and only if $T \vdash \forall x_1 \dots x_n(\varphi(x_1, \dots, x_n) \leftrightarrow \psi(x_1, \dots, x_n))$.

Proof.

($\Longleftarrow$) Suppose $\varphi \in p$. Then by consistency with T and completeness of p, we have $\psi \in p$, and thus that $[\varphi] \subseteq [\psi]$. By symmetry, we get $[\varphi] = [\psi]$.

($\Longrightarrow$) Suppose $T \nvdash \forall x(\varphi(x) \leftrightarrow \psi(x))$ (where $x = (x_1, \dots, x_n)$). Then there is a model of T with a tuple realizing (say) $\varphi(x)$ but not $\psi(x)$. i.e. $\{ \varphi(x), \neg\psi(x) \}$ is consistent with T. By a Zorn's lemma argument, we can extend it to a complete n-type in T, say $p(x_1, \dots, x_n)$. Then $p \in [\varphi] \setminus [\psi]$.

$\square$ Remark 19

Lemma 20 (4.2.2)**.** *$S_n(T)$ is Hausdorff and compact.*

Proof. We check that it's Hausdorff. Suppose $p \neq q$. Thus there is $\varphi \in p$ with $\varphi \notin q$, and thus that $\neg\varphi \in q$. But

$$[\varphi] \cap [\neg\varphi] = [\varphi \wedge \neg\varphi] = \emptyset$$

So we can separate p and q by disjoint open sets.

We check compactness. Suppose

$$S_n(T) = \bigcup_{i \in I} U_i$$

is an open cover, with each

$$U_i = \bigcup_j [\varphi_{ij}]$$

Thus

$$S_n(T) = \bigcup_{i,j} [\varphi_{ij}]$$

Then

$$\Sigma = \{ \neg\varphi_{ij} : i, j \}$$

is not consistent with T. Then, by compactness of partial types, we have some finite subset of Σ is inconsistent with T. Thus

$$T \vdash \forall x_1 \dots x_n(\varphi_{i_0 j_0}(x_1, \dots, x_n) \vee \dots \vee \varphi_{i_\ell, j_\ell}(x_1, \dots, x_n))$$

So

$$S_n(T) \subseteq \bigcup_{k=0}^{\ell} [\varphi_{i_k, j_k}]$$

and $S_n(T)$ is compact. $\square$ Lemma 20

Remark 21. One could also use the compactness of the Stone topology to check compactness of first-order logic by taking T to be the empty theory.

Lemma 22 (4.2.3)**.** *Every clopen set in $S_n(T)$ is of the form $[\varphi]$ for some L-formula $\varphi(x_1, \dots, x_n)$.*

Proof. We prove the following more general statement.

Claim 23. *Suppose C_1, C_2 are disjoint closed subsets of $S_n(T)$. Then there is a basic open set separating them. i.e. there is $\varphi(x_1, \dots, x_n)$ such that $C_1 \subseteq [\varphi]$ but $C_2 \cap [\varphi] = \emptyset$.*

Proof. Set $\mathcal{F} = \{ [\varphi] : C_1 \subseteq [\varphi] \}$. Note then that $S_n(T) = [\top] \in \mathcal{F}$. If $p \in C_2$, then there is $[\psi] \ni p$ with $[\psi] \cap C_1 = \emptyset$ since $C_2 \cap C_1 = \emptyset$. (In particular, C_1^c is open and contains p, so there is a basic open subset of C_1^c containing p.) Note then that $[\neg\psi] \in \mathcal{F}$ and $p \notin [\neg\psi]$.

Thus C_2 is covered by the complements of the elements of $\mathcal{F}$. But C_2 is closed, and $S_n(T)$ is compact and Hausdorff. So C_2 is covered by finitely many complements of elements of $\mathcal{F}$; i.e. we have

$$[\varphi_1], \dots, [\varphi_\ell] \in \mathcal{F}$$

such that

$$\bigcap_{i=1}^{\ell} [\varphi_i] \cap C_2 = \emptyset$$

Then

$$\left[\bigwedge_{i=1}^{\ell} \varphi_i \right] = \bigcap_{i=1}^{\ell} [\varphi_i]$$

is our desired set, as it contains C_1 as a subset. $\square$ Claim 23

Let $C \subseteq S_n(T)$ be clopen. Let $C_1 = C$; let $C_2 = S_n(T) \setminus C$. Then C_1, C_2 are closed and disjoint. By the claim, we then have that they are separated by a basic clopen set, and thus that C is clopen. $\square$ Lemma 22

Lemma 24 (4.2.6)**.** *An n-type p is isolated in T if and only if p is isolated in $S_n(T)$. (i.e. $\{ p \}$ is an open set). In fact, φ isolates p in T if and only if $\{ p \} = [\varphi]$.*

Proof.

($\implies$) Suppose φ isolates p. Then

$$T \vdash \forall x(\varphi(x) \to \psi(x))$$

for each $\psi \in p$. Then comleteness and consistency of p implies that $\varphi \in p$. Thus $p \subseteq [\varphi]$. Suppose $q \in S_n(T)$ satisfies $q \neq p$. Then there is $\psi \in p$ with $\neg\psi \in q$. Then $\{ \varphi, \neg\psi \}$ is inconsistent with T, and thus $q \notin [\varphi]$. So $\{ p \} = [\varphi]$.

($\impliedby$) Suppose $p \in S_n(T)$ is isolated. Then $\{ p \}$ is clopen. So, by the previous lemma (4.2.3), we have that it is a basic open set, and there is φ such that $\{ p \} = [\varphi]$. Let $\psi \in p$. If $\{ \varphi, \neg\psi \}$ were consistent with T then we can extend it to q to get $q \in [\varphi]$ with $q \neq p$, a contradiction. So $\{ \varphi, \neg\psi \}$ is inconsistent with T. Thus

$$T \vdash \forall x(\varphi(x) \to \psi(x))$$

and φ isolates p in T.

$\square$ Lemma 24

2.3 Types over parameters

Definition 25. Suppose $\mathcal{A}$ is an L-structure. Suppose $B \subseteq A$. An *n-type over B in $\mathcal{A}$* is a maximal set of $L(B)$-formulae (where $L(B) = L \cup \{ \underline{b} : b \in B \}$) that is finitely satisfiable in $\mathcal{A}$. The set of such is denoted $S_n^{\mathcal{A}}(B)$.

Example 26. Suppose $a_1, \dots, a_n \in A$. We define

$$\operatorname{tp}(a_1, \dots, a_n/B) = \operatorname{tp}^{\mathcal{A}}(a_1, \dots, a_n/B) = \{ \varphi(x_1, \dots, x_n) \text{ an } L_B\text{-formula} : \mathcal{A} \models \varphi(a_1, \dots, a_n) \}$$

These are precisely the realized types in $\mathcal{A}$. Indeed, if $p(x_1, \dots, x_n) \in S_n^{\mathcal{A}}(B)$ is realized in $\mathcal{A}$ by $(a_1, \dots, a_n) \in A^n$, then $\operatorname{tp}(a_1, \dots, a_n/B) \supseteq p(x_1, \dots, x_n)$. But by maximality of p, we have

$$p(x_1, \dots, x_n) = \operatorname{tp}(a_1, \dots, a_n/B)$$

Remark 27.

1. If $\mathcal{A} \preceq \mathcal{A}'$ and $B \subseteq A$, then $S_n^{\mathcal{A}}(B) = S_n^{\mathcal{A}'}(B)$.

2. If $p \in S_n^{\mathcal{A}}(B)$, then p is realized in some $\mathcal{A}' \succeq \mathcal{A}$. To see this, observe that

$$T = \mathrm{Th}(\mathcal{A}_A) \cup p(c_1, \ldots, c_n)$$

   is consistent by compactness (where $c_1, \ldots, c_n$ are new constant symbols). Then use PMATH 733, fall 2015 notes, 4.45:

   Theorem 28. *$\mathcal{A}$ embeds elementarily into every model of* $\mathrm{Th}(\mathcal{A}_A)$.

   Then if $\mathcal{C} \models T$, we have $\mathcal{C}$ is of the form

$$\mathcal{C} = (\mathcal{A}'_A, a_1, \ldots, a_n)$$

   for some $\mathcal{A}' \succeq \mathcal{A}$, where $c_i^{\mathcal{C}} = a_i$. Hence $(a_1, \ldots, a_n)$ realizes $p(x_1, \ldots, x_n)$ in $\mathcal{A}'$.

3. In fact, there is an elementary extension of $\mathcal{A}$ in which all types from $S_n^{\mathcal{A}}(B)$ are realized. To see this, observe that

$$\mathrm{Th}(\mathcal{A}_A) \cup \{\, p(c_p) : p \in S_n^{\mathcal{A}}(B) \,\}$$

   is consistent, where for each $p \in S_n^{\mathcal{A}}(B)$ we let c_p be an n-tuple of new constant symbols.

4. $S_n^{\mathcal{A}}(B) = S_n(\mathrm{Th}(\mathcal{A}_B))$ since for partial types, we have finite satisfiability in $\mathcal{A}$ is equivalent to consistency with $\mathrm{Th}(\mathcal{A}_B)$. We can use this to endow the former with a Stone topology.

Theorem 29 (4.2.5)**.** *Suppose $\mathcal{A}, \mathcal{B}$ are L-structures. Suppose $A_0 \subseteq A$, $B_0 \subseteq B$. Suppose $f\colon A_0 \to B_0$ is a partial elementary map; i.e. suppose for any $m \geq 0$, any L-formulae $\varphi(x_1, \ldots, x_m)$ and any $a_1, \ldots, a_m \in A_0$, we have*

$$\mathcal{A} \models \varphi(a_1, \ldots, a_m) \iff \mathcal{B} \models \varphi(f(a_1), \ldots, f(a_m))$$

Then there exists a surjective continuous map

$$S_n(f)\colon S_n^{\mathcal{B}}(B_0) \to S_n^{\mathcal{A}}(A_0)$$

i.e. Stone spaces constitute a contravariant functor

Proof. Suppose $x = (x_1, \ldots, x_n)$. Then every $L(A_0)$-formula in x takes the form $\varphi(x, a)$ where $\varphi(x, y_1, \ldots, y_\ell)$ is an L-formula and $a = (a_1, \ldots, a_\ell) \in A_0^\ell$. We can then define $f(\varphi) = \varphi(x, f(a))$ an $L(B_0)$-formula.

For $p \in S_n^{\mathcal{A}}(A_0)$, one could imagine defining

$$f(p) = \{\, f(\varphi) : \varphi \in p \,\}$$

We then have $f(p)$ is a partial type in $\mathrm{Th}(\mathcal{B}_{B_0})$, since f is a partial elementary map; however, it may not be maximal, since f might not be surjective.

For $q \in S_n^{\mathcal{B}}(B_0)$, we instead define

$$S_n(f)(q) = \{\, \varphi : \varphi \text{ an } L(A_0)\text{-formula}, f(\varphi) \in q \,\}$$

Claim 30. $S_n(f)(q) \in S_n^{\mathcal{A}}(A_0)$.

Proof. It's finitely satisfiable in $\mathcal{A}$ since q is finitely satisfiable in $\mathcal{B}$ and f is a partial elementary map. Completeness follows since for all a either $\varphi(x, f(a)) \in q$ or $\neg\varphi(x, f(a)) \in q$. □ Claim 30

We now check continuity. Suppose $\varphi(x, a)$ is an L_{A_0}-formula. Then

$$S_n(f)^{-1}([\varphi(x, a)]) = [\varphi(x, f(a))]$$

since given $q \in S_n^{\mathcal{B}}(B_0)$, we have

$$\begin{aligned} S_n(f)(q) \in [\varphi(x, a)] &\iff \varphi(x, a) \in S_n(f)(q) \\ &\iff \varphi(x, f(a)) \in q \\ &\iff q \in [\varphi(x, f(a))] \end{aligned}$$

We now check surjectivity. Given $p \in S_n^{\mathcal{A}}(A_0)$, let $q \in S_n^{\mathcal{B}}(B_0)$ extend $f(p)$. Then

$$\begin{aligned} S_n(f)(q) &= \{\, \varphi(x,a) : \varphi(x, f(a)) \in q \,\} \\ &\supseteq \{\, \varphi(x,a) : \varphi(x, f(a)) \in f(p) \,\} \\ &= p \end{aligned}$$

Then $S_n(f)(q) \supseteq p$, and p is maximal. So $S_n(f)(q) = p$. □ Theorem 29

Remark 31.

1. If $f\colon A_0 \to B_0$ is a bijective partial elementary map, then $p \mapsto f(p)$ is a continuous map $S_n^{\mathcal{A}}(A_0) \to S_n^{\mathcal{B}}(B_0)$ and it will be the inverse of $S_n(f)$. So $S_n^{\mathcal{A}}(A_0)$ is homeomorphic to $S_n^{\mathcal{B}}(B_0)$.

2. If $\mathcal{A} = \mathcal{B}$ and $A_0 \subseteq B_0$ and $f\colon A_0 \to B_0$ is the containment, then

$$S_n(f)\colon S_n^{\mathcal{A}}(B_0) \to S_n^{\mathcal{A}}(A_0)$$

   is the restriction map

$$p(x) \mapsto p(x) \upharpoonright A_0 = \text{set of formulae in } p(x) \text{ over } A_0$$

   So restriction is a continuous, surjective homomorphism.

Some examples:

Remark 32. Suppose T admits quantifier elimination. Suppose $\mathcal{A} \models T$, $B \subseteq A$, and $a, a' \in A^n$. If a and a' realize the same atomic L_B-formulae, then $\mathrm{tp}(a/B) = \mathrm{tp}(a'/B)$.

Exercise 33. If every type in T is determined by its atomic part, then T admits quantifier elimination.

Example 34. Recall that DLO is the theory of dense linear orderings without endpoints (in the language $L = \{ < \}$); further recall that DLO admits quantifier elimination. What are the 1-types? Well, there are only 2 atomic L-formula: $x < x$ and $x = x$. But the former is never satisfied, and the latter never is; so

$$|S_1(\mathrm{DLO})| = 1$$

More interesting in the case of parameters. Suppose $(A, <) \models \mathrm{DLO}$. Let $B \subseteq A$. What is $S_1(B)$? Well, there are $\mathrm{tp}(b/B)$ for $b \in B$, and there are *cuts*; i.e. partitions $B = L \cup U$ such that $\ell < u$ for all $\ell \in L$, all $u \in U$. This is everything: given any $p(x) \in S_1(B)$ not realized in B, define

$$\begin{aligned} L_p &= \{\, b \in B : p(x) \in [b < x] \,\} \\ U_p &= \{\, b \in B : p(x) \in [x < b] \,\} \end{aligned}$$

Which types are isolated in $S_1(B)$? They are

- Those realized in B
- Cuts (L, U) where $L = \emptyset$ or has a maximum and $U = \emptyset$ or has a minimum.

Example 35. $(\mathbb{Q}, <) \models \mathrm{DLO}$. Then

$$S_1(\mathbb{Q}) = \mathbb{R} \cup \{\, \pm\infty \,\}$$

(Not topologically!) In particular, over countable sets, there may be $2^{\aleph_0}$-many 1-types. (This is, of course, the maximum number of types in a countable set over a countable theory.)

Example 36. Recall that ACF is the theory of algebraically closed fields in the language $L = \{\, 0, 1, +, -, \times \,\}$; further recall that ACF admits quantifier elimination. We'd like to work over subfields of algebraically closed fields as parameter sets. We can, in fact, do this: suppose $K \models \mathrm{ACF}$, $A \subseteq K$. Let k be the subfield of K generated by A. Then the restriction map

$$S_n^K(k) \to S_n^K(A)$$

is surjective and continuous; it is, in fact, bijective.

The point is that every L_k-formula is equivalent to an L_A-formula. To see this, note that the atomic formulae over k are $P(x) = 0$ for $P \in k[x_1, \dots, x_n]$, $x = (x_1, \dots, x_n)$, and then use the fact that elements of k are of the form $f(a)$ where $f \in \mathbb{Z}(Y_1, \dots, Y_\ell)$ and $a \in A^\ell$.

Then $S_n^k(k)$ is in bijective correspondence with $\operatorname{Spec}(k[X_1, \dots, X_n])$, the set of prime ideals in $k[x_1, \dots, x_n]$. The correspondence is given by

$$p(x) \mapsto I_p = \{\, f \in k[X_1, \dots, X_n] : p(x) \in [f(x_1, \dots, x_n)] \,\}$$

The inverse is given by sending I to the type defined by $f(x) = 0 \iff f \in I$. This, too, is not a topological correspondence, though we think the forward map is continuous.

2.4 Section 4.3

Definition 37. Let κ be an infinite cardinal. We say $\mathcal{A}$ is *κ-saturated* if all 1-types over sets of size $< \kappa$ are realized.

Remark 38. If $\mathcal{A}$ is infinite, then

$$\Phi(x) = \{\, x \neq a : a \in A \,\}$$

is a partial 1-type over A, and can thus be extended to a complete type over A. So, if $\mathcal{A}$ is κ-saturated, then $\kappa \leq |A|$.

Remark 39. If $\mathcal{A}$ is κ-saturated, then every type in $S_n^{\mathcal{A}}(B)$ for $|B| < \kappa$ is realized in $\mathcal{A}$, for all $n \geq 1$.

Proof. Apply induction on n. $n = 1$ is the definition of κ-saturation. Suppose $n > 1$, $x = (x_1, \dots, x_n)$, and $p(x) \in S_n^{\mathcal{A}}(B)$, with $|B| < \kappa$. Let $q(x_1, \dots, x_{n-1})$ be the collection of formulae in $p(x)$ in which x_n does not appear. Then $q \in S_{n-1}^{\mathcal{A}}(B)$. The induction hypothesis then implies that there are $a_1, \dots, a_{n-1} \in A$ with $\mathcal{A} \models q(a_1, \dots, a_{n-1})$. Let

$$r(x_n) = \{\, \varphi(a_1, \dots, a_{n-1}, x_n) : \varphi \in p \,\}$$

Claim 40. $r(x_n) \in S_1^{\mathcal{A}}(B \cup \{\, a_1, \dots, a_{n-1} \,\})$.

Proof. We first check finite satisfiability. Suppose $\varphi(a_1, \dots, a_{n-1}, x_n) \in r(x_n)$. So $\varphi(x) \in p(x)$.

$$\begin{aligned} & \exists x_n \varphi(x) \in p(x) \\ \implies & \exists x_n \varphi(x) \in q(x_1, \dots, x_{n-1}) \\ & \implies \mathcal{A} \models \exists x_n \varphi(a_1, \dots, a_{n-1} x_n) \end{aligned}$$

So $\varphi(a_1, \dots, a_{n-1}, x_n)$ is satisfiable in $\mathcal{A}$. But $r(x_n)$ is closed under conjunction. So $r(x)$ is finitely satisfiable in $\mathcal{A}$.

Completeness of $r(x_n)$ follows from completeness of p. □ Claim 40

By κ-saturation there is $b \in A$ such that $\mathcal{A} \models r(b)$ (since $|B \cup \{\, b_1, \dots, b_n \,\}| < \kappa$). Then $(a_1, \dots, a_{n-1}, b)$ realizes $p(x)$. □ Remark 39

Lemma 41 (4.3.1)**.** *Suppose $\mathcal{A}, \mathcal{B}$ are L-structures that are countably infinite and ω-saturated. If $\mathcal{A} \equiv \mathcal{B}$, then $\mathcal{A} \cong \mathcal{B}$.*

Remark 42. In general $\equiv$ does not imply $\cong$; Lowenheim-Skolem says that structures have arbitrarily large elementary extensions. Even in the same cardinality, $\equiv$ does not imply $\cong$.

Example 43. $\mathbb{Q}^{\mathrm{alg}} \equiv \mathbb{Q}(t)^{\mathrm{alg}}$ in the language of rings, as ACF_0 is complete. They are both countably infinite, but they are not isomorphic as the latter has a transcendental element over $\mathbb{Q}$, and the former does not.

In fact, neither of these is ω-saturated. Let $p(x) \in S_1^{\mathbb{Q}^{\mathrm{alg}}}(\mathbb{Q}) = S_1^{\mathbb{Q}^{\mathrm{alg}}}(\emptyset)$ be the type corresponding to $(0) \subseteq \mathbb{Q}[x]$. Then $p(x)$ says $f(x) \neq 0$ for any $f \in \mathbb{Q}[x] \setminus \{\, 0 \,\}$. This is not realized in $\mathbb{Q}^{\mathrm{alg}}$.

For $\mathbb{Q}(t)^{\mathrm{alg}}$, consider $(0) \subseteq \mathbb{Q}(t)[x]$, which corresponds to $q(x) \in S_1^{\mathbb{Q}(t)^{\mathrm{alg}}}(\mathbb{Q}(t)) = S_1^{\mathbb{Q}(t)^{\mathrm{alg}}}(t)$. This is over finitely many parameters but is not realized in $\mathbb{Q}(t)^{\mathrm{alg}}$.

In fact, 4.3.1 implies that ACF_0 has at most one countably ω-saturated model; namely $\mathbb{Q}(t_0, t_1, \dots)^{\mathrm{alg}}$.

Proof of Lemma 41. Back-and-forth argument, generalizing $\aleph_0$-categoricity of DLO. Construct chains of *finite* sets

$$\begin{tikzcd} A_0 \arrow[r, "\subseteq"] \arrow[d, "f_0"] & A_1 \arrow[r, "\subseteq"] \arrow[d, "f_1"] & \dots \\ B_0 \arrow[r, "\subseteq"] & B_1 \arrow[r, "\subseteq"] & \dots \end{tikzcd}$$

with each f_i a bijective partial elementary map and such that

$$\bigcup_i A_i = A$$
$$\bigcup_i B_i = B$$

Then

$$f = \bigcup_i f_i$$

is an isomorphism $\mathcal{A} \cong \mathcal{B}$.

Enumerate

$$A = \{ a_0, a_1, \dots \}$$
$$B = \{ b_0, b_1, \dots \}$$

Recursively construct A_i, B_i, and f_i, making sure at odd stages that

$$\bigcup_i A_i = A$$

and at even stages that

$$\bigcup_i B_i = B$$

Set $A_0 = B_0 = f_0 = \emptyset$. Then f_0 is a partial elementary map since $\mathcal{A} \equiv \mathcal{B}$.

Suppose we have constructed

$$f_i \colon A_i \to B_i$$

a bijective partial elementary map for $i = 2n$. Set $A_{i+1} = A_i \cup \{ a_n \}$. Let $p(x) = \operatorname{tp}(a_n/A_i)$. Then $f_i(p) \in S_1^{\mathcal{B}}(B_i)$. By ω-saturation of $\mathcal{B}$ there is $b \in B$ such that $\mathcal{B} \models f_i(p)(b)$. Set $B_{i+1} = B_i \cup \{ b \}$ and extend f_i to f_{i+1} by $f_{i+1}(a_n) = b$. Check that f_{i+1} is a bijective partial elementary map.

Suppose $i = 2n+1$. Set $B_{i+1} = B_i \cup \{ b_n \}$. Let $q(x) = \operatorname{tp}(b_n/B_i)$. Then $S_1(f_i)(q) = f_i^{-1}(q) \in S_1^{\mathcal{A}}(A_i)$; this has a realization a by ω-saturation of $\mathcal{A}$. Set $A_{i+1} = A \cup \{ a \}$; extend f_i to f_{i+1} by $f_{i+1}(a) = b_n$. This will then be a bijective partial elementary map. □ Lemma 41

Definition 44. Recall that for an infinite cardinal κ, we say T is *κ-categorical* if it has a unique model of size κ.

We are interested in $\aleph_0$-categoricity.

Theorem 45 (Ryll-Nardzewski theorem)**.** *Suppose T is a countable, complete theory. Then T is $\aleph_0$-categorical if and only if for each $n < \omega$ there are only finitely many L-formulae $\varphi(x_1, \dots, x_n)$ modulo T.*

Proof.

($\Longleftarrow$) By Lemma 41, it suffices to show that every countably infinite model of T is ω-saturated. Let $\mathcal{M} \models T$ be countably infinite. Suppose $A \subseteq M$ is finite, say $A = \{ a_1, \dots, a_n \}$. Then every $L(A)$-formula in 1 variable is of the form $\varphi(a_1, \dots, a_n, x)$ where $\varphi(y_1, \dots, y_n, x)$ is an L-formula. So in $T = \operatorname{Th}(\mathcal{M})$ there are only finitely many $L(A)$-formulae. So any $p(x) \in S_1^{\mathcal{M}}(A)$ is equivalent to a single $L(A)$-formula; hence $p(x)$ is realized in $\mathcal{M}$. So $\mathcal{M}$ is ω-saturated.

($\implies$) We begin with a claim.

Claim 46. *All n-types are isolated.*

Proof. If $p(x)$ is not isolated, then by the omitting types theorem, we have $\mathcal{M} \models T$ omitting $p(x)$. By downward Löwenheim-Skolem, we may assume that $\mathcal{M}$ is countable.

Since $p(x) \in S_n(T)$, it is realized in some $\mathcal{N} \models T$; by downward Löwenheim-Skolem, we may assume $\mathcal{N}$ is countable.

Thus $\mathcal{M}$ has no realization of $p(x)$, and $\mathcal{N}$ does; so $\mathcal{M} \not\cong \mathcal{N}$, contradicting the $\aleph_0$-categoricity of T. ☐ Claim 46

So $S_n(T)$ is compact, with every point isolated; thus $S_n(T)$ is finite. Thus there are finitely many clopen sets in $S_n(T)$. Thus, by Lemma 22, we have that modulo T there are only finitely many L-formulae in n variables. (Since $[\varphi] = [\psi]$ if and only if $T \models \forall x(\varphi(x) \leftrightarrow \psi(x))$.)

☐ Theorem 45

Remark 47. The proof of Ryll-Nardzewski shows more. If T is countable and complete, then the following are equivalent:

- T is $\aleph_0$-categorical.
- $S_n(T)$ is finite for all $n \geq 0$.
- All countable models are ω-saturated.

We also get

Corollary 48 (4.3.7)**.** Th$(\mathcal{A})$ *is $\aleph_0$-categorical if and only if* Th$(\mathcal{A}_B)$ *is $\aleph_0$-categorical for any finite $B \subseteq A$.*

Definition 49. A theory T is *small* if $S_n(T)$ is countable for all $n < \omega$.

Lemma 50 (4.3.9)**.** *T is small if and only if there is a countable, ω-saturated model.*

Example 51. ACF_0 is not $\aleph_0$-categorical, as remarked before. It is, however, small, since $S_n(\mathrm{ACF}_0)$ is in bijection with $\mathrm{Spec}(\mathbb{Q}[x_1, \dots, x_n])$, and the latter is countable by the Hilbert basis theorem. We will see in the homework that $\mathbb{Q}(t_1, \dots)^{\mathrm{alg}}$ is a countable ω-saturated model.

Proof of Lemma 50.

($\impliedby$) If $\mathcal{M} \models T$ is ω-saturated, then any type in $S_n(T)$ is realized in $\mathcal{M}$. But $\mathcal{M}$ is countable; so $|S_n(T)| \leq \aleph_0$.

($\implies$) Let $\mathcal{A}_0 \models T$ be countable. Recursively construct an elementary chain of countable models $\mathcal{A} = \mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \dots$ such that $\mathcal{A}_{i+1}$ realizes every 1-type over finitely many parameters in $\mathcal{A}_i$.

Claim 52. *There are only countably many 1-types over finite sets in $\mathcal{A}_i$; i.e.*

$$\left| \bigcup_{B \subseteq_{\mathrm{fin}} A_i} S_1^{\mathcal{A}_i}(B) \right| \leq \aleph_0$$

Proof. Suppose $B \subseteq_{\mathrm{fin}} A_i$.

Claim 53. Th$((\mathcal{A}_i)_B)$ *is also small.*

Proof. Suppose $q(x_1, \dots, x_n) \in S_n^{\mathcal{A}_i}(B)$ where $B = \{ b_1, \dots, b_\ell \}$. Then $q(x_1, \dots, x_n) = p(x_1, \dots, x_n, b_1, \dots, b_\ell)$ for some $p(x_1, \dots, x_n, y_1, \dots, y_\ell) \in S_{n+\ell}(T)$. ☐ Claim 53

This

$$\bigcup_{B \subseteq_{\text{fin}} A_i} S_1^{\mathcal{A}_i}(B)$$

is countable. $\square$ Claim 52

Let this set be $\{ p_1, \ldots, p_n \}$. Use downward Löwenheim-Skolem to realize them:

$$\mathcal{A}_i \preceq \mathcal{A}_i^{(1)} \preceq \ldots$$

where $\mathcal{A}_i^{(j)}$ is countable and realizes p_j. Let

$$\mathcal{A}_{i+1} = \bigcup_j \mathcal{A}_i^{(j)}$$

So $\mathcal{A}_{i+1} \succeq \mathcal{A}_i$ is countable, and satisfies the desired properties. Finally, set

$$\mathcal{A} = \bigcup_i \mathcal{A}_i$$

Then $\mathcal{A}$ is countable, and $\mathcal{A} \models T$ as $\mathcal{A} \succeq \mathcal{A}_0$; furthermore, $\mathcal{A}$ is ω-saturated by construction.

$\square$ Lemma 50

Example 54.

1. DLO is $\aleph_0$-categorical. The unique countable model is $(\mathbb{Q}, <)$; it is then ω-saturated.

2. For F a finite field, let $L = \{ 0, +, -, \lambda_f : f \in F \}$. Let T be the theory of infinite vector spaces over F. Then T is $\aleph_0$-categorical, and its unique countable model is

$$F^\omega = \oplus_{i<\omega} F$$

   which is then ω-saturated.

3. Let F be countably infinite; then this doesn't work, as $F \ncong F \times F$. It has a countably ω-saturated model: namely, the one of dimension $\aleph_0$. (This follows from the homework problem.) Thus the theory of infinite vector spaces over F is small.

4. ACF_0 is not $\aleph_0$-categorical, as seen previously, but it is small.

5. RCF is not small.

Theorem 55 (Vaught)**.** *Suppose T is a countable, complete theory. Then T cannot have precisely 2 countable models.*

Proof. If there were such a theory T, it would have to be small, since every type in $S_n(T)$ is realized in some countable model, and there are only 2 countable models; so there are only countably many n-types. Furthermore, T is not $\aleph_0$-categorical.

Claim 56. *Every small theory T that is small and not $\aleph_0$-categorical has at least three models.*

Proof. By smallness, there is a countable, ω-saturated $\mathcal{A} \models T$. Since T is not $\aleph_0$-categorical, Ryll-Nardzewski yields that there is a non-isolated n-type $p(x) \in S_n(T)$. By the omitting types theorem and downward Löwenheim-Skolem, we have a countable $\mathcal{B} \models T$ omitting $p(x)$; then $\mathcal{B} \ncong \mathcal{A}$.

Let $a = (a_1, \ldots, a_n) \in A^n$ realize $p(x)$. Then $\text{Th}(\mathcal{A}, a_0, \ldots, a_n)$ is not $\aleph_0$-categorical, since $\text{Th}(\mathcal{A}) = T$ is not. (This follows from Ryll-Nardzewski.) Let $(\mathcal{C}, c_1, \ldots, c_n) \equiv (\mathcal{A}, a_1, \ldots, a_n)$ satisfy $(\mathcal{C}, c_1, \ldots, c_n)$ is countable and not ω-saturated. So $\mathcal{C}$ is not ω-saturated. So $\mathcal{C} \ncong \mathcal{A}$. But $(c_1, \ldots, c_n)$ realize $p(x)$; so $\mathcal{C} \ncong \mathcal{B}$. $\square$ Claim 56

$\square$ Theorem 55

2.5 Section 4.5

We assume throughout that T is countable and consistent.

Definition 57. $\mathcal{A} \models T$ is *atomic* if for all $n \in \mathbb{N}$, we have that all the n-types over $\emptyset$ realized in $\mathcal{A}$ are isolated.

Remark 58. When T is complete, this says that $\mathcal{A}$ is "minimal" in the sense that it only realizes the types that it has to.

Definition 59. A *prime model* of T is one which elementarily embeds into every model of T.

Remark 60. This is a "minimum" model with respect to $\preceq$.

Remark 61.

1. Prime models need not exist.
2. Suppose $\mathcal{A}$ is a prime model of T. Then
   (a) $\mathcal{A}$ is countable since downward Löwenheim-Skolem implies that T has a countable model.
   (b) $\mathcal{A}$ is atomic since every non-isolated type is omitted in some model of T, and hence in $\mathcal{A}$.

Theorem 62 (4.5.2)**.** *Suppose T is complete. Then a model of T is prime if and only if it is countable and atomic.*

Proof.

($\Longrightarrow$) Done.

($\Longleftarrow$) Suppose $\mathcal{M}_0 \models T$ is countable and atomic. Suppose $\mathcal{M} \models T$. Let $\mathcal{F}$ be the set of all finite partial elementary maps $f\colon B \to M$ from $\mathcal{M}_0$ to $\mathcal{M}$ where $B \subseteq_{\text{fin}} M_0$. Since $\mathcal{M}_0 \equiv \mathcal{M}$ as T is complete, we have that the empty function is in $\mathcal{F}$. Note also that if $f_0 \subseteq f_1 \subseteq \ldots$ are in $\mathcal{F}$, then

$$\bigcup_{i\in\mathbb{N}} f_i$$

is a partial elementary map. So, as M_0 is countable, it suffices to show that given $f\colon B \to M$ in $\mathcal{F}$ and $a \in M_0$, we can extend f to a partial elementary map on $B \cup \{ a \}$.

Exercise 63. If $\mathcal{A}$ is an atomic model of T then all n-types over finite sets that are realized in $\mathcal{A}$ are isolated.

Consider $p(x) = \operatorname{tp}(a/B)$; this is realized, so the above exercise implies that it is isolated. Thus $f(p)$ is isolated in $\mathcal{M}$, and it is realized in $\mathcal{M}$, say by c; we then extend f by $a \mapsto c$. This completes our construction of an elementary embedding $\mathcal{M}_0 \to \mathcal{M}$.

$\square$ Theorem 62

Remark 64. There is something common in the proofs of 4.3.3 and 4.5.2. In both cases, we had a finite partial elementary map $f\colon A \to N$ from $\mathcal{M} \to \mathcal{N}$ with $A \subseteq_{\text{fin}} M$ and $a \in M$, and we needed to extend f to $A \cup \{ a \}$. This is equivalent to finding a realization of $f(\operatorname{tp}(a/A))$. There are two extreme reasons why this might be possible:

1. $\mathcal{N}$ realizes all types over finite sets; i.e. $\mathcal{N}$ is ω-saturated.
2. $\operatorname{tp}(a/A)$, and hence $f(\operatorname{tp}(a/A))$ are isolated; i.e. $\mathcal{M}$ is atomic.

So prime models and countable ω-saturated models are opposites, but in some ways behave similarly.

Definition 65. An L-structure $\mathcal{M}$ is called *ω-homogeneous* if every finite partial elementary map (i.e. whose domain is finite) $f\colon A \to M$ from $\mathcal{M} \to \mathcal{M}$ and any $a \in M$, we can extend f to a partial elementary map on $A \cup \{ a \}$.

Remark 66. If $\mathcal{M}$ is countable, then ω-homogeneity implies that we can extend f to an automorphism of $\mathcal{M}$. ($\mathcal{M}$ is *strongly ω-homogeneous.*) The proof of 4.3.3 shows that ω-saturated structures are ω-homogeneous.

TODO 1. *Am I confusing 4.3.1 and 4.3.3?*

Remark 67. The proof of Theorem 62 shows that prime models of countable, complete theories are also ω-homogeneous.

Theorem 68 (4.5.3)**.** *All prime models are isomorphic.*

Proof. We use back-and-forth as in 4.3.3 but using the fact that all the types that need to be realized are isolated because our models are atomic. $\square$ Theorem 68

What of the existence of prime models?

Remark 69. For T a countable, complete, $\aleph_0$-categorical theory, we have that the unique countably infinite $\mathcal{M} \models T$ is prime.

Proof. $S_n(T)$ is finite; so all n-types are isolated, and $\mathcal{M}$ is atomic. But $\mathcal{M}$ is countable. So $\mathcal{M}$ is prime. $\square$ Remark 69

Theorem 70 (4.5.7)**.** *A countable, complete theory T has a prime model if and only if the isolated types in $S_n(T)$ are dense for all $n \geq 1$.*

Proof.

($\implies$) Suppose $\mathcal{M} \models T$ is a prime model. Suppose $[\varphi(x)]$ is a non-empty basic clopen set, where $x = (x_1, \dots, x_n)$. We need to show that $[\varphi]$ contains an isolated type.

Well, since $[\varphi] \neq \emptyset$, we have that $\varphi(x)$ is consistent with T. So $T \models \exists x(\varphi(x))$, and we have a realization $a = (a_1, \dots, a_n) \in M^n$ of $\varphi(x)$. Then $\varphi(x) \in \mathrm{tp}(a)$, and $\mathrm{tp}(a) \in [\varphi]$. But $\mathrm{tp}(a)$ is isolated as $\mathcal{M}$ is atomic. So $[\varphi]$ contains an isolated type.

($\impliedby$) Suppose the isolated types are dense for all $n \geq 1$. Fix n, and consider $\Sigma_n(x)$ where $x = (x_1, \dots, x_n)$ given by

$$\Sigma_n(x) = \{ \neg\varphi(x) : \varphi(x) \text{ isolates a type in } S_n(T) \}$$

Claim 71. *Suppose $\mathcal{M} \models T$ omits all the $\Sigma_n(x)$; then every type realized in $\mathcal{M}$ is isolated.*

Proof. Suppose $a \in M^n$. Then a does not realize Σ_n, so a realizes $\varphi(x)$ for some $\varphi(x)$ isolating a type $q(x)$. But $\varphi(x) \in \mathrm{tp}(a)$; so $q(x) \subseteq \mathrm{tp}(a)$. So $q = \mathrm{tp}(a)$, and $\mathrm{tp}(a)$ is isolated. $\square$ Claim 71

Then such an $\mathcal{M}$ is atomic; downward Löwenheim-Skolem then yields a countable atomic model, which is then a prime model. It remains to find $\mathcal{M}$ omitting all Σ_n. We use a generalized form of the omitting types theorem that allows us to simultaneously omit countably many times; we then simply need to show that Σ_n is not isolated.

Let $\psi(x)$ be an L-formula consistent with T. We need to show that $\psi(x)$ does not isolate Σ_n. Consider $[\psi]$; by hypothesis, it contains an isolated type $p(x)$, say by $\varphi(x)$. Then $\psi(x) \in p(x)$, so $T \vdash \forall x(\varphi(x) \to \psi(x))$. Then, if $\psi(x)$ isolated $\Sigma_n(x)$, then $T \vdash \forall x(\psi(x) \to \neg\psi(x))$ since $\neg\varphi(x) \in \Sigma_n$. So $T \vdash \forall x(\varphi(x) \to \neg\varphi(x))$, contradicting our requirement that an isolating formula must be consistent. So $\psi(x)$ does not isolate Σ_n. So each $\Sigma_n(x)$ is not isolated.

Exercise 72. Generalize the proof of the omitting types theorem to simultaneously omit countably many types. Better yet, generalize the Baire category theorem proof.

$\square$ Theorem 70

Definition 73. We say a formula is *complete* if it isolates a type.

Corollary 74. *Suppose T is a countable, complete theory. If T is small, then T has a prime model. Thus $\aleph_0$-categorical implies smallness, which in turn implies the existence of a prime mode.*

Proof. Suppose T has no prime model. Then there is $n \geq 1$ such that the isolated types in $S_n(T)$ are not dense. Then there is an L-formula $\varphi(x_1, \ldots, x_n)$ such that $[\varphi(x)]$ contains no isolated types.

Claim 75. *$\varphi(x)$ is not implied by any formula which isolates a type.*

Proof. Suppose $\psi(x)$ isolates $q(x)$ and $T \vdash \forall x(\psi(x) \to \varphi(x))$. Then if $\varphi(x) \notin q(x)$, we would have $\neg\varphi(x) \in q(x)$, and thus $\psi(x) \to \neg\varphi(x)$, a contradiction. So $\varphi(x) \in q(x)$, and $q \in [\varphi]$, another contradiction. □ Claim 75

We now construct a tree of consistent formulae $\{ \varphi_s(x_1, \ldots, x_n) : s \in 2^{<\omega} \}$ such that

- $$T \vdash \forall x_1 \ldots x_n (\varphi_s(x_1, \ldots, x_n) \leftrightarrow (\varphi_{s\hat{}0}(x_1, \ldots, x_n) \vee \varphi_{s\hat{}1}(x_1, \ldots, x_n))$$
- $$T \vdash \neg\exists x_1 \ldots x_n (\varphi_{s\hat{}0}(x_1, \ldots, x_n \wedge \varphi_{s\hat{}1}(x_1, \ldots, x_n))$$

For each $\alpha \in 2^{<\omega}$, let

$$\Sigma_\alpha(x) = \{ \varphi_{\alpha\restriction n} : n < \omega \}$$

This is consistent with T as it is a nested sequence of formulae each consistent with T with

$$T \vdash \forall x_1 \ldots x_n (\varphi_{a\restriction(n-1)}(x_1, \ldots, x_n) \to \varphi_{a\restriction n}(x_1, \ldots, x_n)$$

Extend Σ_α to $p_\alpha \in S_n(T)$. If $\alpha \neq \beta$ then $p_\alpha \neq p_\beta$ because of the second condition. So

$$|S_n(T)| = 2^{\aleph_0}$$

and T is not small □ Corollary 74

Example 76. Let $L = \{ P_s : s \in 2^{<\omega} \}$ be a collection of unary predicates. Let T consist of the sentences

- $\forall x(P_\varepsilon(x))$
- $\exists^\infty x(P_s(x)$
- $\forall x((P_{s\hat{}0}(x) \vee P_{s\hat{}1}(x)) \iff P_s(x))$
- $\neg\exists x(P_{S\hat{}0}(x) \wedge P_{s\hat{}1}(x))$

for each $s \in 2^{<\omega}$. Then T is complete and has no prime model. (For this we need to show quantifier elimination.)

3 Chapter 5

We look at $\aleph_1$-categorical theories. A useful technique is indiscernible sequences.

Definition 77. Suppose $\mathcal{M}$ is an L-structure; suppose $A \subseteq M$. Suppose I is an infinite linear ordering. A sequence of k-tuples $(a_i : i \in I)$ is *indiscernible over A in $\mathcal{M}$* if

$$\mathrm{tp}(a_{i_1}, \ldots, a_{i_n}/A) = \mathrm{tp}(a_{j_1}, \ldots, a_{j_n}/A)$$

for all $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ and all $n < \omega$. This is sometimes called *order-indiscernible*. If we omit A, we mean $A = \emptyset$.

Remark 78. If $a_i = a_j$ for some $i < j$, then $a_i = a_j$ for all i and j.

Definition 79. Suppose I is an infinite linear order. Suppose $(a_i : i \in I)$ is a sequence of k-tuples in $\mathcal{M}$. The *Ehrenfeucht-Mostowski type* is

$$\begin{aligned}\mathrm{EM}((a_i : i \in I)/A) = \{&\varphi(x_1, \ldots, x_n) : n < \omega, \varphi \text{ an } L(A)\text{-formula}, \\ &\mathcal{M} \models \varphi(a_{i_1}, \ldots, a_{i_n}) \text{ for all } i_1 < \cdots < i_n \text{ in } I\}\end{aligned}$$

Remark 80. $(a_i : i \in I)$ is indiscernible over A if and only if

$$\mathrm{EM}((a_i : i \in I)/A) = \bigcup_{n<\omega} \mathrm{tp}(a_0 \dots a_{n-1}/A)$$

(We have to be a bit careful if $I \not\supseteq \mathbb{N}$, but the point is to pick any sequence in I.)

Lemma 81 (Standard lemma)**.** *Suppose $\mathcal{N}$ is an L-structure; suppose J is an infinite linear ordering. Suppose $(b_j : j \in S)$ is a sequence of k-tuples in N. Given an infinite linear ordering I, there exists $\mathcal{M} \equiv \mathcal{N}$ with an indiscernible sequence $(a_i : i \in I)$ in $\mathcal{M}$ realizing* $\mathrm{EM}((b_j : j \in J))$. *That is, if $\varphi(x_1, \dots, x_n)$ is true in $\mathcal{N}$ of all $(b_{j_1}, \dots, b_{j_n})$ with $j_1 < \dots < j_n$, then $\varphi(x_1, \dots, x_n)$ is true of all (equivalently, some) increasing $(a_{i_1}, \dots, a_{i_n})$.*

Remark 82.

- We can do this over parameters by working in $L(A)$.
- In particular, if T is a theory with an infinite model, then for any infinite linear ordering I, we have that there is a model of T with an indiscernible sequence $(a_i : i \in I)$ with all a_i distinct.

  Proof. Suppose $\mathcal{N} \models T$ is infinite. Let $(b_i : i < \omega)$ be a sequence of distinct elements of N. Applying the standard lemma, we get $\mathcal{M} \equiv \mathcal{N}$ (so $\mathcal{M} \models T$) and $(a_i : i \in I)$ is indiscernible. Furthermore, we have $a_i \neq a_j$ for all $i < j$ in I since $(x_1 \neq x_2) \in \mathrm{EM}((b_j : j < \omega))$. □

The main tool in proving Lemma 81 is the following:

Theorem 83 (Ramsey's theorem)**.** *Suppose A is an infinite set; suppose $n < \omega$. Let $[A]^n = \{ B \subseteq A : |B| = n \}$. Suppose $[A]^n = C_1 \sqcup \dots \sqcup C_k$. Then there is infinite $B \subseteq A$ such that $[B]^n \subseteq C_i$ for some $i \in \{ 1, \dots, k \}$.*

Proof of Lemma 81. We assume $k = 1$; that is, we are dealing with indiscernible sequences of elements, not tuples. Let $C = (c_i : i \in I)$ be new constant symbols. It suffices to prove that the following $L(C)$-theory is consistent:

$$\begin{aligned}\mathrm{Th}(\mathcal{N}) &\cup \{ \varphi(c_{i_1}, \dots, c_{i_n}) \leftrightarrow \varphi(c_{k_1}, \dots, c_{k_n}) : i_1 < \dots < i_n, k_1 < \dots < k_n \text{ in } I, n < \omega \} \\ &\cup \{ \psi(c_{i_1}, \dots, c_{i_n}) : i_1 < \dots < i_n \text{ in } I, \psi(x_1, \dots, x_n) \in \mathrm{EM}((b_j : j \in J)), n < \omega \}\end{aligned}$$

We use a compactness argument. We are then given

- $\mathcal{N}$ an L-structure
- $(b_j : j \in J)$ a linearly ordered sequence in N
- Finitely many new constant symbols $c_1, \dots, c_\ell$
- $\Delta(x_1, \dots, x_n)$ a finite collection of L-formulae

and we wish to prove that

$$\begin{aligned}T = \mathrm{Th}(\mathcal{N}) &\cup \{ \psi(c_{i_1}, \dots, c_{i_n}) : \psi \in \mathrm{EM}_n^{\mathcal{N}}((b_j : j \in J)) 1 \le i_1 < \dots < i_n \le \ell \} \\ &\cup \{ \varphi(c_{i_1}, \dots, c_{i_n}) \leftrightarrow \varphi(c_{k_1}, \dots, c_{k_n}) : \varphi \in \Delta(x), 1 \le i_1 < \dots < i_n \le \ell, 1 \le k_1 < \dots < k_n \le \ell \}\end{aligned}$$

(where EM_n is the Ehrenfeucht-Mostowski type restricted to formulae in n free variables).

Case 1. Suppose the b_j are distinct. Let $B = \{ b_j : j \in J \}$; then this is infinite. Define on $[B]^n$ a relation $\sim$ by $\bar{b} \sim \bar{c}$ if $\mathcal{N} \models \varphi(\bar{b}) \leftrightarrow \varphi(\bar{c})$ for all $\varphi \in \Delta$, all increasing enumerations $\bar{b}$, $\bar{c}$ of n-element subsets of B. This is then an equivalence relation with at most $2^{|\Delta|}$-many classes. Then, by Ramsey's theorem, there is $B' = \{ b_{j_1}, \dots, b_{j_\ell} \} \subseteq B$ such that any two increasing n-tuples from B' realize the same formulae from Δ. So

$$(\mathcal{N}, b_{j_1}, \dots, b_{j_\ell}) \models T$$

Case 2. Suppose the b_j are not distinct but B is infinite. Then we can throw away the repetitions and apply the previous case.

Case 3. Suppose B is finite. Then there exists $j_1 < \cdots < j_\ell$ in J such that $b_{j_1} = \cdots = b_{j_\ell} = b$. So $(\mathcal{N}, b, \ldots, b) \models T$.

$\square$ Lemma 81

Lemma 84 (5.1.6)**.** *Suppose L is countable; suppose $\mathcal{A}$ is an L-structure generated by a well-ordered indiscernible sequence $(a_i : i \in I)$. Then for all $n \geq 1$, we have that $\mathcal{A}$ realizes only countably many n-types over any countable set.*

Proof. Every element of A is of the form $t(a^\alpha)$ where t is an n-ary L-term and $a^\alpha = (a_{\alpha_1}, \ldots, a_{\alpha_\ell}) \in I^\ell$. Suppose $B \subseteq A$ is countable. Let $A_0 = \{ a_i : a_i \in B \}$. Then A_0 is countable, and $A_0 = \{ a_i : i \in I_0 \}$ for some $I_0 \subseteq I$.

Note that a type over A_0 has a unique extension to $A_0 \cup B$, as every $L(A_0 \cup B)$-formula is equivalent to some $L(A)$-formula. So it suffices to count the n-types over A_0 realized in $\mathcal{A}$.

Assume $n = 1$. Let $\mathrm{tp}^{\mathcal{A}}(c/A_0)$ be such a type. Then $c \in A$, so $c = t(a^\alpha)$ for some t, α as above. Then $\mathrm{tp}(c/A_0)$ is determined by $\mathrm{tp}(a^\alpha, A_0)$ and t. But there are countably many L-terms t; so it suffices to count the $\mathrm{tp}(a^\alpha/A_0)$. By indiscernibility, we have that $\mathrm{tp}(a^\alpha/A_0)$ is determined by:

- $\mathrm{tp}_{\mathrm{qf}}(\alpha)$ in the structure $(I, <)$
- $\mathrm{tp}_{\mathrm{qf}}(\alpha_i/I_0)$ in the structure $(I, <)$

But there are finitely many of the first, and countably many of the second. So there are only countably many of these. $\square$ Lemma 84

Corollary 85 (5.1.9)**.** *Suppose T is a countable theory with an infinite model. Suppose κ is an infinite cardinal. Then there is $\mathcal{M} \models T$ with $|M| = \kappa$ such that $\mathcal{M}$ realizes only countably many 1-types over any countable set.*

The proof uses *Skolemization.* Given a language L and an L-theory T, we construct $L = L_0 \subseteq L_1 \subseteq \ldots$ such that for each quantifier-free L_i-formula $\varphi(x, y)$ with y a single variable, $x = (x_1, \ldots, x_n)$, we let

$$L_{i+1} = L_i \cup \{ f_\varphi(x) : \varphi(x, y) \text{ a quantifier-free } L_i\text{-formula} \}$$

where f_φ is an n-ary function symbol. We let

$$L_{\mathrm{Skolem}} = \bigcup_{i<\omega} L_i$$

Let

$$T^* = T \cup \{ \forall x (\exists y \varphi(x, y) \to \varphi(x, f_\varphi(x))) : \varphi(x, y) \in L_{\mathrm{Skolem}} \}$$

Remark 86 (Properties of T^*)*.*

- T^* admits quantifier elimination.
- Every model of T can be expanded to a model of T^*.
- T^* is a universal theory, as the new axioms are universal and modulo the new axioms we have that T is quantifier-free.
- T^* is countable.

Proof of Corollary 85. Let T^* be the Skolemization of T. By the standard lemma, there is $\mathcal{M} \models T^*$ with an indiscernible sequence $(a_i : i < \kappa)$ of distinct elements indexed by κ. Let $\mathcal{N}^* = \langle a_i : i < \kappa \rangle \subseteq \mathcal{M}^*$. Then T^* is universal, so $\mathcal{N}^* \models T^*$. (Note that $\mathcal{N}^*$ is only generated by $(a_i : i < k)$ as an L^*-structure; not as an L-structure.) Then, by the previous theorem, we get that $\mathcal{N}^*$ realizes only countably many types over countably many parameters. But complete types in $\mathcal{N}$ are partial types of $\mathcal{N}^*$, which can then be extended to distinct complete types in $\mathcal{N}^*$. So $\mathcal{N}$ realizes only countably many types. $\square$ Corollary 85

Definition 87. Suppose κ is an infinite cardinal. Suppose T is a complete theory with infinite models. We say T is *κ-stable* if for any $\mathcal{M} \models T$ and any $A \subseteq M$ with $|A| \leq \kappa$, we have that $|S_n(A)| \leq \kappa$ for all $n < \omega$.

Remark 88. ω-stable implies small.

Example 89. ACF_0 are ω-stable, as $S_n(A)$ is in bijection with $\mathrm{Spec}(\mathbb{Q}(A)[x_1, \ldots, x_n]$. Thus if $|A| \leq \aleph_0$, then $|\mathbb{Q}(A)| \leq \aleph_0$; so $|\mathbb{Q}(A)[x_1, \ldots, x_n]| \leq \aleph_0$, and $|S_n(A)| \leq \aleph_0$.

Theorem 90 (5.2.2)**.** *T is κ-stable if and only if for any $\mathcal{M} \models T$ and any $A \subseteq M$ with $|A| \leq \kappa$, we have $|S_1(A)| \leq \kappa$.*

Proof. Induction on n. Suppose $n \geq 1$. Consider the restriction map $\pi \colon S_n(A) \to S_1(A)$. Let $p \in S_1(A)$. Then for some $\mathcal{N} \succeq \mathcal{M}$, we have $p = \mathrm{tp}(b/A)$ for some $b \in N$. Note that $S_n^{\mathcal{M}}(A) = S_n^{\mathcal{N}}(A)$. Then, by homework the first, we have

$$\pi^{-1}(p(x)) \cong S_{n-1}(bA)$$

which has cardinality $\leq \kappa$, by induction hypothesis. Also, by assumption, we have that the image of π has size $\leq \kappa$. So the fibres and image of π have size $\leq \kappa$. So $|S_n(A)| \leq \kappa$. □ Theorem 90

Example 91. DLO is small (in fact, $\aleph_0$-categorical) but not ω-stable: $S_1^{\mathbb{Q}}(\mathbb{Q})$ is in bijection with $\mathbb{R}$.

Example 92. The theory of infinite vector spaces over a field F is ω-stable if F is countable.

Theorem 93 (5.2.4)**.** *Suppose T is countable and complete and has infinite models. If T is κ-categorical for $\kappa > \aleph_0$, then T is ω-stable.*

Proof. Suppose T is not ω-stable; we get $\mathcal{M} \models T$ and $A \subseteq M$ with $|A| \leq \aleph_0$ but $|S_1(A)| > \aleph_0$. Let $\mathcal{N} \succeq \mathcal{M}$ realizes $\aleph_1$-many distinct 1-types over A; say we have $b_i \in N$ for $i < \aleph_1$ with $\mathrm{tp}(b_i/A) \neq \mathrm{tp}(b_j/A)$ for $i < j < \aleph_1$. By upward Löwenheim-Skolem, we may assume $|\mathcal{N}| \geq \kappa$. By downward Löwenheim-Skolem, we have $\mathcal{N}_0 \preceq \mathcal{N}$ with $|N_0| = \kappa$ and $A \subseteq N_0$, $b_i \in N_0$ for all $i < \aleph_1$. (Possible since $\kappa > \aleph_0$ and $|A \cup \{ b_i : i < \aleph_1 \}| = \aleph_1$.) So we have a model of size κ realizing $\aleph_1$-many types over a countable set (namely A). But by Corollary 85, we have $\mathcal{B} \models T$ of size κ such that over any countable subset of B, there are only countably many realized types. So $\mathcal{B} \not\cong \mathcal{N}_0$, and T is not κ-categorical. □ Theorem 93

Assignment 2. *Homework 2, due Wednesday October 21, is the following exercises from the book: 4.3.1, 4.3.7, 4.5.1, 5.1.1, and 5.2.2.*

From now on, when we say T is a complete theory, it is implied that T has only infinite models.

Theorem 94 (5.2.6)**.** *Suppose T is countable and complete. Then the following are equivalent:*

1. *T is ω-stable.*
2. *No model $\mathcal{M} \models T$ has an infinite binary tree of consistent $L(M)$-formulae.*
3. *T is κ-stable for any cardinal $\kappa \geq \aleph_0$.*

Proof.

(1) $\implies$ **(2)** Let S be such a tree; let A be a countable set of parameters such that all the formulae in S are over A. (Possible since S is countable.) Each branch is a partial n-type over A that extends to an element of $S_n(A)$. They are all distinct; so there are $2^{\aleph_0}$-many of them. So T is not ω-stable.

(2) $\implies$ **(3)** Suppose T is not κ-stable for some $\kappa \geq \aleph_0$. Then we have $\mathcal{M} \models T$ and $A \subseteq M$ with $|A| \leq \kappa$ and $|S_1(A)| > \kappa$. But there are only κ-many $L(A)$-formulae. So there is an $L(A)$-formula $\varphi(x)$ such that $\varphi(x)$ is contained in $> \kappa$-many distinct 1-types over A. We call such a formula *big*.

Remark 95. If

$$\Gamma = \{ p \in S_1(A) : p \text{ contains a formula that is not big} \}$$

then $|\Gamma| \leq \kappa$.

So there are $p, q \in S_1(A)$ such that $p \neq q$, $\varphi(x) \in p \cap q$, and every formula in $p(x)$ or in $q(x)$ is big. So we get $\varphi_0(x)$ and $\varphi_1(x)$ both big such that $\mathcal{M} \models \varphi(x) \leftrightarrow \varphi_0(x) \vee \varphi_1(x)$ and $\mathcal{M} \models \neg\exists x(\varphi_0(x) \wedge \varphi_1(x))$. Iterate to get an infinite binary tree of big formulae over A.

(3) $\implies$ (1) Clear.

$\square$ Theorem 94

Recall from Ryll-Nardzewski that $\aleph_0$-categoricity is equivalent to all countable models being ω-saturated.

Theorem 96 (5.2.11)**.** *Suppose T is countable, κ an infinite cardinal. Then T is κ-categorical if and only if all models of size κ are κ-saturated.*

We need some lemmata.

Definition 97. An L-structure $\mathcal{A}$ is *saturated* if it is $|A|$-saturated.

Lemma 98 (5.2.9)**.** *Suppose T is countable, complete, and ω-stable. For all κ and all regular $\lambda \leq \kappa$, we have that T has a model of size κ that is λ-saturated.*

Proof. We try to construct as usual a λ-saturated model. Let $\mathcal{M}_0 \models T$, $|M_0| = \kappa$. Let $\mathcal{M}_1 \succeq \mathcal{M}_0$ realize all types in $S_1(M_0)$. But since ω-stability implies κ-stability, we know that $|S_1(M_0)| = \kappa$. By downward Löwenheim-Skolem, we may assume that $|M_1| = \kappa$; now iterate λ-many times, where for limit ordinal β we let

$$\mathcal{M}_\beta = \bigcup_{\gamma<\beta} \mathcal{M}_\gamma$$

We then obtain $(\mathcal{M}_\alpha : \alpha < \lambda)$ an elemntary chain of models of T, all of size κ, such that every type in $S_1(M_\alpha)$ is realized in $\mathcal{M}_{\alpha+1}$. Let

$$\mathcal{M} = \bigcup_{\alpha<\lambda} \mathcal{M}_\alpha$$

Then $\mathcal{M} \models T$, and $|M| = \kappa$, since $\lambda \leq \kappa$. Let $A \subseteq M$ satisfy $|A| < \lambda$; let $p \in S_1(A)$. By regularity of λ, we have that $A \subseteq M_\alpha$ for some $\alpha < \lambda$. So p is realized in $\mathcal{M}_{\alpha+1}$, and hence in $\mathcal{M}$. So $\mathcal{M}$ is λ-saturated.
$\square$ Lemma 98

Proof of Theorem 96.

($\Longleftarrow$) Suppose all models of size κ are saturated. In general, if $\mathcal{A} \equiv \mathcal{B}$, $|A| = |B| = \kappa$, and $\mathcal{A}$ and $\mathcal{B}$ are κ-saturated, then $\mathcal{A} \cong \mathcal{B}$. This is proven by a back-and-forth argument as in the case of $\kappa = \omega$ (4.3.3); the only difference is that the partial elementary maps we must extend have domains of size $< \kappa$ (rather than finite). So T is κ-categorical.

($\Longrightarrow$) Suppose T is κ-saturated; let $\mathcal{M}$ be the model of T of cardinality κ. We need to show that $\mathcal{M}$ is κ-saturated. If $\kappa = \aleph_0$, we are done by Ryll-Nardzewski. We may thus assume $\kappa > \aleph_0$. By Theorem 93, we have that T is ω-stable. By 5.2.9, we have that T has a model of size κ that is λ-saturated for all regular $\lambda \leq \kappa$. So $\mathcal{M}$ is λ-saturated for all regular $\lambda \leq \kappa$.

Case 1. Suppose κ is a successor cardinal. Then κ is regular, and we may take $\lambda = \kappa$ to get that $\mathcal{M}$ is κ-saturated.

Case 2. Suppose κ is a limit cardinal. Let $A \subseteq M$, $|A| < \kappa$, $p \in S_1(A)$. So $|a| < \lambda$ for some $\lambda < \kappa$. So $|A| < \lambda^+ < \kappa$, and λ^+ is regular. So $\mathcal{M}$ is λ^+-saturated, so p is realized in $\mathcal{M}$.

$\square$ Theorem 96

Definition 99. Suppose $\mathcal{B}$ is an L-structure; suppose $A \subseteq B$. We say $\mathcal{B}$ is *prime over* A (or a *prime extension of* A) if every partial elementary map $A \to \mathcal{M}$ extends to an elementary embedding $\mathcal{B} \to \mathcal{M}$.

Remark 100. $\mathcal{B}$ is prime over A if and only if $\mathcal{B}_A$ is a prime model of $\mathrm{Th}(\mathcal{B}_A)$. (Recall $\mathcal{M}$ expands to a model of $\mathrm{Th}(\mathcal{B}_A)$ if and only if there exists a partial elementary map $A \to \mathcal{M}$.)

Example 101. Suppose $(K, 0, 1, +, -, \times) \models \mathrm{ACF}_0$; suppose $A \subseteq K$. Then $\mathbb{Q}(A)^{\mathrm{alg}}$ is prime over A.

Theorem 102 (5.3.3)**.** *Suppose T is countable, complete, and ω-stable. Then, given any $\mathcal{M} \models T$ and $A \subseteq M$, there is a model of T that is prime over A.*

Proof. We will construct $\mathcal{B} \preceq \mathcal{M}$ with $A \subseteq B$ such that B has an enumeration $(b_\alpha : \alpha < \lambda)$ with $\mathrm{tp}(b_\alpha / A \cup \{ b_\mu : \mu < \alpha \})$ is isolated. Such a structure is called *constructible over A*.

Claim 103. *Constructible extensions are prime. (Compare to "atomic implies prime".)*

Proof. Suppose $f : A \to \mathcal{N}$ is a partial elementary map, where $\mathcal{N}$ is any L-structure. We wish to extend f to B. We do so recursively to all the b_μ with $\mu < \alpha$ with $\alpha < \lambda$. Suppose we have extended f to act on $A \cup \{ b_\mu : \mu < \alpha \}$. Well,

$$p(x) = \mathrm{tp}(b_\alpha / A \cup \{ b_\mu : \mu < \alpha \})$$

is isolated in $\mathcal{B}$. So $f(p)$ is isolated in $\mathcal{N}$, as f is a partial elementary map; so it is realized in $\mathcal{N}$, say by c. We then extend f by $b_\alpha \mapsto c$. □ Claim 103

Note that the above claim doesn't require ω-stability; by contrast, the following claim relies on ω-stability.

Claim 104. *For any $C \subseteq M$ and any $n \geq 0$, we have that the isolated types are dense in $S_n(C)$. (Compare to "small implies the existence of a prime model".)*

Proof. Suppose $C \subseteq M$; suppose $n \geq 0$. Consider $\mathrm{Th}(\mathcal{M}_C)$. Since T is ω-stable, 5.2.6 yields that there is no infinite binary tree of consistent $L(C)$-formulae. Then, by 4.5.9, we have that the isolated types are dense in $S_n(\mathrm{Th}(\mathcal{M}_C))$. (Despite how it was done in class, the step above doesn't need the language to be countable.) So the isolated types are dense in $S_n(C)$. □ Claim 104

We now construct the constructible $\mathcal{B}$ over A. By Zorn's lemma, there is $B = (b_\alpha : \alpha < \lambda)$ with $\mathrm{tp}(b_\alpha / A \cup \{ b_\mu : \mu < \alpha \})$ is isolated and maximal; i.e. whenever $a \in \mathcal{M} \setminus B$, we have that $\mathrm{tp}(a / A \cup B)$ is not isolated. Clearly $A \subseteq B$. We wish to prove that B is the universe of an elementary substructure of $\mathcal{M}$. We use Tarski-Vaught. Let $\varphi(x)$ be an $L(B)$-formula in 1 variable such that $\mathcal{M} \models \exists x \varphi(x)$. We need to show that there is $b \in B$ with $\mathcal{M} \models \varphi(b)$. By the second claim, we have that $[\varphi(x)]$ contains an isolated type $p(x) \in S_1(B)$. Let $a \in M$ realize $p(x)$. So $\mathrm{tp}(a / A \cup b) = \mathrm{tp}(a/B) = p(x)$ is isolated. Then, by maximality, we have $a \in B$, and $\mathcal{M} \models \varphi(a)$. So we have constructed our constructible $\mathcal{B}$ over A. Then by the first claim, we have that B is prime over A. □ Theorem 102

Actually, the proof gave us a *constructible* model over any subset of a model (if T is ω-stable), not just a prime one.

Theorem 105 (5.3.6)**.** *A constructible extension $\mathcal{B}$ over A is atomic over A; i.e. for every $n \geq 0$, we have that every n-type over A realized in $\mathcal{B}$ is isolated.*

In fact, "constructible over A" and "atomic over A" are the same; this uses

Lemma 106 (5.3.5)**.** *In any L-structure, we have that* $\mathrm{tp}(ab)$ *is isolated if and only if* $\mathrm{tp}(a/b)$ *and* $\mathrm{tp}(b)$ *are isolated.*

Proof. ($\Longrightarrow$) If $\varphi(x, y)$ isolated $\mathrm{tp}(ab)$ then $\varphi(x, b)$ isolates $\mathrm{tp}(a/b)$, and $\exists x \varphi(x, y)$ isolates $\mathrm{tp}(b)$.

($\Longleftarrow$) If $\varphi(x, b)$ isolates $\mathrm{tp}(a/b)$ and $\psi(y)$ isolates $\mathrm{tp}(b)$, then $\varphi(x, y) \wedge \psi(y)$ isolates $\mathrm{tp}(ab)$. □ Lemma 106

Proof of Theorem 105. Suppose $\mathcal{B} = (b_\alpha : \alpha < \lambda)$ is a constructible extension of A. Given $b = (b_{\alpha_1}, \ldots, b_{\alpha_n})$ with $\alpha_1 < \cdots < \alpha_n$, we need to show that $\mathrm{tp}(b/A)$ is isolated. Well,

$$\mathrm{tp}(b_{\alpha_n} / A \cup \{ b_\mu : \mu < \alpha_n \})$$

is isolated, say by $\varphi(x, c)$ where c is a tuple from $A \cup \{ b_\mu : \mu < \alpha_n \}$. So

$$\mathrm{tp}(b_{\alpha_n} / A_c \cup \{ b_{\alpha_1}, \ldots, b_{\alpha_{n-1}} \})$$

By induction on α_n, we know that $\mathrm{tp}(c, b_{\alpha_1}, \dots, b_{\alpha_{n-1}}/A)$ is isolated. (Formally, we're doing induction on the highest index α_n.) By 5.3.5 for $L(A)$-structure, we have

$$\mathrm{tp}((c, (b_{\alpha_1}, \dots, b_{\alpha_{n-1}}, b_{\alpha_n})$$

is isolated. Again by 5.3.5, we have that $\mathrm{tp}(b/A)$ is isolated. $\square$ Theorem 105

Definition 107. A theory T is *totally transcendental* if for every $\mathcal{M} \models T$ there does not exist an infinite binary tree of $L(M)$-formulae realized in $\mathcal{M}$. (T may be incomplete, and L may be uncountable.)

Remark 108. We know that when L is countable and T is complete, then total transcendence is equivalent to ω-stability.

Rephrasing the previous theorem, we have

Theorem 109. *Suppose T is complete and totally transcendental; suppose $\mathcal{M} \models T$ and $A \subseteq M$. Then there exists $\mathcal{B} \preceq \mathcal{M}$ such that $\mathcal{B}$ is a prime extension of A. (This is stronger than the analogous statement in Tent and Ziegler.)*

Remark 110. The proof actually found $\mathcal{B} \preceq \mathcal{M}$ constructible over A; we saw that this is the atomic over A.

Corollary 111 (3.5.7)**.** *Suppose T is complete and totally transcendental. Suppose $\mathcal{B} \models T$, $A \subseteq B$, and $\mathcal{B}$ is prime over A. Then $\mathcal{B}$ is atomic over A.*

Proof. We know there is $\mathcal{B}_0 \preceq \mathcal{B}$ such that $\mathcal{B}_0$ is atomic over A. So $\mathrm{id}\colon A \to \mathcal{B}$ is a partial elementary map $\mathcal{B}_0 \to \mathcal{B}$, since $\mathcal{B}_0 \preceq \mathcal{B}$. Since $\mathcal{B}$ is prime over A, we have that id_A extends to an elementary embedding $f\colon \mathcal{B} \to \mathcal{B}_0$. So $\mathcal{B}$ is isomorphic to A to an elementary substructure of $\mathcal{B}_0$. So $\mathcal{B}$ is atomic over A. $\square$ Corollary 111

Theorem 112 (Lachlan's theorem)**.** *Suppose T is a complete, totally transcendental theory; suppose $\mathcal{M} \models T$ is uncountable. Then $\mathcal{M}$ has arbitrarily large elementary extensions which omit any countable partial 1-type over M that $\mathcal{M}$ omits. (i.e. for any κ there is $\mathcal{N} \succeq \mathcal{M}$ with $|N| \geq \kappa$ having the desired property.)*

Proof. By iteration, it suffices to show that there is a proper elementary extension of $\mathcal{M}$ omitting all countable partial types omitted by $\mathcal{M}$.

We call an $L(M)$-formula $\varphi(x)$ is *large* if $\varphi(\mathcal{M})$ is uncountable. By total transcendentality, there is a "minimal" large formula: there is large $\varphi_0(x)$ large such that for any $L(M)$-formula $\psi(x)$, we have either $\varphi_0 \wedge \psi$ or $\varphi_0 \wedge \neg\psi$ is not large (and hence the other is). Let $p(x) = \{\, \psi(x) : \varphi_0 \wedge \psi \text{ is large} \,\}$.

Claim 113. $p(x) \in S_1(M)$.

Proof. Observe that it is closed under conjunction, since if $\psi_1(x), \psi_2(x) \in p(x)$, then $\varphi_0 \wedge \psi_1$ and $\varphi_0 \wedge \psi_2$ are large. So $\varphi_0 \wedge \neg\psi_1$ and $\varphi_0 \wedge \neg\psi_2$ are not large. So $\varphi_0 \wedge (\neg\psi_1 \vee \neg\psi_2)$ is not large. So $\varphi_0 \wedge \psi_1 \wedge \psi_2$ is large.

Furthermore, $p(x)$ is consistent and complete. So $p(x) \in S_1(M)$. $\square$ Claim 113

Claim 114. *$p(x)$ is not realized in $\mathcal{M}$, but every countable subset of $p(x)$ is realized in $\mathcal{M}$.*

Proof. If $p(x)$ were realized, say by $a \in M$, then $(x = a) \in p(x)$. But $\varphi_0 \wedge (x = a)$ is not large, a contradiction. So $p(x)$ is not realized in M.

Suppose $\Pi(x) \subseteq p(x)$ is countable. For all $\psi \in \Pi$, we ahve $\varphi_0(\mathcal{M}) \setminus \psi(\mathcal{M})$ is countable. So $\varphi_0(\mathcal{M}) \setminus \Pi(\mathcal{M})$ is countable. So $\Pi(\mathcal{M})$ is uncountable, and hence non-empty. $\square$ Claim 114

Let $\mathcal{N} \succeq \mathcal{M}$ with $a \in N$ realizing $p(x)$. By total transcendentality, we may assume that $\mathcal{N}$ is atomic over $M \cup \{\, a \,\}$. This $\mathcal{N}$ is our desired extension; certainly by the claim, we have that $\mathcal{N} \neq \mathcal{M}$. It then suffices to show that given $b \in N$, every countable subset of $\Sigma(y) \subseteq \mathrm{tp}(b/M)$ is realized in $\mathcal{M}$. Since $\mathcal{N}$ is atomic over $M \cup \{\, a \,\}$, we have that $\mathrm{tp}(b/M \cup \{\, a \,\})$ is isolated, say by $\chi(a, y)$ where $\chi(x, y)$ is an $L(M)$-formula. Let

$$\Pi(x) = \{\, \forall y (\chi(x, y) \to \sigma(y)) : \sigma \in \Sigma \,\} \cup \{\, \exists y \chi(x, y) \,\}$$

Then $\Pi(x) \subseteq p(x)$ is countable as Σ is countable. By the claim, we have $\Pi(x)$ is realized in $\mathcal{M}$ by $a' \in M$. Let $b' \in M$ satisfy

$$\mathcal{M} \models \chi(a', b')$$

Then $\mathcal{M} \models \sigma(b')$ for all $\sigma \in \Sigma$, since $(\forall y(\chi(x,y) \to \sigma(y)) \in \Pi(x)$. So b' realizes $\Sigma(y)$ in $\mathcal{M}$. □ Theorem 112

Theorem 115 (Downward Morley's theorem, 5.4.2)**.** *Suppose T is countable and κ-categorical for some uncountable κ. Then T is $\aleph_1$-categorical.*

Proof. Suppose T is not $\aleph_1$-categorical. Then there is $\mathcal{M} \models T$ with $|M| = \aleph_1$ with $\mathcal{M}$ not $\aleph_1$-saturated. Suppose $A \subseteq M$ is countable with $p(x) \in S_1(A)$ not realized in $\mathcal{M}$. By 5.2.4, we have that T is ω-stable; so, by Lachlan's theorem there is $\mathcal{N} \succeq \mathcal{M}$ of cardinality $\geq \kappa$ omitting $p(x)$. Since $\kappa \geq |M|$, we may use downward Löwenheim-Skolem to produce such an $\mathcal{N}$ with $|N| = \kappa$.

But T is κ-categorical; so $\mathcal{N}$ is κ-saturated. But $\mathcal{N}$ does not realize $p(x)$ over countably many parameters, a contradiction. So T is $\aleph_1$-categorical. □ Theorem 115

(We use here that for infinite κ, κ-categoricity is equivalent to the saturation of all models of size κ.)

Remark 116. The uncountability of $\mathcal{M} \models T$ is necessary for Lachlan's theorem. To see this, note that ACF_0 is totally transcendental and complete, and $\mathbb{Q}^{\mathrm{alg}} \models \mathrm{ACF}_0$. The type $p(x)$ saying "x is transcendental" is a countable type omitted in $\mathbb{Q}^{\mathrm{alg}}$. But it is realized in every uncountable $\mathcal{N} \models \mathrm{ACF}_0$.

For upward Morley's theorem, we will need more than total transcendentality.

Definition 117. A *vaughtian pair* for a theory T is a pair of models $\mathcal{M} \prec \mathcal{N}$ and an $L(M)$-formula $\varphi(x)$ such that

- $\mathcal{N} \neq \mathcal{M}$
- $\varphi(\mathcal{M})$ is infinite
- $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$

Remark 118. If we allowed $\varphi(\mathcal{M})$ to be finite, then $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$ for all elementary extensions $\mathcal{N} \succeq \mathcal{M}$.

One way this can happen is if $\mathcal{N} \models T$ and $\aleph_0 \leq |\varphi(\mathcal{N})| < |N|$.

Aside 119. In a κ-saturated structure, every infinite definable set has cardinality $\geq \kappa$.

Given such φ and $\mathcal{N}$, we can use downward Löwenheim-Skolem to get $\mathcal{M} \preceq \mathcal{N}$ such that $\varphi(\mathcal{N}) \subseteq \mathcal{M}$ and $|M| = |\varphi(\mathcal{N})| < |N|$. Then $\mathcal{M} \neq \mathcal{N}$ and $\varphi(\mathcal{M}) = \varphi(\mathcal{N}) \cap M = \varphi(\mathcal{N})$. So this is a vaughtian pair.

Lemma 120 (5.5.3)**.** *Suppose T is countable and complete.*

1. *Every countable model of T has a countable ω-homogeneous elementary extension.*

   Remark 121. If T is not small, there may not be a countable ω-saturated model; this says that there is *always* a countable ω-homogeneous model.

2. *If $\mathcal{M}$ and $\mathcal{N}$ are countable ω-homogeneous models of T structures that realize the same n-types over $\emptyset$ for all n, then $\mathcal{M} \cong \mathcal{N}$.*

Proof.

1. Build it by iterating the following process: suppose $\mathcal{M} \models T$ is countable. Let $\mathcal{M}_1 \succeq \mathcal{M}$ realize

   $$\{\, f(\mathrm{tp}(a/A)) : A \subseteq_{\mathrm{fin}} M, a \in M, f\colon A \to \mathcal{M} \text{ a partial elementary map} \,\}$$

   But the above set is countable; so by downward Löwenheim-Skolem, we can get $\mathcal{M}_1$ to be countable. We iterate this $\aleph_0$-many times and take unions to get a countable, ω-homogeneous elementary extension.

2. Perform back-and-forth. Given a partial elementary map $\mathcal{M} \to \mathcal{N}$, say

$$f\colon \{ a_1, \ldots, a_m \} \to \mathcal{N}$$

We wish to extend it to $a \in M$. Let $(b_1, \ldots, b_m, b) \in N^{m+1}$ realize $\operatorname{tp}(a_1, \ldots, a_m, a) = p(x_1, \ldots, x_n, y)$. (Such a realization exists by assumption.) So $\operatorname{tp}(b_1, \ldots, b_m) = \operatorname{tp}(a_1, \ldots, a_m) = \operatorname{tp}(f(a_1, \ldots, f9a_n))$ as f is a partial elementary map. If we define $g\colon \{ b_1, \ldots, b_m \} \to \mathcal{N}$ by $g(b_i) = f(a_i)$, then this a partial elementary map from $\mathcal{N}$ to $\mathcal{N}$. As $\mathcal{N}$ is ω-homogeneous, we have that g extends to an automorphism $g\colon \mathcal{N} \to \mathcal{N}$. Then

$$\begin{aligned}\operatorname{tp}(a_1, \ldots, a_m, a) &= \operatorname{tp}(b_1, \ldots, b_m, b) \\ &= \operatorname{tp}(g(b_1), \ldots, g(b_m), g(b)) \\ &= \operatorname{tp}(f(a_1), \ldots, f(a_m), g(b))\end{aligned}$$

i.e. f extends to a partial elementary map on $\{ a_1, \ldots, a_m, a \}$ by $a \mapsto g(b)$.

□ Remark 121

Theorem 122 (Vaught's 2-cardinal theorem)**.** *Suppose T is complete and countable. If T has a vaughtian pair, then it has an $\aleph_1$-sized model with a countable infinite definable set.*

Proof.

Claim 123. *T has a vaughtian pair where $\mathcal{M}$ and $\mathcal{N}$ are countable.*

Proof. Suppose $\mathcal{M} \prec \mathcal{N}$ with $\varphi(x)$ is a vaughtian pair. Define $L(P) = L \cup \{ P \}$ where P is a unary predicate symbol. View $(\mathcal{N}, M)$ as an $L(P)$-structure where P is interpreted as M. The facts

- M is the universe of $\mathcal{M} \preceq \mathcal{N}$.
- $\mathcal{M} \neq \mathcal{N}$
- $\varphi(\mathcal{M})$ is infinite
- $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$

are part of the $L(P)$-theory of $(\mathcal{N}, M)$. Applying downward Löwenheim-Skolem, we get $(\mathcal{N}_0, M_0) \preceq (\mathcal{N}, M)$ with N_0 and M_0 countable. We then have that $\mathcal{M}_0 \preceq \mathcal{N}_0$ is a vaughtian pair for T with $\varphi(x)$.

□ Claim 123

Claim 124. *T has a countable vaughtian pair with $\mathcal{M} \cong \mathcal{N}$ and $\mathcal{M}$ and $\mathcal{N}$ are ω-homogeneous.*

Proof. By the previous claim, we have $\mathcal{M}_0 \prec \mathcal{N}_0$ a countable vaughtian pair with $\varphi(x)$. We work in $L(P)$, the language of pairs. Let $(\mathcal{N}_0, M_0) \preceq (\mathcal{N}_0', M_0')$ be countable such that every n-type (over $\emptyset$) realized by $\mathcal{N}_0$ is realized by $\mathcal{M}_0'$. We do this by taking

$$\Sigma = \operatorname{Th}(\mathcal{N}_0, M_0)_{N_0} \cup \{ p(c_1^{(p)}, \ldots, c_n^{(p)}) : p(x_1, \ldots, x_n) \in S_n(\emptyset) \text{ realized in } \mathcal{N}_0 \} \cup \{ P(c_i^{(p)}) : \text{ all } c_i^{(p)} \}$$

where the $c_i^{(p)}$ are new constant symbols. Then Σ is consistent since if $\psi(x_1, \ldots, x_n) \in \operatorname{tp}^{\mathcal{N}_0}(a_1, \ldots, a_n)$ with $a_1, \ldots, a_n \in N_0$, then $\exists x_1 \ldots x_n \psi(x_1, \ldots, x_n)$ is in the theory. So there are $b_1, \ldots, b_n \in M_0$ realizing ψ. Then

$$\mathcal{A} = (\mathcal{N}_0, M_0, b_1, \ldots, b_n) \models \operatorname{Th}(\mathcal{N}_0, M_0)_{N_0} \cup \{ \psi(c_1, \ldots, c_n) \}$$

(Of course, one needs to check that this generalizes to taking finitely many formulae.) Furthermore, we can make $(\mathcal{N}_0', M_0')$ countable since $\mathcal{N}_0$ only realizes countably many types (since N_0 is countable).

Now let $(\mathcal{N}_0', M_0') \preceq (\mathcal{N}_1, \mathcal{M}_1)$ also be countable such that $\mathcal{N}_1$ and $\mathcal{M}_1$ are ω-homogeneous as L-structures. We saw how to do this for $\mathcal{N}_0'$ and $\mathcal{M}_0'$ separately; we then just add $\operatorname{Th}(\mathcal{N}_0', \mathcal{M}_0')$ to the set of sentences we wish to realize. (As in 5.5.3 (a).)

We now iterate $\aleph_0$-many times:

$$(\mathcal{N}_0, M_0) \preceq (\mathcal{N}_0', M_0') \preceq (\mathcal{N}_1, M_1) \preceq (\mathcal{N}_1', M_1') \preceq (\mathcal{N}_2, M_2) \preceq \dots$$

Let $(\mathcal{N}, M)$ be the union of this elementary chain. Then $(\mathcal{N}, M) \succeq (\mathcal{N}_0, M_0)$, so in particular $(\mathcal{N}, M)$ is a vaughtian pair with $\varphi(x)$. We also have that $(\mathcal{N}, M)$ is countable. To see that $\mathcal{N}$ and $\mathcal{M}$ are ω-homogeneous, we refer to the non-primed stages:

$$\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i$$
$$\mathcal{N} = \bigcup_{i<\omega} \mathcal{N}_i$$

and thus both are ω-homogeneous as the union of ω-homogeneous structures. Finally, since $\mathcal{M} \preceq \mathcal{N}$, we have that $\mathcal{N}$ realizes every type that $\mathcal{M}$ does; conversely, since

$$\mathcal{M} = \bigcup_{i<\omega} \mathcal{M}_i'$$

we have that $\mathcal{M}$ realizes every type that $\mathcal{N}$ does. So, by 5.5.3 (b), we have $\mathcal{M} \cong \mathcal{N}$. □ Claim 124

Let $\mathcal{M} \prec \mathcal{N}$ and φ be as in the claim. We build a chain

$$\mathcal{M}_0 \preceq \mathcal{M}_1 \preceq \mathcal{M}_2 \preceq \dots$$

of length $\aleph_1$ such that for all $\alpha < \aleph_1$, we have $(\mathcal{M}_{\alpha+1}, M_\alpha) \cong (\mathcal{N}, M)$. We let $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{M}_1 = \mathcal{N}$. Having produced $\mathcal{M}_\alpha$, we are then given $f_\alpha \colon \mathcal{M} \to \mathcal{M}_\alpha$ an isomorphism (since $\mathcal{M} \cong \mathcal{N}$); we then extend

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{f_\alpha} & \mathcal{M}_\alpha \\ \downarrow \preceq & & \downarrow \preceq \\ \mathcal{N} & \xrightarrow{f_{\alpha+1}} & \mathcal{M}_{\alpha+1} \end{array}$$

If $\lambda < \aleph_1$ is a limit ordinal, we let

$$\mathcal{M}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{M}_\alpha$$

But $\mathcal{M}$ is ω-homogeneous; so each $\mathcal{M}_\alpha$ is as well for each $\alpha < \lambda$, and $\mathcal{M}_\lambda$ is ω-homogeneous and countable. Also, since $\mathcal{M}_\alpha \cong \mathcal{M}$, we have that $\mathcal{M}_\alpha$ realizes the same types as $\mathcal{M}$. So $\mathcal{M}_\lambda$ realizes the same types that $\mathcal{M}$ realizes. So, by 5.5.3 (b), we have an isomorphism $f_\lambda \colon \mathcal{M} \to \mathcal{M}_\lambda$.

Having constructed the above chain, let

$$\overline{\mathcal{M}} = \bigcup_{\alpha<\aleph_1} \mathcal{M}_\alpha$$

Then $\overline{\mathcal{M}}$ is of cardinality $\aleph_1$ since $\mathcal{M}_\alpha \prec \mathcal{M}_{\alpha+1}$ (since every $(\mathcal{M}_{\alpha+1}, M_\alpha) \cong (\mathcal{N}, M)$). Well, $\varphi(\mathcal{N}) = \varphi(\mathcal{M})$ since we started with a vaughtian pair. Then, again since $(\mathcal{M}_{\alpha+1}, M_\alpha) \cong (\mathcal{N}, M)$, we have

$$\varphi(\mathcal{M}_\alpha) = \varphi(\mathcal{M}_{\alpha+1})$$
$$\varphi(\mathcal{M}_\lambda) = \varphi(\mathcal{M}_\alpha) \text{ for any } \alpha < \lambda$$

where λ is a limit ordinal. So $\varphi(\overline{\mathcal{M}}) = \varphi(\mathcal{M}_0)$ is countable, as $\mathcal{M}_0$ is countable, and infinite as it forms a vaughtian pair. □ Theorem 122

Corollary 125 (5.5.4)**.** *Suppose T is countable and complete. If T is categorical in some uncountable cardinality, then T has no vaughtian pair.*

Proof. Suppose $\kappa > \aleph_0$ and T is κ-categorical. By the downward Morley's theorem, we have that T is $\aleph_1$-categorical. So there is only one model of T of size $\aleph_1$, say $\mathcal{M}$, and it is $\aleph_1$-saturated. Then, by saturation, we have that every infinite definable set in $\mathcal{M}$ is of size $\aleph_1$. Then, by Vaught's 2-cardinal theorem, we have that T has no vaughtian pair. $\square$ Corollary 125

Corollary 126 (5.5.5)**.** *Suppose T is countable and complete. Suppose T is categorical in an uncountable cardinal. Then every model of T over any infinite definable set is prime. More precisely, suppose $\mathcal{M} \models T$, $A \subseteq M$, and $\varphi(x)$ is an $L(A)$-formula has $\varphi(\mathcal{M})$ is infinite. Then $\mathcal{M}$ is prime over $\varphi(\mathcal{M}) \cup A$.*

Proof. By 5.3.3, there is $\mathcal{M}_0 \preceq \mathcal{M}$ such that $A \cup \varphi(\mathcal{M}) \subseteq \mathcal{M}_0$ that is a prime extension. But then $\varphi(\mathcal{M}_0) = \varphi(\mathcal{M}) \cap \mathcal{M}_0 = \varphi(\mathcal{M})$. (We use that $A \subseteq M_0$.) So $\mathcal{M}_0 \prec \mathcal{M}$ with φ form a vaughtian pair unless $\mathcal{M}_0 = \mathcal{M}$. So $\mathcal{M}$ is prime over $\varphi(\mathcal{M}) \cup A$. $\square$ Corollary 126

Remark 127. The proof used ω-stability to get a prime model, and then the fact that there are no vaughtian pairs to get that it was $\mathcal{M}$. The proof then shows that it is the *unique* prime model over $\varphi(\mathcal{M}) \cup A$.

Remark 128. Prime models are unique only up to isomorphism. i.e. it is possible in general for there to be $A \subseteq M$ and $\mathcal{M} \prec \mathcal{N}$ with $\mathcal{M} \neq \mathcal{N}$ both prime over A. In some examples, this doesn't happen:

- In ACF_0, the prime model over $A \subseteq K$ is $\mathbb{Q}(A)^{\mathrm{alg}}$.
- In VS_F, the prime model over $A \subseteq V$ is $\mathrm{span}_F(A)$.

Definition 129. Suppose $\mathcal{M}$ is an L-structure; suppose $A \subseteq M$.

- An $L(A)$-formula $\varphi(x)$ is *algebraic* if $\varphi(\mathcal{M})$ is finite.
- We say $a \in M$ is *algebraic over* A if it realizes an algebraic formula over A.
- We set $\mathrm{acl}(A) = \{ a \in M : a \text{ is algebraic over } A \}$.
- We say A is *algebraically closed* if $A = \mathrm{acl}(A)$.

Remark 130.

- These notions seem to depend on $\mathcal{M}$, but in fact the notion is preserved if you pass to $\mathcal{N} \succeq \mathcal{M}$; i.e. $\mathrm{acl}_{\mathcal{M}}(A) = \mathrm{acl}_{\mathcal{N}}(A)$ for all $\mathcal{N} \succeq \mathcal{M}$.
- $|\mathrm{acl}(A)| \leq |L| + |A| + \aleph_0$.

Example 131.

1. Suppose $K \models \mathrm{ACF}$ with $L = \{0, 1, +, -, \times\}$. Suppose $A \subseteq K$. Then $\mathrm{acl}(A) = \mathbb{F}(A)^{\mathrm{alg}}$ where
$$\mathbb{F} = \begin{cases} \mathbb{Q} & \mathrm{char}(K) = 0 \\ \mathbb{F}_p & \mathrm{char}(K) = p \end{cases}$$
2. Suppose $V \models \mathrm{VS}_F$ with $L = \{0, +\} \cup \{\lambda_f : f \in F\}$. Suppose $A \subseteq V$. Then $\mathrm{acl}(A) = \mathrm{span}_F(A)$.
3. Let $L = \emptyset$; let X be an infinite set; take $A \subseteq X$. Then $\mathrm{acl}(A) = A$.

Definition 132. A type $p(x) \in S_1(A)$ is *algebraic* if it contains an algebraic formula.

Lemma 133. *If $\varphi(x) \in p(x) \in S(A)$ is algebraic with $|\varphi(\mathcal{M})|$ minimal over all formulae in $p(x)$, then $\varphi(x)$ isolates $p(x)$.*

Proof. Take $\psi(x) \in p(x)$. Then $\varphi(x) \wedge \psi(x) \in p(x)$; so $|\varphi(\mathcal{M})| = |(\varphi \wedge \psi)(\mathcal{M})|$ by minimality. So $(\varphi \wedge \psi)(\mathcal{M}) = \varphi(\mathcal{M})$, and $\varphi(\mathcal{M}) \subseteq \psi(\mathcal{M})$. So $\mathcal{M} \models \forall x(\varphi(x) \to \psi(x))$. So $\varphi(x)$ isolates $p(x)$. $\square$ Lemma 133

Definition 134. If $p(x)$ is an algebraic type and $\varphi(x) \in p(x)$ is algebraic such that $|\varphi(\mathcal{M})|$ is minimal, then we call $|\varphi(\mathcal{M})|$ the *degree* of $p(x)$.

Corollary 135. *Suppose $p(x) \in S_1(A)$ is algebraic. Then $|p(\mathcal{N})| = \deg(p)$ for any $\mathcal{N} \succeq \mathcal{M}$.*

Proof. $p(x)$ is isolated by some $\varphi(x)$; so $p(\mathcal{N}) = \varphi(\mathcal{N})$ for all $\mathcal{N} \geq \mathcal{M}$; so $\deg(p) = |\varphi(\mathcal{M})| = |\varphi(\mathcal{N})|$. □ Corollary 135

Remark 136. If $p(\mathcal{N})$ is finite in all $\mathcal{N} \succeq \mathcal{M}$, then $p(x)$ is algebraic.

Proof. Suppose $p(x)$ is not algebraic. Then each $\varphi(x) \in p(x)$ has $\varphi(\mathcal{M})$ infinite. So

$$\mathrm{Th}(\mathcal{M}_M) \cup \{\, \varphi(c_n) : n < \omega, \varphi(x) \in p(x) \,\} \cup \{\, c_n \neq c_m : n < m < \omega \,\}$$

is consistent by compactness and because no formula in $p(x)$ is algebraic. So there is $\mathcal{N}$ a model of this theory; then $\mathcal{N} \succeq \mathcal{M}$ and $p(\mathcal{N})$ is infinite. □ Remark 136

Lemma 137 (5.6.2)**.** *Suppose $\mathcal{M}$ is an L-structure; suppose $A \subseteq M$. Suppose $p \in S_1(A)$ is non-algebraic and $B \supseteq A$. Then there is a non-algebraic extension of $p(x)$ to $S(B)$.*

Proof. Let

$$q(x) = p(x) \cup \{\, \neg\psi(x) : \psi(x) \text{ an algebraic } L(B)\text{-formula} \,\}$$

If $q(x)$ were not finitely satisfiable in $\mathcal{M}$, then for some $\varphi(x) \in p(x)$ we have $\mathcal{M} \models \forall x(\varphi(x) \to \psi(x))$ an algebraic $L(B)$-formula, and $\varphi(x)$ is algebraic, a contradiction. Extend $q(x)$ to $\widehat{q}(x) \in S_1(B)$; this is non-algebraic because it contains the negation of every algebraic $L(B)$-formula. □ Lemma 137

Lemma 138 (5.6.4)**.** *Every partial elementary bijection $f\colon A \to B$ extends to a partial elementary bijection $f\colon \mathrm{acl}(A) \to \mathrm{acl}(B)$.*

Proof. Suppose $a \in \mathrm{acl}(A)$. Then $\mathrm{tp}(a/A)$ is algebraic; so $f(\mathrm{tp}(a/A))$ is algebraic, and hence isolated. So it has a realization in $\mathrm{acl}(B)$; we can then extend f by mapping a to said realization. Similarly, we can extend f to hit any given $b \in \mathrm{acl}(B)$ by something in $\mathrm{acl}(A)$ using f^{-1}. Let $f\colon A' \to B'$ be a maximal (with respect to the domain) partial elementary bijection extending f with $A' \subseteq \mathrm{acl}(A)$ and $B' \subseteq \mathrm{acl}(B)$. Then by the above arguments, we get $A' = \mathrm{acl}(A)$ and $B' = \mathrm{acl}(B)$. □ Lemma 138

We can view acl as a closure operator $\mathrm{acl}\colon \mathcal{P}(M) \to \mathcal{P}(M)$. Properties:

- acl is *reflexive*: $A \subseteq \mathrm{acl}(A)$.
- acl has *finite character*:

  $$\mathrm{acl}(A) = \bigcup_{A' \subseteq_{\mathrm{fin}} A} \mathrm{acl}(A')$$

  since any algebraic formula uses only finitely many parameters from A.
- acl is *transitive*: $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$.

  Proof. Suppose $c \in \mathrm{acl}\{\, b_1, \dots, b_n \,\}$ with $b_i \in \mathrm{acl}(A)$. We wish to show $c \in \mathrm{acl}(A)$. Let $\varphi(x, y_1, \dots, y_n)$ be an L-formula such that $\varphi(x, b_1, \dots, b_n)$ witnesses $c \in \mathrm{acl}\{\, b_1, \dots, b_n \,\}$. Let $\varphi_i(y_i)$ be an algebraic $L(A)$-formula witnessing $b_i \in \mathrm{acl}(A)$. Let

  $$\theta(x) = \exists y_1 \dots y_n \left(\bigwedge_{i=1}^{n} \varphi_i(y_i) \wedge \varphi(x, y_1, \dots, y_n) \wedge \exists^{\leq k} z \varphi(z, y_1, \dots, y_n) \right)$$

  where $k = |\varphi(\mathcal{M}, b_1, \dots, b_n)|$. Then $\theta(x)$ holds of c, witnessed by $y_i = b_i$ and $\theta(x)$ is over A and is algebraic. So $c \in \mathrm{acl}(A)$. □

We can extend the notion of acl to n-space:

Definition 139. We say $\varphi(x_1, \dots, x_n)$ is *algebraic* if $\varphi(\mathcal{M}) \subseteq M^n$ is finite. We say $a = (a_1, \dots, a_n) \in M^n$ is *algebraic* over $A \subseteq M$ if it realizes an algebraic formula. We write $a \in \mathrm{acl}(A)$. (Note that this is a slight abuse of notation, as $a \in M^n$ and $\mathrm{acl}(A) \subseteq M$.)

Exercise 140. $a \in \mathrm{acl}(A)$ if and only if each $a_i \in \mathrm{acl}(A)$.

So we can talk about *algebraic* n-types, etc.

3.1 Strong minimality

Definition 141. Suppose T is a complete theory. Suppose $\mathcal{M} \models T$ and $\varphi(x)$ is an $L(M)$-formula (with $x = (x_1, \dots, x_n)$). The definable set $\varphi(M)$ is *minimal in* $\mathcal{M}$ if $\varphi(x)$ is non-algebraic and for every other $L(M)$-formula $\psi(x)$ we have that one of $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ is algebraic. i.e. every definable subset of $\varphi(\mathcal{M})$ is finite or cofinite.

Definition 142. The $L(M)$-formula $\varphi(x)$ is *strongly minimal* if for every elementary extension $\mathcal{N} \succeq \mathcal{M}$, we have that $\varphi(\mathcal{N})$ is minimal in $\mathcal{N}$. In this case we also say that $\varphi(\mathcal{M})$ is strongly minimal.

Definition 143. The theory T is *strongly minimal* if and only if the formula "$x = x$" is strongly minimal in some $\mathcal{M} \models T$. i.e. The universe M is strongly minimal. (i.e. N is minimal for all $\mathcal{N} \succeq \mathcal{M}$).

Example 144.

- The theory of infinite sets in $L = \emptyset$ is strongly minimal.
- If F is a field, then VS_F is strongly minimal.
- If p is prime or 0, then ACF_p is strongly minimal. (Note that if $K \models \mathrm{ACF}_p$ then K^2 is not minimal.)
- Suppose $K \models \mathrm{ACF}_p$ where p is prime or 0. Suppose C is an irreducible algebraic curve. Then C is strongly minimal. e.g. Say $C = \{\, (x, y) \in K^2 : y = ax + b \,\}$ with $a \neq 0$. Consider $C \to K$ given by $(x, y) \mapsto x$; this is a definable bijection (i.e. a bijection whose graph is definable).

  Exercise 145. Strong minimality is preserved under definable bijections.

Proposition 146. *Suppose T is complete and totally transcendental. Suppose $\mathcal{M} \models T$. Then every definable set in $\mathcal{M}$ has a minimal definable subset.*

Proof. If $\varphi(\mathcal{M})$ is not minimal, then it can be split into two infinite, disjoint, definable subsets $\varphi_0(\mathcal{M})$ and $\varphi_1(\mathcal{M})$. If neither of these is minimal, iterate. Since T is totally transcendental, we have that this process stops; i.e. there is a minimal definable subset. □ Proposition 146

Remark 147. Write $\varphi(x)$ as $\varphi(x, a)$ where $\varphi(x, y)$ is an L-formula and $a = (a_1, \dots, a_m)$. Whether $\varphi(x, a)$ is strongly minimal depends only on $\mathrm{tp}(a) \in S_m(T)$. i.e. If $\mathcal{N} \models T$ and $b \in N^m$ with $\mathrm{tp}(b) = \mathrm{tp}(a)$, then $\varphi(x, b)$ is strongly minimal if $\varphi(x, a)$ is. In particular, if $m = 0$, then strong minimality depends only on φ.

Proof. $\varphi(x, a)$ is strongly minimal if and only if for any L-formula $\psi(x, z)$ (where $z = (z_1, \dots, z_\ell)$), we have that the set of $L(a)$-formulae

$$\Sigma_\psi(z) = \{\, \exists^{>k} x(\varphi(x, a) \wedge \psi(x, z)) \wedge \exists^{>k} x(\varphi(x, a) \wedge \neg\psi(x, z)) : k \in \mathbb{N} \,\}$$

has no realization in any $\mathcal{N} \succeq \mathcal{M}$.

Aside 148. $\varphi(\mathcal{M})$ is minimal if and only if for all ψ, we have Σ_ψ is not realized in $\mathcal{M}$.

But this holds if and only if $\Sigma_\psi(z)$ is not finitely satisfiable in $\mathcal{M}$ for any ψ; i.e. for every ψ there is some k_ψ such that, if

$$\theta_\psi(y) = \forall z(\exists^{\leq k_\psi} x(\varphi(x, y) \wedge \psi(x, z)) \vee \exists^{\leq k_\psi} x(\varphi(x, y) \wedge \neg\psi(x, z)))$$

then $\mathcal{M} \models \theta_\psi(a)$. Then $\varphi(x, a)$ is strongly minimal if and only if $\mathcal{M} \models \theta_\psi(a)$ for all ψ; i.e. if and only if $\theta_\psi(y) \in \mathrm{tp}(a)$ for all ψ. □ Remark 147

Lemma 149. *If $\mathcal{M}$ is ω-saturated, then minimal in $\mathcal{M}$ implies strongly minimal.*

Proof. Suppose $\varphi(x, a)$ is not strongly minimal; then there is some $\psi(x, z)$ such that $\Sigma_\psi(z)$ is realized in some $\mathcal{N} \succeq \mathcal{M}$. So $\Sigma_\psi(z)$ is a partial ℓ-type over a. So $\Sigma_\psi(z)$ is realized in $\mathcal{M}$ by ω-saturation. So, by Aside 148, we have that $\varphi(\mathcal{M})$ is not minimal. □ Lemma 149

Assignment 3. *Due Monday November 16. Do 5.2.5, 3.3.1 (prove random graph has quantifier elimination and is complete) + 5.5.3, 5.6.1, 5.7.3, 5.7.4.*

Definition 150. We say T *eliminates $\exists^\infty x$ quantifier* if for every L-formula $\varphi(x, y)$ where $y = (y_1, \ldots, y_n)$ there is a bound $N_\varphi \geq 1$ such that for any $\mathcal{M} \models T$ and any $a \in M^n$, we have that $\varphi(\mathcal{M}, a)$ is either of size $\leq N_\varphi$ or is infinite.

The point is that for every φ there is a formula $\psi(y)$ such that for any $\mathcal{M} \models T$ and any $a \in M^n$, we have

$$\mathcal{M} \models \psi(a) \iff \varphi(\mathcal{M}, a) \text{ is infinite}$$

Thus $T \models \forall y(\psi(y) \leftrightarrow \exists^\infty x(\varphi(x, y)))$. In particular, we take $\psi(y)$ to be

$$\exists x_1 \ldots x_{N_\varphi+1} \left(\bigwedge_{i \neq j} (x_i \neq x_j) \wedge \varphi(x_i) \right)$$

Lemma 151. *If T has no vaughtian pair then T eliminates $\exists^\infty x$.*

Proof. Fix $\varphi(x, y)$. Suppose T does not eliminate $\exists^\infty x \varphi(x, y)$. Let $L^* = L \cup \{ P, c \}$ where P is a unary predicate symbol and $c = (c_1, \ldots, c_n)$ are new constant symbols with $n = |y|$. Let

$$T^* = T \cup \{ \text{“}P \text{ is an elementary } L\text{-substructure”} \} \cup \{ \forall x(\varphi(x, c) \to P(x)) \} \cup \{ P(c_i) : i \in \{ 1, \ldots, n \} \}$$

Note that except for the possibility that $\varphi(x, c)$ is algebraic, we have that T^* is the theory of a vaughtian pair for T. To actually get a vaughtian pair, we use the theory

$$S = T^* \cup \{ \exists^{\geq k} x \varphi(x, c) : k \in \mathbb{N} \}$$

Claim 152. *S is consistent.*

Proof. We use compactness. For any k there is a model $\mathcal{M} \models T$ with $a \in M^n$ such that $\varphi(\mathcal{M}, a)$ is finite of size $\geq k$. (Since T does not eliminate $\exists^\infty x(\varphi(x, y))$.) Pick $\mathcal{N} \succ \mathcal{M}$. Since $\varphi(x, a)$ is algebraic, we have that $\varphi(\mathcal{N}, a) \subseteq M$. So $(\mathcal{N}, \mathcal{M}, a) \models T^* \cup \{ \exists^{\geq k} x \varphi(x, c) \}$. By compactness, we have S is consistent. □ Claim 152

Then any model of S is a vaughtian pair. □ Lemma 151

Lemma 153. *Suppose T is a complete theory that eliminates $\exists^\infty x$. Suppose $\mathcal{M} \models T$ and φ is an $L(M)$-formula with $\varphi(\mathcal{M})$ minimal. Then $\varphi(x)$ is strongly minimal.*

Proof. If $\varphi(x)$ were not strongly minimal, then in some $\mathcal{N} \succeq \mathcal{M}$ there is some $\psi(x, z)$ and some $b \in N^\ell$ (where $\ell = |z|$) such that $\varphi(\mathcal{N}) \wedge \psi(\mathcal{N}, b)$ and $\varphi(\mathcal{N}) \wedge \neg\psi(\mathcal{N}, b)$ are infinite. Then

$$\mathcal{N} \models \exists^\infty x(\varphi(x) \wedge \psi(x, b)) \wedge \exists^\infty x(\varphi(x) \wedge \neg\psi(x, b))$$

Since T eliminates $\exists^\infty x$, this can be expressed as a first-order statement. So

$$\exists^\infty x(\varphi(x) \wedge \psi(x, z)) \wedge \exists^\infty x(\varphi(x, \wedge \neg\psi(x, z)))$$

is realized in $\mathcal{M}$. So $\varphi(\mathcal{M})$ is not minimal in $\mathcal{M}$. □ Lemma 153

Exercise 154. If T eliminates $\exists^\infty x$ for x a single variable then it eliminates $\exists^\infty x$ for x an n-tuple of variables.

Corollary 155. *Suppose T is countable, complete, and uncountably categorical. Then every definable set (in any model) contains a strongly minimal definable set.*

Proof. Fix $\mathcal{M} \models T$; suppose $X \subseteq M^n$ is definable. By total transcendentality we have that X contains a minimal definable set Y. Since T has no vaughtian pair, we have that Y is strongly minimal. □ Corollary 155

Lemma 156. *Suppose $\mathcal{M}$ is an L-structure; suppose $\varphi(x)$ is an $L(M)$-formula where $x = (x_1, \ldots, x_n)$. Then $\varphi(\mathcal{M})$ is minimal if and only if there is a unique $p(x) \in S_n(M)$ that is non-algebraic and contains $\varphi(x)$.*

Proof.

($\implies$) Let
$$p(x) = \{\, \psi(x) : \psi(x) \text{ is an } L(M)\text{-formula such that } \varphi \wedge \neg\psi \text{ is algebraic} \,\}$$

Then $p(x)$ is complete since $\varphi(\mathcal{M})$ is minimal, and $p(x)$ is non-algebraic since $\varphi(x)$ is non-algebraic. Furthermore, $p(x)$ is clearly the unique such type.

($\impliedby$) Suppose $\varphi(\mathcal{M})$ is not minimal, witnessed by $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ both non-algebraic. Let

$$p_1(x) = \{\, \varphi \wedge \psi \,\} \cup \{\, \neg\theta : \theta \text{ an algebraic } L(M)\text{-formula} \,\}$$
$$p_2(x) = \{\, \varphi \wedge \neg\psi \,\} \cup \{\, \neg\theta : \theta \text{ an algebraic } L(M)\text{-formula} \,\}$$

Then these are distinct partial types (check), and any completion is non-algebraic and contains φ.

□ Lemma 156

We view this as saying that $\varphi(x)$ has a unique "generic" extension.

Corollary 157. *Suppose $p(x) \in S_n(A)$ is strongly minimal. Then for any $\mathcal{N} \succeq \mathcal{M}$ and any $A \subseteq B \subseteq N$, we have that $p(x)$ has a unique non-algebraic extension to B.*

Proof. Existence is by 5.6.2 (does not use strong minimality). Suppose $q_1(x), q_2(x) \in S_n(B)$ are non-algebraic types extending $p(x)$. Let $\varphi(x) \in p(x)$ be strongly minimal. So $\varphi(\mathcal{N})$ is minimal. Let $q_1(x) \subseteq \widehat{q_1}(x) \in S_n(N)$ be non-algebraic; let $q_2(x) \subseteq \widehat{q_2}(x) \in S_n(N)$ be non-algebraic (again by 5.6.2). Now $\varphi \in \widehat{q_1} \cap \widehat{q_2}$. So, by lemma applied to $\varphi(N)$, we have $\widehat{q_1} = \widehat{q_2}$. So $q_1 = q_2$. □ Corollary 157

Definition 158. We say a type $p(x)$ is *strongly minimal* if it is non-algebraic and and contains a strongly minimal formula.

Corollary 159 (5.7.4)**.** *Suppose $\mathcal{M}$ is an L-structure with $A \subseteq M$. Suppose $p(x) \in S_n(A)$ is strongly minimal; suppose $m > 0$. Then there is a unique type over A of an m-tuple $(a_1, \dots, a_m)$ of realizations of $p(x)$ with $a_i \notin \operatorname{acl}(Aa_1 \dots a_{i-1})$ for all $i \in \{\, 1, \dots, m \,\}$. (i.e. if $(b_1, \dots, b_m) \models p(x)$ with $b_i \notin \operatorname{acl}(Ab_1 \dots b_{i-i})$, then $\operatorname{tp}(a_1 \dots a_m/A) = \operatorname{tp}(b_1 \dots b_m/A)$.)*

Recall that an n-tuple is in $\operatorname{acl}(B)$ if every coordinate is.

Remark 160. Since $p(x)$ is strongly minimal, we have that there always exist such m-tuples. (We call such an m-tuple an *m-tuple of* acl*-independent realizations of* $p(x)$.) Indeed, take $a_1 \models p(x)$ such that $a_1 \notin \operatorname{acl}(A)$. Extend $p(x)$ to a non-algebraic type over Aa_1; let a_2 realize it. Then $a_2 \models p(x)$ and $a_2 \notin \operatorname{acl}(Aa_1)$.

Proof of Corollary 159. Induction on m. The case $m = 1$ is simply because $p(x)$ is complete. Suppose then that $m > 1$. Suppose $(b_1, \dots, b_m)$ and $(a_1, \dots, a_m)$ are acl-independent sequences of realizations of $p(x)$. By the induction hypothesis we have $\operatorname{tp}(b_1 \dots b_{m-1}/A) = \operatorname{tp}(a_1 \dots a_{m-1}/A)$. Let $f\colon A \cup \{\, b_1, \dots, b_{m-1} \,\} \to A \cup \{\, a_1, \dots, a_{m-1} \,\}$ be given by $f(b_i) = a_i$ and $f \restriction A = \mathrm{id}$; then f is a partial elementary map. Let $q(x) = f(\operatorname{tp}(b_m/Ab_1 \dots b_{m-1})$; then $q(x)$ is non-algebraic since $b_m \notin \operatorname{acl}(Ab_1 \dots b_{m-1})$ and f is a partial elementary map. Note that as $f \restriction A = \mathrm{id}$, we have that b_m and a_m both realize $p(x)$. Then $q(x)$ and $\operatorname{tp}(a_m/Aa_1 \dots a_{m-1})$ are both non-algebraic extensions of $p(x)$ to $A \cup \{\, a_1, \dots, a_{m-1} \,\}$; so, by the last corollary, we have
$$f(\operatorname{tp}(b_m/Ab_1 \dots b_{m-1})) = q(x) = \operatorname{tp}(a_m/Aa_1 \dots a_{m-1})$$
So we can extend f to a partial elementary map taking b_m to a_m. So $\operatorname{tp}(b_1 \dots b_m/A) = \operatorname{tp}(a_1 \dots a_m/A)$. □ Corollary 159

Definition 161. A *pregeometry* or *matroid* is a set X together with a function $\operatorname{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying

Reflexivity $A \subseteq \operatorname{cl}(A)$

Transitivity $\operatorname{cl}(\operatorname{cl}(A)) = \operatorname{cl}(A)$

Finite character

$$\mathrm{cl}(A) = \bigcup_{A' \subseteq_{\mathrm{fin}} A} \mathrm{cl}(A')$$

Steinitz exchange If $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$ then $b \in \mathrm{cl}(Aa)$.

Example 162.

- If X is any set, we can set $\mathrm{cl}(A) = A$.
- If F is a field and V is a vector space over F, we can set $\mathrm{cl}(A) = \mathrm{span}_F(A)$.
- If K is an algebraically closed field, we can set $\mathrm{cl}(A) = \mathbb{F}(A)^{\mathrm{alg}}$.

In every pregeometry there is a notion of independence:

Definition 163. Suppose (X, cl) is a pregeometry; suppose $A \subseteq X$. We say $C \subseteq X$ is an *independent set over* A if for all $c \in C$ we have $c \notin \mathrm{cl}(A \cup (C \setminus \{ c \}))$.

Fact 164. *Suppose* (X, cl) *is a pregeometry and* $A \subseteq X$.

1. $C \subseteq X$ *is independent over* A *if and only if given any enumeration* $C = \{ c_\alpha : \alpha < \kappa \}$ *and any* $\alpha < \kappa$ *we have* $c_\alpha \notin \mathrm{cl}(A \cup \{ c_\beta : \beta < \alpha \})$.
2. *If* $C \subseteq X$ *and* $D \subseteq X$ *are both maximal independent sets over* A*, then* $|C| = |D|$.
3. $C \subseteq X$ *is maximally independent over* A *if and only if* C *is independent over* A *and* $\mathrm{cl}(C) = X$.

Proof. The usual proof in linear algebra for span works in pregeometries. □ Fact 164

Definition 165. We call a maximally independent set $C \subseteq X$ over A a *basis* for X over A; we set $\dim(X) = |C|$.

Theorem 166 (5.7.5)**.** *Suppose* T *is a complete theory,* $\varphi(x)$ *an* L*-formula with* $x = (x_1, \dots, x_n)$*, and* $\mathcal{M} \models T$*. Suppose* $\varphi(x)$ *is strongly minimal. Then*

$$\begin{aligned} \mathrm{cl}\colon \mathcal{P}(\varphi(\mathcal{M})) &\to \mathcal{P}(\varphi(\mathcal{M})) \\ A &\mapsto \mathrm{acl}(A) \cap \varphi(\mathcal{M}) \end{aligned}$$

is a pregeometry on $\varphi(\mathcal{M})$.

Remark 167. If $n > 1$ and $A \subseteq M^n$, we set

$$\mathrm{acl}(A) = \mathrm{acl}(\{ a \in M : a \text{ is a co-ordinate of some } n\text{-tuple in } A \})$$

and we write $(c_1, \dots, c_n) \in \mathrm{acl}(A) \subseteq M$ to mean every $c_i \in \mathrm{acl}(A)$.

Proof of Theorem 166. We have proved the first three axioms for (M, acl); they then follow easily for $(\varphi(\mathcal{M}), \mathrm{cl})$. It remains to show exchange. Suppose $a, b \in \varphi(\mathcal{M})$ and $A \subseteq \varphi(M)$. Suppose $b \notin \mathrm{acl}(Aa)$ and $a \notin \mathrm{acl}(A)$. It remains to show that $a \notin \mathrm{acl}(Ab)$. Let $p(x) \in S_n(A)$ be the (unique by 5.7.3) non-algebraic type containing $\varphi(x)$. Then $a \models p(x)$ since $\mathrm{tp}(a/A)$ is non-algebraic and contains $\varphi(x)$. Also $b \models p(x)$ and $b \notin \mathrm{acl}(Aa)$; so (a, b) is an independent pair of realizations of $p(x)$. So its type over A is completely determined by $b \notin \mathrm{acl}(Aa)$ and $a \notin \mathrm{acl}(A)$.

Now, let $\mathcal{N} \succeq \mathcal{M}$ such that $p(\mathcal{N})$. (Possible since $p(x)$ is non-algebraic.) Let $q(x) \in S_n(Ap(\mathcal{N}))$ be the unique non-algebraic extension of $p(x)$. Let $\mathcal{K} \succeq \mathcal{N}$ have a realization b' of $q(x)$. Now, for all $a' \in p(\mathcal{N})$, we have that

$$\mathrm{tp}(a', b'/A) = \mathrm{tp}(a, b/A)$$

since (a', b') satisfies $b' \notin \mathrm{acl}(Aa')$ and $a' \notin \mathrm{acl}(A)$. In particular, fixing $a' \in p(\mathcal{N})$, we have that every element of $p(\mathcal{N})$ realizes $\mathrm{tp}(a'/Ab')$; so $a' \notin \mathrm{acl}(Ab')$. So $a \notin \mathrm{acl}(Ab)$. □ Theorem 166

We thus get notions of independence, basis, and dimension; we use the notation $\text{acl-dim}_\varphi(\mathcal{M}) = \dim(\varphi(\mathcal{M}))$ in the sense of the above pregeometry.

This extends to parameters simply by working in $L(A)$. We use the notation $\text{acl-dim}_\varphi(\mathcal{M}/A) = \text{acl-dim}_\varphi(\mathcal{M}_A)$. Note that the closure operator is now $\text{cl}(B) = \text{acl}(B \cup A) \cap \varphi(\mathcal{M})$.

Lemma 168 (5.7.6)**.** *Suppose $\mathcal{M}, \mathcal{N}$ are L-structures with $A \subseteq M$ and $A \subseteq N$ with $\mathcal{M}_A \equiv \mathcal{N}_A$. Let $\varphi(x)$ be an A-definable strongly minimal formula (with x is a single variable). Then there exists a bijective partial elementary map $f\colon A \cup \varphi(\mathcal{M}) \to A \cup \varphi(\mathcal{N})$ such that $f \restriction A = \text{id}$ if and only if $\dim_\varphi(\mathcal{M}/A) = \dim_\varphi(\mathcal{N}/A)$. (Such a map is called a partial elementary map over A.)*

Remark 169. If φ is $x = x$, i.e. we are in a strongly minimal theory, then this says that models are determined by dimension.

Proof of Lemma 168.

($\implies$) The property of being an acl-basis is preserved by bijective partial elementary maps.

($\impliedby$) Let $U \subseteq \varphi(\mathcal{M})$ and $V \subseteq \varphi(\mathcal{N})$ be acl-bases over A of $\varphi(\mathcal{M})$ and $\varphi(\mathcal{N})$, respectively. Let $f\colon A \cup U \to A \cup V$ be any bijection with $f \restriction A = \text{id}$. (Note that $A \cap U = A \cap V = \emptyset$, so this is possible.) 5.7.4 then says that each distinct m-tuple from U has the same type over A as its image under f. Suppose $a_1, \dots, a_m \in U$. Then $\text{tp}(a_1 \dots a_m/A)$ says only that $a_1 \notin \text{acl}(A)$, $a_2 \notin \text{acl}(Aa_1)$, $\dots$, $a_m \notin \text{acl}(Aa_1 \dots a_{m-1})$; i.e. f is a partial elementary map. By 5.6.4, we have that f extends to a partial elementary map $\text{acl}(A \cup U) \to \text{acl}(A \cup V)$, and thus $\text{acl}(A \cup U) \cap \varphi(\mathcal{M}) \to \text{acl}(A \cup V) \cap \varphi(\mathcal{N})$; i.e. $\text{cl}(U) \to \text{cl}(V)$, i.e. $\varphi(\mathcal{M}) \to \varphi(\mathcal{N})$.

□ Lemma 168

Remark 170. A better formulation of the statement: there is a bijective partial elementary map $f\colon \varphi(\mathcal{M}) \to \varphi(\mathcal{N})$ in $L(A)$ if and only if $\dim_\varphi(\mathcal{M}/A) = \dim_\varphi(\mathcal{N}/A)$.

Consider in particular a strongly minimal theory T; so we have some $\mathcal{M} \models T$ such that (M, acl) is a pregeometry. Then $\text{acl-dim}(\mathcal{M})$ is the dimension of this pregeometry. We see that models of T are determined up to isomorphism by acl-dim.

Theorem 171 (Baldwin-Lachlan)**.** *Suppose $\kappa > \aleph_0$. Suppose T is countable and complete. Then T is κ-categorical if and only if T is ω-stable and has no vaughtian pairs.*

Proof.

($\implies$) Done. (5.5.4).

($\impliedby$) T is ω-stable; so it is small, and thus has a prime model $\mathcal{M}_0$. Then $\mathcal{M}_0$ is countable. We also know that there exists a strongly minimal $L(M_0)$-formula $\varphi(x)$ with x a single variable. Indeed, by total transcendentality we have M_0 contains a minimal definable set. Since T has no vaughtian pair, we have that $\exists^\infty x$ is eliminated; thus minimal implies strongly minimal. Let $\mathcal{M}_1, \mathcal{M}_2$ be κ-sized models. By primality we may assume $\mathcal{M}_0 \preceq \mathcal{M}_1$ and $\mathcal{M}_0 \preceq \mathcal{M}_2$.

Now, for each $i \in \{1, 2\}$, we have $|\varphi(\mathcal{M}_i)| = \kappa$ since T has no vaughtian pairs. Let $B_i \subseteq \varphi(\mathcal{M}_i)$ be an acl-basis over M_0. Then $\text{acl}(M_0 \cup B_i) = \varphi(\mathcal{M}_i)$ for $i \in \{1, 2\}$. Then

$$\begin{aligned} \kappa &= |\text{acl}(M_0 \cup B_i| \\ &= |M_0 \cup B_i| \text{ (since } L \text{ is countable)} \\ &\leq |M_0| + |B_i| \\ &= \aleph_0 + |B_i| \end{aligned}$$

So $|B_i| = \kappa$. So $\text{acl-dim}_\varphi(\mathcal{M}_i/M_0) = \kappa$. By the lemma there is a bijective partial elementary map $f\colon \varphi(\mathcal{M}_1) \to \varphi(\mathcal{M}_2)$ in the language $L(M_0)$. We thus get a bijective partial elementary map in L: $g\colon M_0 \cup \varphi(\mathcal{M}_1) \to M_0 \cup \varphi(\mathcal{M}_2)$ with $g \restriction M_0 = \text{id}$ and $g \restriction \varphi(\mathcal{M}_1) = f$. Since T has no vaughtian pairs, we have that $\mathcal{M}_1$ is prime over $M_0 \cup \varphi(\mathcal{M}_1)$; then g extends to an elementary embedding $\mathcal{M}_1 \to \mathcal{M}_2$. So $\mathcal{M}_1 \cong g(\mathcal{M}_1) = \mathcal{M}_2' \preceq \mathcal{M}_2$, and $g(\mathcal{M}_1)$ contains $M_0 \cup \varphi(\mathcal{M}_2)$. So $\varphi(\mathcal{M}_1) \subseteq M_2'$ with $\mathcal{M}_2' \preceq \mathcal{M}_2$; since T has no vaughtian pairs, we have that $\mathcal{M}_2' = \mathcal{M}_2$, and g is an isomorphism.

□ Theorem 171

Corollary 172 (Morley's theorem)**.** *Suppose T is countable and complete; suppose $\kappa > \aleph_0$. Then T is κ-categorical if and only if T is $\aleph_1$-categorical.*

Final exams: oral, individually scheduled, done before December 17.

3.2 Loose ends in strongly minimal theories

Recall that T is strongly minimal theory if "$x = x$" is strongly minimal in some (equivalently, any) $\mathcal{M} \models T$; in this case, we have (M, acl) is a pregeometry.

Theorem 173. *Suppose T is strongly minimal and complete. Then*

1. *T is κ-categorical for any $\kappa \geq \aleph_0 + |L|$.*
2. *Every infinite κ is the* acl-dim *of some model of T. The finite cardinals that are possible* acl-dim *of models of T form an end segment.*
3. *If $\mathcal{M} \models T$, then* $\mathrm{acl\text{-}dim}(\mathcal{M})$ *is infinite if and only if $\mathcal{M}$ is ω-saturated.*
4. *All models of T are ω-homogeneous.*

Proof. We begin with a claim.

Claim 174. *Suppose $\mathcal{M} \models T$, $A \subseteq M$ is infinite and $A = \mathrm{acl}(A)$. Then A is the universe of an elementary substructure of $\mathcal{M}$.*

Proof. Given an $L(A)$-formula $\varphi(x)$, we need to show that if $\varphi(\mathcal{M})$ is non-empty, then there is $a \in A$ with $\mathcal{M} \models \varphi(A)$. If $\varphi(\mathcal{M})$ is finite, then all its members are in $\mathrm{acl}(A) = A$ by definition of algebraic closure. If $\varphi(\mathcal{M})$ is infinite, then by strong minimality of T we have that $\varphi(\mathcal{M})$ is cofinite, and $A \cap \varphi(\mathcal{M}) \neq \emptyset$ since A is infinite. □ Claim 174

1. Suppose $\kappa > \aleph_0 + |L|$; suppose $\mathcal{M}_1, \mathcal{M}_2 \models T$ with $|M_1| = |M_2| = \kappa$. Let $B_i \subseteq M_i$ be an acl-basis for M_i. Then $\kappa = |M_i| = |\mathrm{acl}(B_i)| \leq |B_i| + \aleph_0 + |L|$. But $\kappa > \aleph_0 + |L|$; so $|B_i| \geq \kappa$. But $B_i \subseteq M_i$, so $|B_i| \leq \kappa$, and $|B_i| = \kappa$. So $\mathrm{acl\text{-}dim}(\mathcal{M}_1) = \mathrm{acl\text{-}dim}(\mathcal{M}_2) = \kappa$; so $\mathcal{M}_1 \cong \mathcal{M}_2$. Let $f \colon B_1 \to B_2$ be any bijection; then this is a partial elementary map. Extend f to acl: we may take $f \colon M_1 \to M_2$ to be a bijective partial elementary map, which is then an isomorphism.

2. Suppose $\kappa > \aleph_0 + |L|$. Let $\mathcal{M} \models T$ be of size κ. By the proof of (a) we have that $\mathrm{acl\text{-}dim}(\mathcal{M}) = \kappa$.

   Suppose $\aleph_0 \leq \kappa \leq \aleph_0 + |L|$. Let $\mathcal{M} \models T$ with $|M| > \aleph_0 + L$. Then $\mathrm{acl\text{-}dim}(\mathcal{M}) = |M| > \kappa$, so we can find an acl-independent set $B \subseteq M$ of size κ. By the claim, since $\kappa \geq \aleph_0$, we have that $\mathrm{acl}(B) \preceq \mathcal{M}$. Then $\mathrm{acl\text{-}dim}((\mathcal{B})) = \kappa$.

   Suppose $\mathcal{M} \models T$ with $\mathrm{acl\text{-}dim}(\mathcal{M}) = n < \omega$. Let $\{ b_1, \dots, b_n \}$ be an acl-basis for M. Let $\mathcal{N} \succeq \mathcal{M}$; let $c \in N \setminus M$. Then $\mathrm{acl}(\{ b_1, \dots, b_n \} = M)$, so $\{ b_1, \dots, b_n, c \}$ is acl-independent. So in (N, acl), we have $\mathrm{acl}(\{ b_1, \dots, b_n, c \}) \preceq \mathcal{N}$ by the claim, since $\mathrm{acl}(\{ b_1, \dots, b_n, c \}) \supseteq M$, and thus is infinite. But then $\mathrm{acl\text{-}dim}(\mathrm{acl}(\{ b_1, \dots, b_n, c \})) = n + 1$.

3. Suppose $A \subseteq M$, $|A| < \omega$, and $p \in S_1(A)$. If p is algebraic, then it is realized in $\mathcal{M}$ as it is isolated. If p is non-algebraic, then it is the unique non-algegraic type, so any $a \in M \setminus \mathrm{acl}(A)$ will realize it. So p will be realized if and only if $\mathrm{acl}(A) \neq M$. So $\mathrm{acl\text{-}dim}(\mathcal{M})$ is infinite if and only if $\mathcal{M}$ is ω-saturated.

4. Suppose $\mathcal{M} \models T$, $f \colon A \to B$ is a partial elementary map with $|A| = |B| < \omega$. Extend f to $f \colon \mathrm{acl}(A) \to \mathrm{acl}(B)$. Let $n = \mathrm{acl\text{-}dim}(\mathrm{acl}(A)) = \mathrm{acl\text{-}dim}(\mathrm{acl}(B))$. If $\mathrm{acl}(A) = M$, we are done. If $\mathrm{acl}(A) \subsetneq M$, then $\dim(\mathcal{M}) > n$; so $\mathrm{acl}(B) \neq M$. Then if $a \in M \setminus \mathrm{acl}(A)$, then $p = \mathrm{tp}(a/(\mathcal{A}))$ is non-algebraic, so $f(p) \in S_1(\mathrm{acl}(B))$ is non-algebraic, and is thus realized by any $b \in M \setminus \mathrm{acl}(B) \neq \emptyset$; we can then extend f by $a \mapsto b$.

□ Theorem 173

3.3 Eschewing the monster model

Proposition 175. *Suppose κ is an infinite cardinal. Then every L-structure has a κ-saturated elementary extension.*

Proof. Replacing κ by κ^+, we may assume κ is regular. Suppose $\mathcal{M}$ is an L-structure. We build a chain

$$\mathcal{M} = \mathcal{M}_0 \preceq \mathcal{M}_1 \preceq \dots$$

of length κ such that $\mathcal{M}_{\alpha+1}$ is an elementary extension of $\mathcal{M}_\alpha$ in which all types over $\mathcal{M}_\alpha$ are realized. For α a limit ordinal, we let

$$\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\beta$$

Let

$$\mathcal{N} = \bigcup_{\alpha<\kappa} \mathcal{M}_\alpha$$

Then, since κ is regular, we have $\mathcal{N} \succeq \mathcal{M}$ is κ-saturated. □ Proposition 175

Remark 176. A more careful proof would show that if $|M| \leq \kappa$, then there is an elementary extension of $\mathcal{M}$ that is κ^+-saturated and of size 2^κ. If we assume GCH, we would actually get a saturated elementary extension. Outright saturation is useful because of its strong homogeneity properties, but we don't wish to assume GCH.

Theorem 177. *Suppose κ is an infinite cardinal. Then every L-structure has an elementary extension that is κ-saturated and strongly κ-homogeneous.*

Proof. Again, we may assume κ is regular. Suppose $\mathcal{M}$ is an L-structure; we build a chain

$$\mathcal{M} = \mathcal{M}_0 \preceq \mathcal{M}_1 \preceq \dots$$

of length κ where $\mathcal{M}_{\alpha+1}$ is $|M_\alpha|^+$-saturated by iterating the above proposition. At a limit ordinal α, we set

$$\mathcal{M}_\alpha = \bigcup_{\beta<\alpha} \mathcal{M}_\alpha$$

Let

$$\mathcal{N} = \bigcup_{\alpha<\kappa} \mathcal{M}_\alpha$$

Clearly $\mathcal{N}$ is κ-saturated. Let $f\colon A \to N$ be a partial elementary map with $|A| < \kappa$. By regularity we have that A and $f(A)$ are contained in M_α for some $\alpha < \kappa$. So $f\colon A \to f(A)$ is a partial elementary map from $\mathcal{M}_{\alpha+1}$ to itself. We work in $\mathcal{M}_{\alpha+1}$.

Claim 178. *f extends to a partial elementary map f_α whose domain and range contain M_α.*

Proof. Enumerate $M_\alpha \setminus A$ and extend f by back-and-forth, using the fact that $\mathcal{M}_{\alpha+1}$ is $|M_\alpha|^+$-saturated. □ Claim 178

Let

$$\widehat{f} = \bigcup_{\alpha<\kappa} f_\alpha$$

Then $\mathrm{dom}(\widehat{f}) \supseteq \mathcal{N}$ and $\mathrm{Ran}(\widehat{f}) \supseteq \mathcal{N}$. So $\widehat{f}$ is an automorphism of $\mathcal{N}$. □ Theorem 177

Hereafter, by "a suffiently saturated model", we mean a structure with sufficiently large saturation and strong homogeneity.

Theorem 179. *Suppose $\mathcal{M}$ is κ-saturated and strongly κ-homogeneous. Then*

1. *(κ^+-universality) If $\mathcal{N} \equiv \mathcal{M}$ and $|N| \leq \kappa$, then there is an elementary embedding $\mathcal{N} \to \mathcal{M}$.*

2. *If $b, b' \in M$ and $A \subseteq M$ with $|A| < \kappa$, then $\operatorname{tp}(b/A) = \operatorname{tp}(b'/A)$ if and only if there is $f \in \operatorname{Aut}_A(\mathcal{M})$ with $f(b) = b'$. (i.e. f is an automorphism of $\mathcal{M}$ with $f \restriction A = \mathrm{id}$.)*
3. *Suppose $X \subseteq M^n$ is definable (over some parameter set). Suppose $A \subseteq M$ with $|A| < \kappa$. Then X is A-definable if and only if X is $\operatorname{Aut}_A(\mathcal{M})$-invariant.*
4. *Suppose $b \in M^n$, $A \subseteq M$, and $|A| < \kappa$. Then the following are equivalent:*
   (a) $b \in \operatorname{acl}(A)$.
   (b) $\operatorname{tp}(b/A)$ *has finitely many realizations in* $\mathcal{M}$.
   (c) The $\operatorname{Aut}_A(\mathcal{M})$*-orbit of b is finite.*
5. *Suppose $b \in M^n$ with $A \subseteq M$ and $|A| < \kappa$. Then the following are equivalent:*
   (a)
   $$b \in \operatorname{dcl}(A) = \{ b' \in M : \{ b' \} \text{ is } A\text{-definable} \}$$
   (We say a tuple b is in $\operatorname{dcl}(A)$ *if every component is; equivalently, if $\{ b \}$ is an A-definable subset of M^n.)*
   (b) $\operatorname{tp}(b/A)$ *has only b as a realization in* $\mathcal{M}$.
   (c) $\{ b \}$ is the $\operatorname{Aut}_A(\mathcal{M})$*-orbit of b.*

Proof.

1. We argue by extending partial elementary maps. Then $\emptyset \to \emptyset$ is a partial elementary map $\mathcal{N} \to \mathcal{M}$ because $\mathcal{N} \equiv \mathcal{M}$.

   Given a partial elementary map $f \colon A \to M$ with $A \subseteq N$ and $|A| < \kappa$, we can extend f to any $b \in N$ by the κ-saturation of $\mathcal{M}$.

   If we enumerate $N = \{ a_\alpha : \alpha < \kappa \}$ and set $A_\alpha = \{ a_\beta : \beta < \alpha \}$, then the A_α form a chain with
   $$N = \bigcup_{\alpha < \kappa} A_\alpha$$
   and $|A_\alpha| < \kappa$. So we get $f \colon \mathcal{N} \to \mathcal{M}$ an elementary embedding. (At limits, take unions.)

   Note that here we didn't use strong κ-homogeneity; it sufficed to assume κ-saturation.
2. ($\Longleftarrow$) Clear.

   ($\Longrightarrow$) If $\operatorname{tp}(b/A) = \operatorname{tp}(b'/A)$ then the map $f \colon A \cup \{ b \} \to A \cup \{ b' \}$ given by
   $$f(x) = \begin{cases} x & x \in A \\ b' & x = b \end{cases}$$
   is a partial elementary map. But $|A \cup \{ b \}| < \kappa$. So, by strong homogeneity, we have that f extends to an automorphism of $\mathcal{M}$.
3. ($\Longrightarrow$) Clear.

   ($\Longleftarrow$) Write $X = \varphi(\mathcal{M}, b)$ for some L-formula $\varphi(x, z)$ where $x = (x_1, \ldots, x_n)$ and $b = (b_1, \ldots, b_m)$. Let $y = (y_1, \ldots, y_n)$. Set
   $$\Phi(x, y) = \{ \psi(x) \leftrightarrow \psi(y) \} \cup \{ \varphi(x, b) \wedge \neg(y, b) \}$$
   Note that these are formulae over Ab. If $\Phi(x, y)$ were finitely realized, then by κ-saturation (since $|Ab| < \kappa$), it would be realized by $d, e \in M^n$. So $\operatorname{tp}(d/A) = \operatorname{tp}(e/A)$ but $d \in X$ and $e \notin X$. So, by

(b), we have some $f \in \mathrm{Aut}_A(\mathcal{M})$ with $f(d) = e$, contradicting the $\mathrm{Aut}_A(\mathcal{M})$-invariance of X. So $\Phi(x, y)$ is not finitely realized in $\mathcal{N}$. So there are $L(A)$-formulae $\psi_1, \ldots, \psi_\ell$ such that

$$\mathcal{M} \models \forall x \forall y \left(\left(\bigwedge_{i=1}^{\ell} \psi_i(x) \leftrightarrow \psi_i(y) \right) \to (\varphi(x, b) \leftrightarrow \varphi(y, b)) \right)$$

But if we partition M^n into finitely many disjoint sets $D_1, \ldots, D_{2^\ell}$ depending on which ψ_i are realized and which are not, then this says that each D_j is either contained in X or disjoint from X. So X is a finite union of D_j. But each D_j is A-definable. So X is A-definable.

Note that this required both κ-saturation and strong κ-homogeneity.

4. **(a)** $\implies$ **(b)** Clear.

   (b) $\implies$ **(c)** By (2).

   (c) $\implies$ **(a)** Let $X = \{ f(b) : f \in \mathrm{Aut}_A(\mathcal{M}) \}$. Then X is finite, and hence definable, and X is $\mathrm{Aut}_A(\mathcal{M})$-invariant. So, by (3), we have that X is A-definable. But $b \in X$ and X is finite; so $b \in \mathrm{acl}(A)$.

5. Similar.

$\square$ Theorem 179

We sometimes say a set X is A-invariant to mean that X is $\mathrm{Aut}_A(\mathcal{M})$-invariant.

As a general convention, if T is a complete theory, by a "sufficiently saturated model", we mean a model $\mathcal{U} \models T$ which is κ-saturated and strongly κ-homogeneous for some sufficiently large κ. Once such is fixed, we have that following additional conventions:

1. All parameter sets are assumed to be in U and of cardinality $< \kappa$.
2. Every type $p(x) \in S(A)$ is assumed to be over $A \subseteq U$ with $|A| < \kappa$; so all types are realized.
3. Every model $\mathcal{N} \models T$ is assumed to be of size $\leq \kappa$ and an elementary substructure of U.
4. We write $\models \varphi(a)$ to mean $\mathcal{U} \models \varphi(a)$.

unless explicitly stated otherwise.

3.4 Morley rank

Fix a complete theory T (not necessarily countable); fix a sufficiently saturated model $\mathcal{U}$.

Definition 180. Suppose $\varphi(x)$ is a formula with parameters where $x = (x_1, \ldots, x_n)$. We recursively define, for any ordinal α, what it means to say $\mathrm{MR}(\varphi) \geq \alpha$:

- $\mathrm{MR}(\varphi) \geq 0$ if φ is consistent.
- Given any ordinal α, we say $\mathrm{MR}(\varphi) \geq \alpha + 1$ if there exist formulae $\psi_0(x), \psi_1(x), \ldots$ with parameters (not necessarily the same parameters as φ) such that
  - $\mathcal{U} \models \forall x(\psi_i(x) \to \varphi(x))$; i.e. $\psi_i(\mathcal{U}) \subseteq \varphi(\mathcal{U})$.
  - For $i \neq j$, we have $\mathcal{U} \models \forall x(\neg(\psi_i(x) \wedge \psi_j(x)))$.
  - For all i, we have $\mathrm{MR}(\psi_i) \geq \alpha$.
- For β a limit ordinal, we say $\mathrm{MR}(\varphi) \geq \beta$ if $\mathrm{MR}(\varphi) \geq \alpha$ for all $\alpha < \beta$.

We now define what it means to say $\mathrm{MR}(\varphi) = \alpha$.

- If φ is inconsistent, we say $\mathrm{MR}(\varphi) = -\infty$.
- If $\mathrm{MR}(\varphi) \geq \alpha$ for all ordinals α, we set $\mathrm{MR}(\varphi) = \infty$.

- If φ is consistent and $\mathrm{MR}(\varphi)$ is not $\geq \alpha$ for all α, then there exists a maximal ordinal β such that $\mathrm{MR}(\varphi) \geq \beta$. (To see this, note that if γ is the least ordinal such that $\mathrm{MR}(\varphi) \not\geq \gamma$; by definition, we have γ is not a limit ordinal, say $\gamma = \beta + 1$, and then β is our desired ordinal.) For this β we define $\mathrm{MR}(\varphi) = \beta$.

If $X = \varphi(\mathcal{U})$ for some formula φ then we define $\mathrm{MR}(X) = \mathrm{MR}(\varphi)$.

Remark 181. If $\models \forall x(\varphi(x) \leftrightarrow \psi(x))$, then $\mathrm{MR}(\varphi) = \mathrm{MR}(\psi)$.

Lemma 182. $\mathrm{MR}(\varphi) = 0$ *if and only if* φ *is algebraic.*

Proof.

($\Longrightarrow$) Suppose $\mathrm{MR}(\varphi) = 0$; then $\mathrm{MR}(\varphi) \geq 0$, and φ is consistent. On the other hand, $\mathrm{MR}(\varphi) = 0$ implies that $\mathrm{MR}(\varphi) \not\geq 1$. So $\varphi(\mathcal{U})$ does not have infinitely many disjoint, definable subsets of Morley rank ≥ 0; i.e. $\varphi(\mathcal{U})$ does not have infinitely many disjoint, non-empty, definable sets. But for $a \in X = \varphi(\mathcal{U})$, we have that $\{ a \}$ is a non-empty, definable subset. So $\varphi(\mathcal{U})$ is finite. So φ is algebraic.

($\Longleftarrow$) Suppose φ is algebraic. Then φ is consistent, so $\mathrm{MR}(\varphi) \geq 0$. If we had $\mathrm{MR}(\varphi) \geq 1$, then $\varphi(\mathcal{U})$ would have infinitely many disjoint, non-empty, definable subsets, and $\varphi(\mathcal{U})$ would be infinite, a contradiction. So $\mathrm{MR}(\varphi) \not\geq 1$, and $\mathrm{MR}(\varphi) = 0$.

□ Lemma 182

Remark 183. This has to be computed in a sufficiently saturated model. (Actually $\aleph_1$-saturation and strong $\aleph_1$-homogeneity suffices; possibly $\aleph_0$ works.)

Lemma 184. *Suppose* $\varphi(x) = \psi(x, a)$ *where* $\psi(x, y)$ *is an* L*-formula and* $a = (a_1, \ldots, a_n) \in U^m$*. If* $a' \models \mathrm{tp}(a)$*, then* $\mathrm{MR}(\psi(x, a')) = \mathrm{MR}(\psi(x, a))$*. i.e.* MR *depends only on the type of the parameters.*

Proof. We show by induction on α that $\mathrm{MR}(\psi(x, a)) \geq \alpha$ implies $\mathrm{MR}(\psi(x, a')) \geq \alpha$.

- Suppose $\mathrm{MR}(\psi(x, a)) \geq 0$; then $\models \exists x \psi(x, a)$, and $\models \exists x \psi(x, a')$, so $\mathrm{MR}(\psi(x, a')) \geq 0$.
- Suppose $\mathrm{MR}(\psi(x, a)) \geq \alpha + 1$. Then there are $\psi_i(x, b_i)$ where $\psi_i(x, z_i)$ are L-formulae with $|z_i| = |b_i|$ such that
  - $\psi_i(\mathcal{U}, b_i) \subseteq \psi(\mathcal{U}, a)$.
  - $\psi_i(\mathcal{U}, b_i) \cap \psi_j(\mathcal{U}, b_j) = \emptyset$ for $i \neq j$.
  - $\mathrm{MR}(\psi_i(\mathcal{U}, b_i)) \geq \alpha$.

  Now, $\mathrm{tp}(a') = \mathrm{tp}(a)$, so $a' = f(a)$ for some $f \in \mathrm{Aut}(\mathcal{U})$. Then

  - $\psi_i(\mathcal{U}, f(b_i)) \subseteq \psi(\mathcal{U}, a')$.
  - $\psi_i(\mathcal{U}, f(b_i)) \cap \psi_j(\mathcal{U}, b_j) = \emptyset$ for $i \neq j$.
  - By the induction hypothesis, since $\mathrm{tp}(b_i) = \mathrm{tp}(f(b_i))$, we have that $\mathrm{MR}(\psi(\mathcal{U}, f(b_i))) = \mathrm{MR}(\psi_i(\mathcal{U}, b_i)) \geq \alpha$.

  So $\mathrm{MR}(\psi(\mathcal{U}, a')) \geq \alpha + 1$.
- Limit case is easy.

□ Lemma 184

Lemma 185.

1. *If* $\varphi \to \psi$ *then* $\mathrm{MR}(\varphi) \leq \mathrm{MR}(\psi)$.
2. *If* $\mathrm{MR}(\varphi) = \alpha$ *for* α *an ordinal, then for any* $\beta < \alpha$ *there is a formula* $\psi \to \varphi$ *such that* $\mathrm{MR}(\psi) = \beta$.

Proof.

1. Clear.

2. We apply induction on α. The case $\alpha = 0$ is vacuous.

   Suppose α is an ordinal with $\mathrm{MR}(\varphi) = \alpha + 1$; suppose $\beta < \alpha + 1$. Then there are $(\varphi_i : i < \omega)$ implying φ that are pairwise inconsistent with each $\mathrm{MR}(\varphi_i) \geq \alpha$. If all $\mathrm{MR}(\varphi_i) \geq \alpha + 1$, then $\mathrm{MR}(\varphi) \geq \alpha + 1$, a contradiction. So there is some i_0 such that $\mathrm{MR}(\varphi_{i_0}) < \alpha + 1$; then $\mathrm{MR}(\varphi_{i_0}) = \alpha$. If $\beta = \alpha$, then φ_{i_0} is our desired ψ. If $\beta < \alpha$, the by induction hypothesis there is $\psi \to \varphi_{i_0}$ with $\mathrm{MR}(\psi) = \beta$. But then $\psi \to \varphi$, and we have our desired ψ.

   The limit case is clear.

□ Lemma 185

Definition 186. We say φ *has Morley rank* if $\mathrm{MR}(\varphi)$ is an ordinal.

Corollary 187. *If φ has Morley rank, then* $\mathrm{MR}(\varphi) < (2^{|L|+\aleph_0})^+$.

Proof. Let

$$O = \{\, \alpha \text{ ordinal} : \mathrm{MR}(\psi(x)) = \alpha \text{ for some } \psi(x) \,\}$$

(This is a set by the axiom of replacement, since the collection of formulae with parameters is a set.) But

$$|O| \leq (|L| + \aleph_0)\left|\bigcup_{\ell<\omega} S_\ell(T)\right| \leq 2^{|L|+\aleph_0}$$

as the Morley rank of $\varphi(x, a)$ depends only on φ and the type of a.

(Note that $\psi(x)$ may have parameters from the big universal domain, so there are too many of them.)

By previous lemma, we have that O is an initial segment of an ordinal. So O is an ordinal with $|O| \leq 2^{|L|+\aleph_0}$. So $O < (2^{|L|+\aleph_0})^+$. So, for every $\alpha \in O$, we have $\alpha < (2^{|L|+\aleph_0})^+$. □ Corollary 187

Corollary 188. *If T is totally transcendental then every consistent formula has Morley rank.*

Proof. Suppose $\mathrm{MR}(\varphi) = \infty$. Let $\lambda = (2^{|L|+\aleph_0})^+$. Then $\mathrm{MR}(\varphi) \geq \lambda + 1$. In particular, there are $\varphi_0 \to \varphi$ and $\varphi_1 \to \varphi$ with $\varphi_0 \wedge \varphi_1$ inconsistent and $\mathrm{MR}(\varphi_0) \geq \lambda$, $\mathrm{MR}(\varphi_1) \geq \lambda$. By part (a) of the previous lemma, we may assume $\varphi_0 \wedge \varphi_1 \leftrightarrow \varphi$; just enlarge φ_0 to make this happen. (In particular, we can take $\varphi_0 = \varphi \wedge \neg\varphi_1$.) But then by the previous corollary, we have $\mathrm{MR}(\varphi_0) = \mathrm{MR}(\varphi_1) = \infty$. Iterating, we build an infinite binary tree. So T is not totally transcendental. □ Corollary 188

Lemma 189. $\mathrm{MR}(\varphi \vee \psi) = \max\{\, \mathrm{MR}(\varphi), \mathrm{MR}(\psi) \,\}$.

Proof. It is easily seen that $\mathrm{MR}(\varphi \vee \psi) \geq \max\{\, \mathrm{MR}(\varphi), \mathrm{MR}(\psi) \,\}$. For the converse, it suffices to show that if $\mathrm{MR}(\varphi \vee \psi) \geq \alpha + 1$, then $\max(\mathrm{MR}(\varphi), \mathrm{MR}(\psi)) \geq \alpha + 1$. Let $(\theta_i : i < \omega)$ witness $\mathrm{MR}(\varphi \vee \psi) \geq \alpha + 1$. For any i, we have $\theta_i \leftrightarrow (\theta_i \wedge \varphi) \vee (\theta_i \wedge \psi)$. By induction hypothesis, we have $\max(\mathrm{MR}(\theta_i \wedge \varphi), \mathrm{MR}(\theta_i \wedge \psi)) \geq \alpha$. So either $\mathrm{MR}(\theta_i \wedge \varphi) \geq \alpha$ or $\mathrm{MR}(\theta_i \wedge \psi) \geq \alpha$. So at least one of these cases happens infinitely often; say $\mathrm{MR}(\theta_i \wedge \varphi) \geq \alpha$ for infinitely many i. Then $(\theta_i \wedge \varphi : i < \omega)$ witnesses that $\mathrm{MR}(\varphi) \geq \alpha + 1$. So $\max(\mathrm{MR}(\varphi), \mathrm{MR}(\psi)) \geq \alpha + 1$.
□ Lemma 189

Definition 190. We say φ and ψ are *α-equivalent* (for α an ordinal) if $\mathrm{MR}((\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)) < \alpha$. (Note that the argument of MR here is the symmetric difference of φ and ψ.)

Exercise 191. This is an equivalence relation.

Proposition 192 (6.7.4)**.** *Suppose* $\mathrm{MR}(\varphi) = \alpha$ *an ordinal. Then φ is T-equivalent to some $\varphi_1 \vee \varphi_2 \vee \dots \varphi_d$ where*

- $\mathrm{MR}(\varphi_i) = \alpha$ *for each* $i \in \{\, 1, \dots, d \,\}$.
- $\varphi_1, \dots, \varphi_d$ *are pairwise disjoint.*
- *Each $\varphi_i(\mathcal{U})$ does not contain two disjoint definable sets of Morley rank α.*

Moreover, d is unique, and the decomposition is unique up to α-equivalence.

This $d = \mathrm{MD}(\varphi)$ is called the *Morley degree* of φ.

Proof. If $\varphi(\mathcal{U})$ can be split into two disjoint definable subsets of Morley rank α, then do so. Iterate. If we get an infinite tree, it must have an infinite branch; say $\varphi = \psi_0 \leftarrow \psi_1 \leftarrow \dots$ such that each ψ_i has Morley rank α and $\mathrm{MR}(\psi_i \wedge \neg\psi_{i+1}) = \alpha$. But then $\psi_0 \wedge \neg\psi_1, \psi_1 \wedge \neg\psi_2, \dots$ witness that $\mathrm{MR}(\varphi) \geq \alpha + 1$, a contradiction.

So the tree is finite. The leaf nodes of this finite tree are the desired $\varphi_1, \dots, \varphi_d$.

We now verify uniqueness of the decomposition. Suppose $\mathrm{MR}(\varphi) = \alpha$. Suppose $\varphi \leftrightarrow \varphi_1 \vee \dots \vee \varphi_d$ and $\varphi \leftrightarrow \psi_1 \vee \dots \vee \psi_\ell$ with each φ_j and ψ_j is of Morley rank α but cannot be split into two definable subsets of Morley rank α. Note that, for fixed i, we have $\psi_i \leftrightarrow (\psi_i \wedge \varphi_1) \vee \dots \vee (\psi_i \wedge \varphi_d)$; furthermore, the $\psi_i \wedge \varphi_j$ are disjoint and paritition $\psi_i(\mathcal{U})$. So there is a unique $1 \leq j_i \leq d$ such that $\mathrm{MR}(\psi_i \wedge \varphi_{j_i}) = \alpha$, and $\mathrm{MR}(\psi_i \wedge \varphi_j) < \alpha$ for $j \neq j_i$. So

$$\psi_i \wedge \neg\varphi_{j_i} = \bigvee_{j \neq j_i} (\psi_i \wedge \varphi_j)$$

So $\mathrm{MR}(\psi_i \wedge \neg\varphi_{j_i}) < \alpha$. So ψ_i is α-equivalent to φ_{j_i}, by a symmetric argument. Applying the same argument to φ_{j_i}, we see that $i \mapsto j_i$ is injective; so $\ell \leq d$, and each ψ_i is α-equivalent to φ_{j_i}. By symmetry, we are done. □ Proposition 192

Notation 193. $(\mathrm{MR}, \mathrm{MD})(\varphi) = (\mathrm{MR}(\varphi), \mathrm{MD}(\varphi))$. We order such pairs by the lexicographical ordering.

Remark 194. φ is strongly minimal if and only if $(\mathrm{MR}, \mathrm{MD})(\varphi) = (1, 1)$.

Remark 195. Suppose $\mathrm{MR}(\varphi) = \alpha$ is an ordinal; suppose ψ is such that $\mathrm{MR}(\varphi \wedge \psi) = \mathrm{MR}(\varphi \wedge \neg\psi) = \alpha$. Then $\mathrm{MD}(\varphi) = \mathrm{MD}(\varphi \wedge \psi) + \mathrm{MD}(\varphi \wedge \neg\psi)$. If, on the other hand, $\mathrm{MR}(\varphi \wedge \neg\psi) < \alpha$, then $\mathrm{MD}(\varphi) = \mathrm{MD}(\varphi \wedge \psi)$.

Theorem 196. *T is totally transcendental if and only if every consistent formula (with parameters) has Morley rank.*

Proof.

($\implies$) Done in Corollary 188.

($\impliedby$) Suppose T is not totally transcendental; let $(\varphi_j : j \in 2^{<\omega})$ be an infinite binary tree of consistent formulae witnessing this.

Claim 197. *If $\mathrm{MR}(\varphi_s) = \alpha$ is an ordinal, then $(\mathrm{MR}, \mathrm{MD})(\varphi_{s\frown i}) < (\mathrm{MR}, \mathrm{MD})(\varphi_s)$ for some $i \in \{0, 1\}$.*

Proof. Suppose $\mathrm{MR}(\varphi_{s0}) = \mathrm{MR}(\varphi_{s1}) = \alpha$. Then $\mathrm{MD}(\varphi) = \mathrm{MD}(\varphi_{s0}) + \mathrm{MD}(\varphi_{s1})$. So one of $\mathrm{MD}(\varphi_{s0})$ and $\mathrm{MD}(\varphi_{s1})$ is $< \mathrm{MD}(\varphi_j)$. □ Claim 197

If φ_ε has Morley rank, then we find an infinite properly descending sequence of (α_i, d_i) where the α_i are ordinals and $d_i \geq 1$. But this is a well-ordering, a contradiction. So $\mathrm{MR}(\varphi_\varepsilon) = \infty$.

□ Theorem 196

Definition 198. A *definable grape* (G, x) in T is a definable set $G \subseteq U^n$ with a definable $\times\colon G \times G \to G$ (i.e. $\Gamma(\times) \subseteq U^{3n}$ is definable) such that $(G, \times)$ is a grape. (Definitions here allow parameters.)

Definition 199. We say $(G, \times)$ is a *totally transcendental grape* if it is definable in a totally transcendental theory.

Corollary 200. *A totally transcendental grape satisfies the descending chain condition on definable subgrapes. i.e. there does not exist an infinite, properly descending chain of definable subgrapes.*

Proof. Suppose $(H, \times)$ is a definable subgrape of $(G, \times)$.

Claim 201. *If* $\mathrm{MR}(H) = \mathrm{MR}(G)$*, then* G/H *is finite and*

$$\mathrm{MD}(G) = \sum_{i=1}^{\ell} \mathrm{MD}(g_i H)$$

where $g_1 H, \ldots, g_\ell H$ *are the distinct left cosets of* H*.*

Proof. Let $g \in G$. Then the map $H \to gH$ given by $h \mapsto gh$ is a definable bijection using the parameter g. So $(\mathrm{MR}, \mathrm{MD})(H) = (\mathrm{MR}, \mathrm{MD})(gH)$. In particular, all cosets have Morley rank $\mathrm{MR}(G)$. But distinct cosets are disjoint; so we must have finitely many of them, else we would have infinitely many disjoint subsets of G of Morley rank $\mathrm{MR}(G)$, a contradiction. Say the distinct cosets are $g_1 H, \ldots, g_\ell H$. Then

$$G = \bigsqcup_{i=1}^{\ell} g_i H$$

So

$$\mathrm{MD}(G) = \sum_{i=1}^{\ell} \mathrm{MD}(g_i H)$$

□ Claim 201

So if $(H, \times)$ is a proper definable subgrape of $(G, \times)$, then $(\mathrm{MR}, \mathrm{MD})(H) < (\mathrm{MR}, \mathrm{MD})(G)$; the descending chain condition follows. □ Corollary 200

Example 202. $(\mathbb{Q}, +)$ is totally transcendental, since $(\mathbb{Q}, +) \models \mathrm{TFDAG}$, and the latter is a strongly minimal (and hence totally transcendental) theory. On the other hand, for $(\mathbb{Z}, +)$, let $(G, +)$ be a sufficiently saturated elementary extension. Then

$$\mathbb{Z} > 2\mathbb{Z} > \cdots > 2^n\mathbb{Z} > \ldots$$

is a definable descending chain that doesn't stabilize. So

$$G > 2G > \ldots$$

is a definable descending chain of subgrapes. So $(G, +)$ is not totally transcendental. So $\mathrm{Th}(\mathbb{Z}, +)$ is not totally transcendental.

Definition 203. Suppose $p \in S_n(A)$. We define $\mathrm{MR}(p) = \min\{\, \mathrm{MR}(\varphi) : \varphi \in p \,\}$. If $\mathrm{MR}(p) = \alpha$ is an ordinal, then we define $\mathrm{MD}(p) = \min\{\, \mathrm{MD}(\varphi) : \varphi \in p, \mathrm{MR}(\varphi) = \alpha \,\}$. If $a \in U^n$, we define $(\mathrm{MR}, \mathrm{MD})(a/A) = (\mathrm{MR}, \mathrm{MD})(\mathrm{tp}(a/A))$.

Remark 204.

1. Algebraic types have Morley rank 0 and Morley degree equal to the number of realizations.

2. $p \in S_n(A)$ is strongly minimal if and only if $(\mathrm{MR}, \mathrm{MD})(p) = (1, 1)$.

Proposition 205. *Suppose* $\varphi(x)$ *is an* $L(A)$*-formula. Then there is* $p \in S_n(A)$ *such that* $\varphi \in p$ *and* $\mathrm{MR}(p) = \mathrm{MR}(\varphi)$*.*

Proof. Consider

$$\Phi(x) = \{\, \varphi \,\} \cup \{\, \neg\psi : \psi \text{ an } L(A)\text{-formula, } \mathrm{MR}(\varphi \wedge \psi) < \mathrm{MR}(\varphi) \,\}$$

Then Φ is finitely satisfiable since $\varphi(\mathcal{U})$ cannot be contained in a finite union of definable subsets of strictly smaller rank. Extend to a complete type $p \in S_n(A)$. Then $\mathrm{MR}(p) \le \mathrm{MR}(\varphi)$ by definition. If $\mathrm{MR}(p) < \mathrm{MR}(\varphi)$, then there is $\psi \in p$ with $\mathrm{MR}(\psi) = \mathrm{MR}(p)$. But then $\psi \wedge \varphi \in p$; so $\mathrm{MR}(\varphi) \le \mathrm{MR}(\psi \wedge \varphi) \le \mathrm{MR}(\psi) = \mathrm{MR}(p) < \mathrm{MR}(\varphi)$, a contradiction.

So $\mathrm{MR}(p) = \mathrm{MR}(\varphi)$. □ Proposition 205

Lemma 206 (6.4.1)**.** *If* $b \in \mathrm{acl}(Aa)$ *then* $\mathrm{MR}(b/A) \le \mathrm{MR}(a/A)$*.*

Proof. We may assume that $\mathrm{MR}(a/A) = \alpha$ is an ordinal. We prove by induction on α that $\mathrm{MR}(b/A) \leq \alpha$.

For the base case, suppose $\alpha = 0$; then $a \in \mathrm{acl}(A)$ and $b \in \mathrm{acl}(Aa)$. So $b \in \mathrm{acl}(A)$, and $\mathrm{MR}(b/A) = 0$.

Now, for the induction step, suppose $\alpha > 0$; then we have $\varphi(x, y) \in \mathrm{tp}(a, b/A)$ such that $\varphi(a, \mathcal{U})$ is finite, say of size d. We can add to $\varphi(x, y)$ so that for all a', we have $|\varphi(a', \mathcal{U})| \leq d$; we do this by replacing $\varphi(x, y)$ with

$$\varphi(x, y) \wedge \exists^{\leq d} y \varphi(x, y)$$

Let $\psi(x) = \exists y(\varphi(x, y)) \in \mathrm{tp}(a/A)$. Replacing $\varphi(x, y)$ by $\varphi(x, y) \wedge \sigma(x)$ where $\sigma(x) \in \mathrm{tp}(a/A)$ with $\mathrm{MR}(\sigma) = \mathrm{MR}(a/A)$, we may assume that $\mathrm{MR}(\psi(x)) = \mathrm{MR}(a/A) = \alpha$. Let $\chi(y) = \exists x \varphi(x, y) \in \mathrm{tp}(b/A)$.

Claim 207. $\mathrm{MR}(\chi) \leq \alpha$.

Proof. Suppose $(\chi_i(y) : i < \omega)$ are pairwise disjoint, definable subsets of $\chi(\mathcal{U})$. Let $\psi_i(x) = \exists y(\varphi(x, y) \wedge \chi_i(y))$. Then each $\psi_i(x) \to \psi(x)$.

Subclaim 208. *Some* ψ_{i_0} *has* $\mathrm{MR}(\psi_{i_0}) = \beta < \alpha$.

Proof. Suppose $a' \in \psi_i(\mathcal{U}) \cap \psi_j(\mathcal{U})$ where $i \neq j$. Then there are b_1, b_2 with $\varphi(a', b_1)$ and $\varphi(a', b_2)$, where $b_1 \in \chi_1(\mathcal{U})$ and $b_2 \in \chi_2(\mathcal{U})$. But $\chi_i(\mathcal{U}) \cap \chi_j(\mathcal{U}) = \emptyset$. So $b_1 \neq b_2$. So any $d + 1$ distinct members of $\{ \psi_i(\mathcal{U}) : i < \omega \}$ has empty intersection.

Now, suppose for contradiction that $\mathrm{MR}(\psi) = \alpha$ for all $i < \omega$.

Case 1. Suppose $\mathrm{MR}(\psi_1 \wedge \psi_0) < \alpha$, then $\mathrm{MR}(\psi_0 \wedge \neg\psi_1) = \alpha$; replace ψ_0 by $\psi_0 \wedge \neg\psi_1$, and similarly replace ψ_1 by $\psi_1 \wedge \neg\psi_0$.

Case 2. Suppose $\mathrm{MR}(\psi_1 \wedge \psi_0) = \alpha$; replace ψ_0 by $\psi_0 \wedge \psi_1$, and drop ψ_1.

The second case cannot happen more than d times, since $\psi_0(\mathcal{U}) \wedge \cdots \wedge \psi_{d+1}(\mathcal{U}) = \emptyset$. Iterating this produces an infinite family of disjoint, definable subsets of $\psi(x)$ of Morley rank α, contradicting our assumption that $\mathrm{MR}(\psi) = \alpha$. □ Subclaim 208

So there is i_0 such that $\mathrm{MR}(\psi_{i_0}(x)) = \beta < \alpha$. Let $b' \in \chi_{i_0}(\mathcal{U})$. Find a' such that $\varphi(a', b')$. Then $b' \in \mathrm{acl}(Aa')$ since $|\varphi(a', \mathcal{U})| \leq d$. Then $a' \in \psi_{i_0}(\mathcal{U})$; so $\mathrm{MR}(a'/A) \leq \beta < \alpha$. Then, by the induction hypothesis, we have $\mathrm{MR}(b'/A) \leq \mathrm{MR}(a'/A) \leq \beta < \alpha$. By the previous proposition, we have that $\chi_{i_0}(\mathcal{U})$ has an element whose Morley rank over A is $\mathrm{MR}(\chi_{i_0})$. So $\mathrm{MR}(\chi_{i_0}) \leq \beta < \alpha$.

So $\mathrm{MR}(\chi) \leq \alpha$. □ Claim 207

Thus $\mathrm{MR}(b/A) \leq \mathrm{MR}(\chi) = \alpha = \mathrm{MR}(a/A)$ since $\chi \in \mathrm{tp}(b/A)$. □ Lemma 206

Proposition 209. *Suppose* $\varphi(x)$ *defined over* B *is strongly minimal. Suppose* $a_1, \ldots, a_\ell \in \varphi(\mathcal{U}) \subseteq U^n$. *Then* $\{ a_1, \ldots, a_\ell \}$ *are* acl*-independent over* B *if and only if* $\mathrm{MR}(a_1, \ldots, a_\ell/B) = \ell$.

(Recall the pregeometry is given by $(\varphi(\mathcal{U}), \mathrm{cl})$ *where* $\mathrm{cl}(A) = \mathrm{acl}(AB) \cap \varphi(\mathcal{U})$*.)*

Proof. We apply induction on ℓ.

Case 1. Suppose $\ell = 1$. Then $\{ a \}$ is acl-independent over B if and only if $a \notin \mathrm{acl}(B)$, which holds if and only if $\mathrm{MR}(a/B) \geq 1$. But $\varphi(x) \in \mathrm{tp}(a/B)$ and $\mathrm{MR}(\varphi) = 1$. So $\mathrm{MR}(a/B) \leq 1$. So $\{ a \}$ is acl-independent if and only if $\mathrm{MR}(a/B) = 1$.

Case 2. Suppose $\ell > 1$.

($\Longleftarrow$) Suppose $\mathrm{MR}(a_1 \ldots a_\ell/B) = \ell$. Let $\{ a_1, \ldots, a_m \}$ for $m \leq \ell$ be an acl-basis (i.e. a maximal acl-independent subset) of $\{ a_1, \ldots, a_\ell \}$ over B. Then $(a_1, \ldots, a_\ell) \in \mathrm{acl}(Ba_1 \ldots a_m)$. So, by 6.4.1, we have $\mathrm{MR}(a_1 \ldots a_\ell/B) \leq \mathrm{MR}(a_1 \ldots a_m/B)$. On the other hand, we have $\mathrm{MR}(a_1 \ldots a_\ell/B) \geq \mathrm{MR}(a_1 \ldots a_m/B)$ since $m \leq \ell$. To see this, we use the following exercise:

Exercise 210. Suppose $X \subseteq U^{n+1}$ is a definable set and $\pi\colon U^{n+1} \to U^n$ is a coordinate projection, then $\mathrm{MR}(\pi X) \leq \mathrm{MR}(X)$.

We then note that if $\psi(x_1,\dots,x_\ell) \in \operatorname{tp}(a_1 \dots a_\ell/B)$, then $\exists x_{m+1} \dots \exists x_\ell \psi(x_1,\dots,x_\ell) \in \operatorname{tp}(a_1 \dots a_m/B)$, and by the exercise, we have $\operatorname{MR}(\exists x_{m+1} \dots \exists x_\ell \psi(x_1,\dots,x_\ell)) \le \operatorname{MR}(\psi(x_1,\dots,x_\ell))$; thus $\operatorname{MR}(a_1 \dots a_\ell/M) \ge \operatorname{MR}(a_1 \dots a_m/B)$.

So $\operatorname{MR}(a_1 \dots a_\ell/B) = \operatorname{MR}(a_1 \dots a_m/B)$. Now, if $\{ a_1,\dots,a_\ell \}$ were acl-dependent over B, then $m < \ell$, so by the induction hypothesis we have $\operatorname{MR}(a_1 \dots a_m/B) = m < \ell = \operatorname{MR}(a_1 \dots a_\ell/B)$, a contradiction. So $\{ a_1,\dots,a_\ell \}$ is acl-independent.

($\Longrightarrow$) Suppose $\{ a_1,\dots,a_\ell \}$ is acl-independent over B.

Claim 211. $\operatorname{MR}(a_1 \dots a_\ell/B) \ge \ell$.

Proof. Let $b_1, b_2, \dots \in \varphi(\mathcal{U}) \setminus \operatorname{acl}(B)$ be distinct. Note that this exists since $\varphi(x)$ has a unique non-algebraic extension $p(x) \in S_n(B)$; we can then take the b_i to be the realizations of $p(x)$. Suppose $\psi(x_1,\dots,x_\ell) \in \operatorname{tp}(a_1 \dots a_\ell/B)$. Let $\psi_i(x_1,\dots,x_\ell) = \psi(x_1,\dots,x_\ell) \wedge (x_1 = b_i)$; then ψ_i is an $L(Bb_i)$-formula. We also have $\psi_i \to \psi$ and $(\psi_i \wedge \psi_j)(\mathcal{U}) = \emptyset$ for $i \ne j$.

We now compute $\operatorname{MR}(\psi_i)$. Fix i. Let $c_2,\dots,c_\ell \in \varphi(\mathcal{U})$ be such that $\{ b_i, c_2, \dots, c_\ell \}$ is acl-independent over B. To see that we can do this, note that $b_i \notin \operatorname{acl}(B)$. Then the unique non-algebraic type $p(x)$ over B containing $\varphi(x)$ is strongly minimal, so it has a unique non-algebraic extension $p_2(x) \in S_n(Bb_i)$. Let $c_2 \models p_2(x)$; then $c_2 \notin \operatorname{acl}(Bb_i)$, so $\{ b_i, c_2 \}$ is acl-independent over B. Now, $p_2(x)$ has a unique non-algebraic extension $p_3(x) \in S_n(Bb_ic_2)$; we proceed inductively.

Now $\{ a_1,\dots,a_\ell \}$ is also acl-independent over B and $\operatorname{tp}(b_ic_2 \dots c_\ell/B) = \operatorname{tp}(a_1 \dots a_\ell/B) \ni \psi$. So $\psi_i \in \operatorname{tp}(b_ic_2 \dots c_\ell/Bb_i)$. So $\operatorname{MR}(\psi_i) \ge \operatorname{MR}(b_1c_2 \dots c_\ell/Bb_i) \ge \operatorname{MR}(c_2 \dots c_\ell/Bb_i) = \ell - 1$ by the induction hypothesis. So $\operatorname{MR}(\psi) \ge \ell$ for all $\psi \in \operatorname{tp}(a_1 \dots a_\ell/B)$; so $\operatorname{MR}(a_1 \dots a_\ell/B) \ge \ell$. □ Claim 211

Claim 212. $\operatorname{MR}(a_1 \dots a_\ell/B) \le \ell$.

Proof. By the previous claim we have $\operatorname{MR}(\varphi(\mathcal{U})^\ell) \ge \ell$ since $\operatorname{MR}(a_1 \dots a_\ell/B) \ge \ell$ and $(a_1,\dots,a_\ell) \in \varphi(\mathcal{U})^\ell$. We show that $\operatorname{MR}(\varphi(\mathcal{U})^\ell) \le \ell$. Suppose otherwise; then $\varphi(\mathcal{U})^\ell$ has two disjoint definable subsets $X, Y \subseteq \varphi(\mathcal{U})^\ell$ over $B' \supseteq B$ with $\operatorname{MR}(X) = \ell = \operatorname{MR}(Y)$. Let $c \in X$ satisfy $\operatorname{MR}(c/B') = \operatorname{MR}(X) \ge \ell$; let $b \in Y$ satisfy $\operatorname{MR}(b/B') = \operatorname{MR}(Y) \ge \ell$. Then by the forward direction of this proposition, if $c = (c_1,\dots,c_\ell)$ and $b = (b_1,\dots,b_\ell)$, then $\{ c_1,\dots,c_\ell \}$ and $\{ b_1,\dots,b_\ell \}$ are acl-independent over B'. So $\operatorname{tp}(c_1 \dots c_\ell/B') = \operatorname{tp}(b_1 \dots b_\ell/B')$, contradicting our assumption that $c \in X$, $b \in Y$, and $X \cap Y = \emptyset$. So $\operatorname{MR}(\varphi(\mathcal{U})^\ell) \le \ell$. □ Claim 212

So $\operatorname{MR}(a_1 \dots a_\ell/B) = \ell$.

□ Proposition 209

Corollary 213 (6.4.2)**.** *If $\varphi(x)$ is strongly mimimal over B and $a_1,\dots,a_m \in \varphi(\mathcal{U})$, then* $\operatorname{MR}(a_1 \dots a_n/B) = \text{acl-dim}(\{ a_1,\dots,a_n \}/B)$.

Proof. Let $\{ a_1,\dots,a_\ell \}$ be an acl-basis over B for $\{ a_1,\dots,a_m \}$ with $\ell \le m$. Then $\text{acl-dim}(\{ a_1,\dots,a_m \}/B) = \ell$. On the other hand, $\operatorname{MR}(a_1,\dots,a_\ell/B) \le \operatorname{MR}(a_1 \dots a_m/B) \le \operatorname{MR}(a_1 \dots a_\ell/B)$ since $a_1,\dots,a_m \in \operatorname{acl}(Ba_1 \dots a_\ell)$. So $\operatorname{MR}(a_1 \dots a_m/B) = \operatorname{MR}(a_1 \dots a_\ell/B) = \ell$ by the previous proposition. □ Corollary 213

Example 214.

1. Consider the theory T of infinite sets. Suppose $a_1,\dots,a_m \in U$ with $B \subseteq U$. Then $\operatorname{MR}(a_1 \dots a_m/B) = |\{ a_1,\dots,a_m \} \setminus B|$.
2. If $T = \mathrm{VS}_F$ with $v_1,\dots,v_m \in V$ and $B \subseteq V$, then $\operatorname{MR}(v_1 \dots v_m/B) = \dim_F(v_1 \dots v_m/B)$ is the relative linear dimension.
3. If $T = \mathrm{ACF}_p$ for p a prime or zero, we have $\operatorname{MR}(a_1 \dots a_m/B) = \operatorname{trdeg}(\mathbb{F}(B, a_1,\dots,a_m)/\mathbb{F}(B))$.

4 Differential fields

All rings are commutative, have unity, and extend $\mathbb{Q}$.

Definition 215. A *derivation* on a ring R is an additive function $\delta\colon R \to R$ (i.e. $\delta(a+b) = \delta a + \delta b$) satisfying the Leibniz rule:

$$\delta(ab) = a\delta b + b\delta a$$

We call $(R, 0, 1, +, -, \times, \delta)$ a *differential ring*. We define the *constants of* (R, δ) to be the subring $\{ x \in R : \delta x = 0 \}$. We let DF_0 be the theory of differential fields of characteristic 0.

Example 216. The natural examples are rings of functions:

- $(\mathbb{C}[z], \frac{d}{dz})$.
- $(\mathbb{C}(z), \frac{d}{dz})$.
- The field of meromorphic functions at the origin on $\mathbb{C}$ with $\frac{d}{dz}$.

Remark 217. Modulo DF_0, we have that every quantifier-free L-formula $\varphi(x)$ (with $x = (x_1, \dots, x_n)$) is equivalent to a finite boolean combination of equations of the form

$$P(x, \delta x, \dots, \delta^k x) = 0$$

where

- $\delta x = (\delta x_1, \dots, \delta x_n)$
- $P \in \mathbb{Z}[X_0, X_1, \dots, X_K]$ with $X_i = (X_{i1}, \dots, X_{in})$.

Definition 218. Suppose (K, δ) is a differential field; suppose $z = (z_1, \dots, z_n)$ are indeterminates. We set $K\{z\} = K[X_0, X_1, \dots]$ (with $X_i = (X_{i1}, \dots, X_{in})$ and where we identify $X_0 = z$) equipped with the derivation $\delta x_i = x_{i+1}$ (extended in the canonical way to all of $K[X_0, \dots]$ using additivity and the Leibniz rule). A typical element of $K\{z\}$ is of the form $P(z, \delta z, \delta^2 z, \delta^k z)$ for some k. We call $K\{z\}$ the *ring of differential polynomials* (sometimes abbreviated *δ-polynomials*).

Aside 219. If $(K, \delta) \models \mathrm{DF}_p$, we have $\delta(a^p) = pa^{p-1}\delta a = 0$ for all $a \in K$; so K^p are constants. But K/K^p is a finite extension, so in some sense "most" of the elements are constants. Better to work with Hasse-Schmidt derivations.

Differential algebraic geometry is an expansion of algebraic geometry. Given $P \in K\{z\}$, we set $\operatorname{ord}(P)$ to be the largest k such that $\delta^k z$ appears in P; the differential polynomials of order 0 are then just ordinary polynomials in z.

Where should we look for solutions to differential polynomial equations?

We go to existentially closed differential fields.

Definition 220. $\mathcal{M} \models T$ is *existentially closed* if for any quantifier-free formula $\varphi(x)$ over $\mathcal{M}$ (with $x = (x_1, \dots, x_n)$) such that φ has a realization in some $\mathcal{N} \models T$ with $\mathcal{M} \subseteq \mathcal{N}$, we have that $\varphi(x)$ has a realization in $\mathcal{M}$.

Example 221. Algebraically closed fields are precisely the existentially closed fields.

We work in existentially closed differential fields. By last term, a theory has existentially closed models if it is universal-existential; so DF_0 has existentially closed models.

Problem: the definition of existentially closed is too unwieldy, and in particular is not first-order.

Definition 222. A *differentially closed field* is a differential field (K, δ) such that given any $P, Q \in K\{x\}$ (where x is a single variable) with $\operatorname{ord} Q < \operatorname{ord} P$, we have $a \in K$ such that $P(a) = 0$ and $Q(a) \neq 0$.

Remark 223. This is first-order: we could say something like, for $M \leq N$,

- For all choices of coefficients $(c_{i_0, \dots, i_n} : i_0 + \cdots + i_n \leq N)$

- For all choices of coefficients $(d_{j_0,\dots,j_m} : j_0 + \dots + j_n \leq M)$
- if some $c_{i_0,\dots,i_n} \neq 0$ with $i_n \neq 0$
- then there exists a such that
  - $$0 = \sum_{i_0+\dots+i_n \leq N} c_{i_0,\dots,i_n} a^{i_0} (\delta a)^{i_1} \dots (\delta^n a)^{i_n}$$
  - $$0 \neq \sum_{j_0+\dots+j_m \leq M} d_{j_0,\dots,j_m} a^{j_0} (\delta a)^{j_1} \dots (\delta^m a)^{j_m}$$

Assignment 4. *Due Monday December 7, questions 6.1.2, 6.2.2, 6.2.3, 6.4.1.*

Lemma 224 (D1)**.** *Suppose (R, δ) is a differential ring. Suppose $P(x_1, \dots, x_n) \in R[x_1, \dots, x_n]$; suppose $a_1, \dots, a_n \in R$. Then*

$$\delta(P(a_1, \dots, a_n)) = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i} \delta a_i + P^\delta(a_1, \dots, a_n)$$

where P^δ is obtained from P by applying δ to the coefficients.

Proof. By example. Let $P = cxy \in R[x, y]$ for $c \in R$. Then

$$\begin{aligned} \delta(P(a, b)) &= \delta(cab) \\ &= \delta(c)ab + c(a\delta b + b\delta a) \\ &= \delta(c)ab + ca\delta(b) + cb\delta(a) \\ &= P^\delta(a, b) + c\frac{\partial P}{\partial y}(a, b)\delta(b) + \frac{\partial P}{\partial x}(a, b)\delta(a) \end{aligned}$$

In general consider $cx_1^{m_1} \dots x_n^{m_n}$. We then apply induction on $m_1 + \dots + m_n$. □ Lemma 224

Lemma 225 (D2)**.** *Suppose (R, δ) is a differential integral domain. Then*

1. *δ extends uniquely to a derivation on $K = \mathrm{Frac}(R)$.*
2. *Suppose $L \supseteq K$ is an extension field. Suppose $a_1, \dots, a_{n-1} \in L$ are algebraically independent over K; suppose $a_n \in L$ has $a_n \in K(a_1, \dots, a_{n-1})^{\mathrm{alg}}$. Then there is a unique derivation δ on $K(a_1, \dots, a_n)$ extending δ on K such that $\delta(a_i) = a_{i+1}$ for $i \in \{1, \dots, n-1\}$.*
3. *δ extends uniquely to K^{alg}.*

Proof.

1. We define
$$\delta\Big(\frac{a}{b}\Big) = \frac{b\delta a - a\delta b}{b^2}$$
for any $a, b \in R$. Check that this is a derivation on K. It is unique as this formula is obtained by the Leibniz rule applied to $\delta(ab^{-1})$.
2. **Case 1.** Suppose $n = 1$; we are given $a \in K^{\mathrm{alg}}$, and we wish to extend δ to $K(a)$. Let $P(x) \in K[x]$ be the minimal polynomial of a over K. Then $0 = P(a)$; so
$$0 = \delta(P(a)) = \frac{\mathrm{d}P}{\mathrm{d}x}(a)\delta a + P^\delta(a)$$
by Lemma 224. But $\frac{\mathrm{d}P}{\mathrm{d}x}$ has strictly smaller degree than P; so $\frac{\mathrm{d}P}{\mathrm{d}x}(a) \neq 0$, and
$$\delta a = \frac{-P^\delta(a)}{\frac{\mathrm{d}P}{\mathrm{d}x}(a)}$$
This proves uniqueness; one checks that this actually defines a derivation on $K(a)$.

Case 2. Suppose $n > 1$. We set

$$\delta(a_n) = \frac{-\sum_{i=1}^{n-1} \frac{\partial P}{\partial x_i}(a_1, \dots, a_n)\delta a_i + P^\delta(a_1, \dots, a_n)}{\frac{\partial P}{\partial x_n}(a_1, \dots, a_n)}$$

where P is obtained as follows: let $Q(x_n) \in K(a_1, \dots, a_{n-1})[x_n]$ be the minimal polynomial of a_n over $K(a_1, \dots, a_{n-1})$. Clearing denominators, we get $Q' \in K[a_1, \dots, a_{n-1}][x_n]$ with $Q'(a_n) = 0$. We then write $Q' = P(a_1, \dots, a_{n-1}, x_n)$ for some $P \in K[x_1, \dots, x_n]$; this is our desired P.

3. Iterate the $n = 1$ case of (2) to extend uniquely all the way to K^{alg}.

□ Lemma 225

Proposition 226 (D3)**.** *Any differential field extends to a differentially closed field.*

Proof. Suppose $(K, \delta) \models \mathrm{DF}_0$. Given $P, Q \in K\{z\}$ with $\mathrm{ord}(P) > \mathrm{ord}(Q)$, we want an extension $(F, \delta) \supseteq (K, \delta)$ with $c \in F$ such that $P(c) = 0$ and $Q(c) \neq 0$. This will suffice by a double-chain-type argument. Take

$$\begin{aligned} P &= f(z, \delta z, \dots, \delta^n z) \\ Q &= g(z, \delta z, \dots, \delta^m z) \end{aligned}$$

where $n = \mathrm{ord}(P) > \mathrm{ord}(Q) = m$ and $f \in K[x_0, \dots, x_n]$ with x_n appearing and $g \in K[x_0, \dots, x_m]$ with x_m appearing. Let $a \in K(x_0, \dots, x_{n-1})$ satisfy $f(x_0, \dots, x_{n-1}, a) = 0$. (Possible because f is non-constant as an element of $K(x_0, \dots, x_{n-1})[x_n]$, and thus has a root in $K(x_0, \dots, x_{n-1})^{\mathrm{alg}}$.) Let $F = K(x_0, \dots, x_{n-1}, a) \supseteq K$. Then by Lemma 225 part (2), we can extend δ to $K(x_0, \dots, x_{n-1}, a)$ so that $\delta x_0 = x_1, \dots, \delta x_{n-1} = a$. So

$$\begin{aligned} 0 &= f(x_0, \dots, x_{n-1}, a) \\ &= f(x_0, \delta x_0, \delta^2 x_0, \dots, \delta^{n-1} x_0, \delta^n x_0) \\ &= P(x_0) \\ 0 &\neq g(x_0, x_1, \dots, x_m) \\ &= g(x_0, \delta x_0, \dots, \delta^m x_0) \\ &= Q(x_0) \end{aligned}$$

So $c = x_0 \in F$ works. □ Proposition 226

Theorem 227 (D4)**.** DCF_0 *admits quantifier elimination.*

Proof. Suppose $(F_i, \delta) \models \mathrm{DCF}_0$ for $i \in \{1, 2\}$. Suppose $(R, \delta) \subseteq (F_i, \delta)$ is a differential subring of F_1 and F_2. Then (R, δ) extends uniquely to $K = \mathrm{Frac}(R)$; we may thus assume that (K, δ) is a differential subfield of (F_i, δ) for $i \in \{1, 2\}$.

Claim 228. *It suffices to prove that for any $a \in F_1$ there is an L-embedding of $K\langle a\rangle = K(a, \delta a, \delta^2 a, \dots)$ (the differential field generated by a over K) into an elementary extension of (F_2, δ) over K.*

Proof. Suppose $\theta(x)$ be a conjunction of literals over K; suppose $a \in F_1$ realizes $\theta(x)$. Then by assumption we have an L-embedding $f\colon (K\langle a\rangle, \delta) \hookrightarrow (\widetilde{F_2}, \delta)$ satisfying

$$\begin{array}{ccc} (K\langle a\rangle, \delta) & \xhookrightarrow{f} & (\widetilde{F_2}, \delta) \\ {\scriptstyle\subseteq}\uparrow & & {\scriptstyle\preceq}\uparrow \\ (K, \delta) & \xrightarrow{\subseteq} & (F_2, \delta) \end{array}$$

where $(\widetilde{F_2}, \delta) \succeq (F_2, \delta)$. Let $b = f(a) \in \widetilde{F_2}$. Then $f\colon K\langle a\rangle \to K\langle b\rangle$ is an L-isomorphism over K with $f(\delta^i a) = \delta^i b$. Then

$$\begin{aligned}
(F_1, \delta) \models \theta(a) &\implies (K\langle a\rangle, \delta) \models \theta(a) \text{ (since } \theta \text{ is quantifier-free and } (K\langle a\rangle, \delta) \subseteq (F_1, \delta))\\
&\implies (K\langle b\rangle, \delta) \models \theta(b) \text{ (since } f \text{ is an } L\text{-isomorphism with } f \restriction K = \mathrm{id} \text{ and } f(a) = b)\\
&\implies (\widetilde{F_2}, \delta) \models \theta(b)\\
&\implies (\widetilde{F_2}, \delta) \models \exists x \theta(x)\\
&\implies (F_2, \delta) \models \exists x \theta(x) \text{ (since } (F_2, \delta) \preceq (\widetilde{F_2}, \delta))
\end{aligned}$$

So our more familiar criterion quantifier elimination holds. $\square$ Claim 228

Remark 229. The above can be made into a general criterion for quantifier elimination.

We verify the claimed condition for quantifier elimination.

Case 1. Suppose $\{ a, \delta a, \delta^2 a, \dots \}$ is algebraically independent in F_1 over K.

Claim 230. *For each $Q \in K\{x\} \setminus \{ 0 \}$, there is $b \in F_2$ such that $Q(b) \neq 0$.*

Proof. By the axioms there is b such that $\delta^{\mathrm{ord}(Q)+1} x = 0$ and $Q(x) \neq 0$. $\square$ Claim 230

Thus $\Phi(x) = \{ Q(x) \neq 0 : Q \in K\{x\}, Q \neq 0 \}$ is finitely realized in (F_2, δ).

Remark 231. Note that

$$\bigwedge_{i=1}^{\ell} (Q_i(b) \neq 0)$$

holds if and only if $(Q_1 Q_2 \dots Q_\ell)(b) \neq 0$.

So there is $(\widetilde{F_2}, \delta) \succeq (F_2, \delta)$ and $b \in \widetilde{F_2}$ such that $\models \Phi(b)$; i.e. $\{ b, \delta b, \dots \}$ is algebraically independent over K in $\widetilde{F_2}$.

Case 2. Suppose $\{ a, \delta a, \dots \}$ is algebraically dependent in F_1 over K. Then there is $n < \omega$ such that $\{ a, \dots, \delta^{n-1} a \}$ is algebraically independent over K but $\delta^n a \in K(a, \delta a, \dots, \delta^{n-1} a)^{\mathrm{alg}}$. Let $f(x_0, \dots, x_n) \in K[x_0, \dots, x_n]$ be such that $f(a, \delta a, \dots, \delta^{n-1} a, x_n)$ is a minimal polynomial for $\delta^n a$ over $K(a, \dots, \delta^{n-1} a)$.

We then know that $K\langle a\rangle = K(a, \dots, \delta^n a)$ by D2 (ii). Let

$$\Phi(x) = \{ f(x, \delta x, \dots, \delta^n x) = 0 \} \cup \{ g(x, \delta x, \dots, \delta^m x) \neq 0 : m < n, g \neq 0 \}$$

Then $\Phi(x)$ is finitely satisfiable in F_2 by the axioms for DCF_0. (Note that $\mathrm{ord}(g_1 g_2) \leq \max\{ \mathrm{ord}(g_1), \mathrm{ord}(g_2) \}$.) Hence there is some $(\widetilde{F_2}, \delta) \succeq (F_2, \delta)$ and $b \in F_2$ such that $(\widetilde{F_2}, \delta) \models \Phi(b)$. Then $\{ b, \delta b, \dots, \delta^{n-1} b \}$ is algebraically independent. We then get $\alpha\colon K(a, \dots, \delta^{n-1} a) \to K(b, \dots, \delta^{n-1} b)$ such that

$$\begin{tikzcd}
K(a, \dots, \delta^{n-1} a) \arrow[rr, "\alpha"] & & K(b, \dots, \delta^{n-1} b) \\
& K \arrow[ul, "\subseteq"] \arrow[ur, "\subseteq"'] &
\end{tikzcd}$$

and $\alpha(\delta^i a) = \delta^i b$. But f is a minimal polynomial of $\delta^n a$ over $K(a, \dots, \delta^{n-1} a)$, and

$$\alpha(f(a, \dots, \delta^{n-1} a, x_n)) = f(b, \delta b, \dots, \delta^{n-1} b, x_n)$$

is a minimal polynomial of $\delta^n b$ over $K(b, \dots, \delta^{n-1} b)$. So we can extend α to a field isomorphism $\alpha'\colon K\langle a\rangle = K(a, \dots, \delta^n a) \to K(b, \dots, \delta^n b) = K\langle b\rangle$ such that $\alpha'(\delta^i a) = \delta^i b$ for $i \leq n$ and $\alpha' \restriction K = \mathrm{id}_K$. So α' is an isomorphism of differential fields. So we have $\alpha'\colon K\langle a\rangle \to K\langle b\rangle \subseteq (\widetilde{F_2}, \delta)$. So we have proven our criterion.

□ Theorem 227

Theorem 232 (D5). DCF_0 *is complete.*

Proof. $(\mathbb{Z}, 0)$ embeds in every differential field, since $1 = 1 \cdot 1$, so $\delta(1) = 1 \cdot \delta(1) + \delta(1) \cdot 1 = 2\delta(1)$. So $\delta(1) = 0$, and $\delta(n) = 0$ for all $n \in \mathbb{Z}$. But DCF_0 admits quantifier elimination; so any statement is equivalent to a quantifier-free statement, which can then be decided in the image of $(\mathbb{Z}, 0)$. So DCF_0 is complete. □ Theorem 232

Theorem 233 (D6). DCF_0 *is the theory of existentially closed differential fields.*

Proof.

($\Longleftarrow$) Suppose (F, δ) is existentially closed. By D3 we can extend (F, δ) to $(\widetilde{F}, \delta) \models \mathrm{DCF}_0$. But (F, δ) is existentially closed, and $(F, \delta) \subseteq (\widetilde{F}, \delta)$; so $(F, \delta) \models \mathrm{DCF}_0$ since DCF_0 is universal-existential. (By checking axioms and using the fact that (F, δ) is existentially closed.)

($\Longrightarrow$) Suppose $(F, \delta) \models \mathrm{DCF}_0$. Suppose $\theta(x)$ is quantifier-free over F with $(F, \delta) \subseteq (F_1, \delta)$ with $\theta(x)$ realized by $a \in F_1$. Then
$$(F, \delta) \subseteq (F_1, \delta) \subseteq (\widetilde{F}, \delta) \models \mathrm{DCF}_0$$
with $(F, \delta) \models \mathrm{DCF}_0$. By quantifier elimination, we have $(F, \delta) \preceq (\widetilde{F_1}, \delta)$. But $\widetilde{F_1} \models \exists x \theta(x)$; so $F \models \exists x \theta(x)$. So (F, δ) is existentially closed.

□ Theorem 233

Theorem 234 (D7). DCF_0 *is ω-stable.*

Proof. Suppose $(K, \delta) \models \mathrm{DCF}_0$ with $A \subseteq K$ countable. We wish to show that $S_1(A)$ is countable. Let $F = \mathbb{Q}\langle A \rangle$ be the differential field generated by A over $\mathbb{Q}$; then $F = \mathbb{Q}(\{\, \delta^i a : i < \omega, a \in A \,\})$. Then $|F| = \aleph_0$. It suffices to show that $S_1(F)$ is countable.

Let $(\overline{K}, \delta) \succeq (K, \delta)$ be $\aleph_1$-saturated. Then $S_1(F) = \{\, \mathrm{tp}(a/F) : a \in \overline{K} \,\}$. By quantifier elimination, we have that $\mathrm{qftp}(q/F) \vdash \mathrm{tp}(a/F)$ for any $a \in \overline{K}$. But $\mathrm{qftp}(a/F) = \mathrm{qftp}_{L_{\mathrm{Ring}}}(a, \delta a, \delta^2 a, \dots /F)$. So it suffices to count $\{\, \mathrm{qftp}_{L_{\mathrm{Ring}}}(a, \delta a, \dots /F) : a \in \overline{K} \,\}$.

Given $a \in \overline{K}$, let
$$n(a/F) = \begin{cases} \text{the least } n < \omega \text{ such that } \delta^n a \in F(a, \dots, \delta^{n-1} a) & \text{such } n \text{ exists} \\ \omega & \text{else} \end{cases}$$

If $n(a/F) = n < \omega$ then set $P_{a/F} \in F[x_0, \dots, x_n]$ such that $P_{a/F}(a, \dots, \delta^{n-1} a, x_n)$ is the miimal polynomial of $\delta^n a$ over $F(a, \dots, \delta^{n-1} a)$.

Suppose $b \in \overline{K}$.

Claim 235. *Suppose $n(a/F) = n(b/F) = n < \omega$ and $P_{a/F} = P_{b/F}$. Then* $\mathrm{qftp}_{L_{\mathrm{Ring}}}(a, \delta a, \dots /F) = \mathrm{qftp}_{L_{\mathrm{Ring}}}(b, \delta b, \dots /F)$.

Proof. Note that $\{\, a, \dots, \delta^{n-1} a \,\}$ and $\{\, b_1, \dots, \delta^{n-1} b \,\}$ are both algebraically independent over F. So we have a field isomorphism $f \colon F(a, \dots, \delta^{n-1} a) \to F(b, \delta b, \dots, \delta^{n-1} b)$ such that $f(\delta^i a) = \delta^i b$ and $f \restriction F = \mathrm{id}_F$. Then

$$\begin{aligned} f(\text{minimal polynomial of } \delta^n a \text{ over } F(a, \dots, \delta^{n-1} a)) &= f(P_{a/F}(a, \dots, \delta^{n-1} a, x_n)) \\ &= P_{a/F}(b, \delta b, \dots, \delta^{n-1} b, x_n) \\ &= P_{b/F}(b, \dots, \delta^{n-1} b, x_n) \\ &= \text{minimal polynomial of } \delta^n b \text{ over } F(b, \dots, \delta^{n-1} b) \end{aligned}$$

Thus we can extend to a field isomorphism $f \colon F(a, \dots, \delta^n a) \to F(b, \dots, \delta^n b)$ with $f(\delta^n a) = \delta^n b$. But by D2 (ii), we have $F(a, \dots, \delta^n a) = F(a, \delta a, \dots)$ and $F(b, \dots, \delta^n b) = F(b, \delta b, \dots)$. So f witnesses $\mathrm{qftp}_{L_{\mathrm{Ring}}}(a, \delta a, \dots /F) = \mathrm{qftp}_{L_{\mathrm{Ring}}}(b, \delta b, \dots /F)$. □ Claim 235

Claim 236. *Suppose $n(a/F) = n(b/F) = \omega$. Then $\mathrm{qftp}_{L_{\mathrm{Ring}}}(a, \delta a, \dots /F) = \mathrm{qftp}_{L_{\mathrm{Ring}}}(b, \delta b, \dots /F)$.*

Proof. Note that $\{ a, \delta a, \dots \}$ and $\{ b, \delta b, \dots \}$ are both algebraically independent over F. So $f \colon F(a, \delta a, \dots) \to F(b, \delta b, \dots)$ given by $f \restriction F = \mathrm{id}_F$ and $f(\delta^i a) = \delta^i b$ is an isomorphism witnessing that $\mathrm{qftp}_{L_{\mathrm{Ring}}}(a, \delta a, \dots /F) = \mathrm{qftp}_{L_{\mathrm{Ring}}}(b, \delta b, \dots /F)$. ☐ Claim 236

So $|S_1(F)| \leq |\{ (n_{a/F}, P_{a/F}) : a \in \overline{K} \}|$. But $n_{a/F} \in \mathbb{N}$ and $P_{a/F} \in F[x_0, \dots, x_n]$; so $|S_1(F)| \leq \aleph_0$. ☐ Theorem 234

So DCF_0 is totally transcendental; so the Morley rank of every definable is ordinal-valued.

We work in a sufficiently saturated $(K, \delta) \models \mathrm{DCF}_0$. Let $C = \{ x \in K : \delta x = 0 \}$ be the field of constants; then C is a definable subset of K.

Claim 237. *C is algebraically closed.*

Proof. By the axioms K is algebraically closed. Suppose $a \in K$ with $a \in C^{\mathrm{alg}}$. Let $P(x)$ be the minimal polynomial of a over C. Then $\delta(P(a)) = 0$. So

$$\frac{\mathrm{d}p}{\mathrm{d}x}(a)\delta a + P^{\delta}(a) = 0$$

But $P^{\delta}(a) = 0$, and $\frac{\mathrm{d}P}{\mathrm{d}x}(a) \neq 0$. So $\delta a = 0$, and $a \in C$. ☐ Claim 237

Claim 238. $\mathrm{MR}(C) = 1$*; in fact, C is a strongly minimal definable set in (K, δ).*

Proof. Suppose $\theta(x)$ is a quantifier-free L-formula such that $\theta(K) \subseteq C$. Replace all occurrences of δx in $\theta(x)$ by 0; we then get $\theta(x) \leftrightarrow \varphi(x) \wedge (\delta x = 0)$ where $\varphi(x)$ is a quantifier-free L_{Ring}-formula. So $\varphi(K)$ is finite or cofinite in K. So $\theta(K) = \varphi(K) \cap C$ is finite or cofinite. ☐ Claim 238

Claim 239. *Let $C_n = \{ x \in K : \delta^n x = 0 \}$; then C_n is a subgrape of K. Then* $\mathrm{MR}(C_n) = n$.

Sketch. C_n is actually closed under multiplication by constants; i.e. C_n is a C-vector subspace of K. But by the theory of linear differential equations, we have that every homogeneous linear differential equation of order n has a fundamental system of solutions $e_1, \dots, e_n$ that are C-linearly independent and such that every other solution is a C-linear combination of these. So $\dim_C(C_n) = n$.

Then the map $C_n \to C^n$ given by $a_1 e_1 + \dots + a_n e_n \mapsto (a_1, \dots, a_n)$ is a vector space isomorphism definable in (K, δ) between sets in (K, δ) definable over $\{ e_1, \dots, e_n \}$. But Morley rank is preserved by definable bijection, and the Morley rank of a product is the sum of the Morley ranks. So $\mathrm{MR}(C_n) = \mathrm{MR}(C^n) = n$. ☐ Claim 239

So $C = C_1 \leq C_2 \leq \dots \leq K$. So $\mathrm{MR}(K) \geq \omega$.